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GROUP ACTIONS ON VON NEUMANN REGULAR RINGS

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Abstract

Let A be a ring with indentity, G a finite group of automorphisms of A. The main result of this paper is that A/A^G is Galois if and only if it is Frobenius and the module ${}_{A*G}A$ (or A_{A*G}) is faithful. Moreover if $|G|$ is invertible the author improves [2, Theorem 8] and [3, Theorem 8].

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Unless otherwise stated, A/B is ring extension with the same identity.

Definition. *An ring extension A/B is separable if the A-bimodule homomorphism μ : $A\otimes_B A\to A, a\otimes b\to ab$, splits.*

It is easy to see that A/B is separable if and only if there exist elements $x_1, x_2,\cdots, x_n,\ y_1, y_2,\cdots, y_n\in A$ such that for $a\in A$

$$\sum_{i=1}^n x_iy_i=1,\quad \text{and}\quad \sum_{i=1}^n ax_i\otimes y_i=\sum_{i=1}^n x_i\otimes y_ia. \qquad (*)$$

We have the following

Lemma 1. *Let A/B be separable, M a (left) A-module.*

(1) *If M is completely reducible as a B-module, then so is M as an A-module.*

(2) *If B is semisimple artinian, so is A.*

Proof. Assume that A/B is separable. By above remark there exists an unique element $\sum\limits_{i=1}^n x_i\otimes y_i\in A\otimes_B A$ such that $(*)$ holds. Let N be an A-submodule of M. Then there is a B -module projection $f: M\to N$ with $f(n)=n,\ n\in N$. Define a map $\tilde f$ as below

$$\tilde f: M\to N,\quad \tilde f(m)=\sum_{i=1}^n x_i f(y_im).$$

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For $a \in A$,

$$\begin{aligned}\tilde{f}(am) &= \sum_{i=1}^{n} x_i f(y_i am) \\ &= \sum_{i=1}^{n} a x_i f(y_i m) \\ &= a f(m).\end{aligned}$$

So $\tilde{f}$ is an A-module homomorphism, and

$$\begin{aligned}\tilde{f}(n) &= \sum_{i=1}^{n} x_i f(y_i n) \\ &= \sum_{i=1}^{n} x_i y_i n = n.\end{aligned}$$

Thus f is an A-module projection of M into N, M_A is completely reducible as an A-module

(2) is obvious by (1).

For von Neumann regular ring, we have

Lemma 2. *Let A/B be separable. If B is von Neumann regular, so is A.*

Proof. It suffice to show that every left A-module is flat. Let (x_i, y_i) satisfy (*), M be a left A-module, $\mu : A \otimes_B M \to M$ the canonical map. Then μ splits as an A-homomorphism, and its inverse is μ^{-1} via $\mu^{-1}(m) = \sum_i x_i \otimes (y_i m)$. Since ${}_BM$ is flat, $A \otimes_B M$ is a flat A-module. So M is flat, and hence A is regular.

Let G be a finite group acting on A as automorphisms,

$$A^G = \{a \in A | g(a) = a, \quad \forall g \in A\},$$

$A * G$ be the trival crossed product of A with G.

Proposition 1. *Let G be a finite group acting on A as automorphisms with $|G|^{-1} \in A$. If A/A^G is separable, then the following are equivalent:*

(1) *A is regular.*

(2) *A^G is regular.*

(3) *$A * G$ is regular.*

Proof. (1) $\Rightarrow$ (3). Since $|G|$ is invertible, $A * G/A$ is separable. Hence $A * G$ is regular by Lemma 2.

(2) $\Rightarrow$ (1). It is clear.

(3) $\Rightarrow$ (2). Since $A * G$ is regular, there exists some $c \in A * G$ such that $tct = t$, where $t = \sum_{g \in G} g$. It follows that $e = tc$ is an idempotent element of $A * G$ such that $e(A * G)e \cong A^G$. Thus A^G is regular.

Notice that the above proposition also holds if we replace regularity by semisimplicity.

Recall that by A/B a Frobenius extension, we mean A is finitely generated projective right B-module, and $A \cong \mathrm{Hom}_{-B}(A, B)$. It is equivalent to the existence of finite pair

$\{x_i, y_i\}$ of elements in A and B-bimodule map $h : A \to B$ such that for any $a \in A$,

$$a = \sum_i x_i h(y_i a) = \sum_i h(a x_i) y_i.$$

$(h; x_i, y_i)$ is called a Frobenius system. In the sequel, we simply say A/B is Frobenius. First we have

Lemma 3.[6, Proposition 2.7] *Let A/B be a Frobenius extension, functors F, G be as below*

$F : A\text{-Mod} \to B\text{-Mod}$, *restriction functor,*

$G : B\text{-mod} \to B\text{-Mod}, \quad N \mapsto A \otimes_B N.$

Then (F, G) is an adjoint pair of functors.

Recall that if finite group G acts on A as automorphisms, then A becomes both an $(A * G, A^G)$-module and an $(A^G, A * G)$-module via

$$a * \sigma \to x = a\sigma(x), \quad x \leftarrow b = xb,$$

$$x \leftarrow a * \sigma = \sigma^{-1}(xa), \quad c \to x = bx,$$

where $a, x \in A, \sigma \in G$ and $b \in A^G$. Let

$$t = \sum_{\sigma \in G} \sigma, \quad B = A^G$$

and

$$[\ ,\ ]: \ A \otimes_B A \to A * G, \quad a \otimes b \mapsto atb,$$

$$(\ ,\ ): \ A \otimes_{A*G} A \to B, \quad a \otimes b \mapsto t(ab).$$

Then we can form a Morita context

$$\{{}_{A*G}A_B, \ {}_BA_{A*G}, \ [\ ,\ ], \ (\ ,\ )\}.$$

Let J and J' denote $\mathrm{Ann}_{A*G-}(A)$ and $\mathrm{Ann}_{-A*G}(A)$, respectely. Now we have

Proposition 2. *The following diagrams are commutative:*

$$\begin{array}{ccc} A \otimes_B A & \xrightarrow{\overline{[\ ,\ ]}} & A * G/J \\ {\scriptstyle 1\otimes\omega}\downarrow & & \downarrow{\scriptstyle \iota} \\ A \otimes_B \mathrm{Hom}_{-B}(A, B) & \xrightarrow{\text{cdno.}} & \mathrm{End}_{-B}(A), \end{array}$$

$$\begin{array}{ccc} A \otimes_B A & \xrightarrow{\overline{[\ ,\ ]}} & A * G/J' \\ {\scriptstyle \omega'\otimes 1}\downarrow & & \downarrow{\scriptstyle \iota'} \\ \mathrm{Hom}_{-B}(A, B) \otimes_B A & \xrightarrow{\text{cdno.}} & \mathrm{End}_{-B}(A), \end{array}$$

where $\omega(a)(x) = (a, x)$, and $\omega'(a)(x) = (x, a)$, $a, x \in A$.

Notice that a composition map of any two maps in the above commutative diagrams is $\mathrm{End}_{-B}(A)$ (or $\mathrm{End}_{B-}(A)$)-linear. Now we can sharpen [4, Theorem 5]. It is easy to prove

Theorem 1. *The following are equivalent:*

(1) A_B *is finite projective and* ω *is surjective.*

(1') ${}_BA$ *is finite projective and* ω' *is surjective.*

(2) $A \otimes_B A \to End_{-B}(A)$, $a \otimes b \longmapsto (x \mapsto at(bx))$, *is surjective.*

(2') $A \otimes_B A \to End_{B-}(A)$, $a \otimes b \longmapsto (x \mapsto t(xa)b)$, *is surjective.*

(3) $A * G = AtA + J$.

(3') $A * G = AtA + J'$.

Furthermore, if any of (,)*'s* (*resp.,*(, ')') *holds, then* $J \supseteq J'$ *and* $\mathrm{End}_{-B}(A) \cong A*G/_J$ (*resp.,* $J' \cong J$ *and* $\mathrm{End}_{B-}(A) \cong A * G/J'$).

Theorem 2. *The following are equivalent:*

(1) A/B *is Frobenius with Frobenius map* t.

(2) $A \otimes_B A \to \mathrm{End}_{-B}(A)$ (*or* $\mathrm{End}_{B-}(A)$) *is bijective.*

(3) $A * G = AtA \oplus J$. *Furthermore, if any of the above conditions holds, then* $J = J'$.

Remark. Let $\Delta = A * G$,

$$I = \sum \{f(A) | f \in \mathrm{Hom}_{-\Delta}(A, \Delta)\}$$

and

$$I' = \sum \{g(A) | g \in \mathrm{Hom}_{\Delta-}(A, \Delta)\}$$

Then both I and I' coincide with AtA . In fact, we have

Lemma 4. *Both of the nature maps*

$$\eta: \ A \to \mathrm{Hom}_{\Delta-}(A, \Delta), \ a \mapsto (a \mapsto xta)$$

and

$$\eta': \ A \to \mathrm{Hom}_{-\Delta}(A, \Delta), \ a \mapsto (a \mapsto atx)$$

are bijective.

Proof. It is easy to see that both η and η' are injective. since $\Delta * G/A$ is a Frobenius extension with Frobenius system $(h; \sigma, \sigma^{-1})_{\sigma \in G}$, where

$$h(\sum a_\sigma \sigma) = a_1,$$

for any $f \in \mathrm{Hom}_{\Delta-}(A, \Delta), \ x \in A$, we have

$$\begin{aligned} f(x) &= \sum_{\sigma \in G} x\sigma h(\sigma^{-1} f(1)) \\ &= \sum_{\sigma \in G} x\sigma h(f(\sigma^{-1} \cdot 1)) \\ &= xth(f(1)) \\ &= \eta(h(f(1))). \end{aligned}$$

So η is surjective. Similarly, η' is surjective.

Recall that A/A^G is Galois if and only if there exist finite pairs x_i, y_i of elements in A such that

$$\sum_i \sigma(x_i) y_i = \begin{cases} 1, & \sigma = 1, \\ 0, & \sigma = 1. \end{cases}$$

The latter is

$$\sum_i x_i t y_i = 1 * 1 \in A * G,$$

$$t = \sum_{\sigma \in G} \sigma,$$

that is, $A * G = AtA$. Thus we have

Theorem 3. *A/A^G is Galois if and only if it is Frobenius and the module ${}_{A*G}A$ (or A_{A*G}) is faithful.*

Proposition 3. *Let A/A^G satisfy any condition of Theorem 1. If there is some $c \in A$ such that $t(c) = 1$, then $A^G \sim A * G/J$ (resp., $A * G/J'$).*

Proof. If $t(c) = 1$ for some $c \in A$, then A^G is a direct summand of A as a left or right A^G-module. It follows that if (,)'s (resp., (, ')'s) of Theorem 1 holds, then A_{A^G} (resp., ${}_{A^G}A$) is a progenerator. In this situation, $A^G \sim A * G/J$ (resp., $A * G/J'$).

Recall that a ring is biregular if any principal ideal is generated by a central idempotent. A is a self-injective ring if its right reglar module A_A is injective. It is well-known that a self-injective biregular ring is a von Neumann regular ring.

Proposition 4. *If $A * G$ is biregular, then A/A^G is a Frobenius extension. Moreover if the trace map is surjective, then A_{A^G} and ${}_{A^G}A$ are progenerators, that is, $A * G/J \sim A^G$.*

Proof. Since $I = AtA$ is a principal ideal of $A * G$, there exists a central idemopent element e in $A * G$ such that $I = e(A * G)$. It follows that

$$\begin{aligned} A * G &= I + (1 - e)(A * G) \\ &= I \oplus J. \end{aligned}$$

So A/A^G is Frobenius by Theorem 2. By Proposition 3, $A^G \sim A * G/J$.

Now we can improve [2, Theorem 8] and [3, Theorem 8]. We have

Theorem 4. *Let A be regular in the sence of von Neumann, G a finite group acting on A as automorphisms with $|G|^{-1} \in A$. Then the following are equivalent:*

(1) *A is biregular self-injective.*

(2) *A^G is biregular self-injective and A/A^G is Frobenius.*

(3) *$A * G$ is biregular self-injective.*

Proof. (1) $\Rightarrow$(3). By [2, Corollary 2].

(3) $\Rightarrow$ (2). A/A^G is Frobenius by Proposition 4. Since $A * G$ is regular, there exists an element

$$x = \sum_{\sigma \in G} a_\sigma \sigma \in A * G$$

such that $txt = t$. Let $c = \sum a_\sigma$. Then

$$t = txt = tct = t(c)t,$$

and hence $t(c) = 1$. It follows that $A * G/J \sim A^G$ by Proposition 4. But $A * G = I \oplus J$, and hence $A * G/J \simeq I$ is an injective $A * G$-module. So $A * G/J$ is self-injective, and hence A^G is self-injective. On the other hand, tc is an idempotent element of $A * G$ such that

$$A^G \cong tc(A * G)tc.$$

It follows that A^G is biregular.

(2) $\Rightarrow$ (1). Since A/A^G is Frobenius, A is self-injective by Lemma 1. Now (1) holds by [2, Theorem 8].

Now if G is finite with $|G|^{-1} \in A$, the converse of [3, Theorem 8] is true.

Corollary. *Let A be a commutative ring, G a finite group of automorphisms of A with $|G|^{-1} \in A$. Then the following are equivalent:*

(1) *A is self-injective von Neumann regular.*

(2) *A^G is self-injective von Neumann regular and A/A^G is Frobenius and separable.*

(3) *$A * G$ is self-injective biregular.*

Proof. Since A is commutative, A is biregular if A is regular. Now it is sufficient to show that A/A^G is separable if $A * G$ is biregular. But this is true by [1, Prosition 4].

Notice that the above corollary also holds if we replace self-injective regular (or biregular) by semisimple artinian.

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