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AN INVARIANCE OF CDF EQUATION**

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Abstract

This paper presents a new invariance for the CDF equation. Using this invariance, the author obtains some new solutions of CDF equation.

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CDF equation[3,4,5]

$$\phi_t = \phi_{xxx} + 2\phi_x^3 + 6\eta^2\phi_x \sin 2\phi \tag{1}$$

(η is an arbitrary constant) is related to the MKdV equation

$$q_t = q_{xxx} + 6q^2 q_x. \tag{2}$$

There is a transformation

$$q = -(\phi_x + \eta \sin 2\phi) \tag{3}$$

between (1) and (2).In fact,subtituting (3) into (2), we have

$$q_t - (q_{xxx} + 6q^2 q_x) = -(\partial_x + 2\eta \cos 2\phi)(\phi_t - (\phi_{xxx} + 2\phi_x^3 + 6\eta^2 \phi_x \sin^2 2\phi)).$$

Since (1) is invariant when we change (ϕ, η) into $(-\phi, -\eta)$, there are two transformations between (1) and (2):

$$\mu_1 : \phi \longrightarrow q = -(\phi_x + \eta \sin 2\phi),$$

$$\mu_2 : \phi \longrightarrow q = \phi_x - \eta \sin 2\phi.$$

In the following, $\int f dx$ (or $\int f dt$) means an arbitrary primitive function of f and it is taken definitely.

Lemma. *If ϕ is a solution of* (1), *then*

$$c \equiv -\frac{1}{2}\Big(\int \cos 2\phi dx\Big)_t - \phi_{xx} \sin 2\phi + \phi_x^2 \cos 2\phi + 3\eta^2 \cos 2\phi - \eta^2 \cos^3 2\phi$$

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and

$$\begin{aligned}h \equiv& 2\eta\Big(\int \sin 2\phi e^{-2\eta\int\cos 2\phi dx}dx\Big)_t - 8\eta c\int \sin 2\phi e^{-2\eta\int\cos 2\phi dx}dx\\&-4\eta(\phi_{xx}\cos 2\phi + \phi_x^2\sin 2\phi + 2\eta\phi_x + 2\eta^2\sin 2\phi + \eta^2\sin^3 2\phi)e^{-2\eta\int\cos 2\phi dx}\end{aligned}$$

are the functions of t *only (i.e.,* $c_x = 0 = h_x$*).*

Proof. When ϕ satisfies (1),

$$\begin{aligned}c_x =& \phi_t\sin 2\phi - \phi_{xxx}\sin 2\phi - 2\phi_x^3\sin 2\phi\\&-6\eta^2\phi_x\sin 2\phi + 6\eta^2\phi_x\cos^2 2\phi\sin 2\phi\\=&\sin 2\phi(\phi_t - \phi_{xxx} - 2\phi_x^3 - 6\eta^2\phi_x\sin^2 2\phi) = 0.\end{aligned}$$

$$\begin{aligned}h_x =& -4\eta(\phi_{xxx}\cos 2\phi + 2\phi_x^3\cos 2\phi + 2\eta\phi_{xx}\\&+4\eta^2\phi_x\cos 2\phi + 6\eta^2\phi_x\sin^2 2\phi\cos 2\phi)e^{-2\eta\int\cos 2\phi dx}\\&-2\eta\cos 2\phi(-4\eta\phi_{xx}\cos 2\phi - 4\eta\phi_x^2\sin 2\phi\\&-8\eta^2\phi_x - 8\eta^3\sin 2\phi - 4\eta^3\sin^3 2\phi)e^{-2\eta\int\cos 2\phi dx}\\&-8\eta^2 c\sin 2\phi e^{-2\eta\int\cos 2\phi dx} + 4\eta\phi_t\cos 2\phi e^{-2\eta\int\cos 2\phi dx}\\&-4\eta^2\Big(\int\cos 2\phi dx\Big)_t\sin 2\phi e^{-2\eta\int\cos 2\phi dx}\\=&4\eta\cos 2\phi(\phi_t - \phi_{xxx} - 2\phi_x^3 - 6\eta^2\phi_x\sin^2 2\phi)e^{-2\eta\int\cos 2\phi dx}\\&-8\eta^2\sin 2\phi\Big(c + \phi_{xx}\sin 2\phi - \phi_x^2\cos 2\phi\\&-2\eta^2\cos 2\phi - \eta^2\cos 2\phi\sin^2 2\phi + \frac{1}{2}\Big(\int\cos 2\phi dx\Big)_t\Big)e^{-2\eta\int\cos 2\phi dx} = 0.\end{aligned}$$

Theorem. *If* ϕ *is a solution of* (1), *then*

$$\overline{\phi} = \arctan(\tan\phi + y) \tag{4}$$

is a solution of (1) *as well,where*

$$y = \sec^2\phi e^{-2\eta\int\cos 2\phi dx}/Q, \tag{5}$$

$$Q = -\tan\phi e^{-2\eta\int\cos 2\phi dx} - 2\eta\int\sin 2\phi e^{-2\eta\int\cos 2\phi dx}dx + \epsilon, \tag{6}$$

$$\epsilon = e^{4\eta\int c(t)dt}\Big(\int h(t)e^{-4\eta\int c(t)dt}dt + \alpha\Big), \tag{7}$$

α *is an arbitrary constant (i.e.,* $\epsilon' = 4\eta c(t)\epsilon + h(t)$*).*

Proof. We check

$$\overline{\phi}_t = \overline{\phi}_{xxx} + 2\overline{\phi}_x^3 + 6\eta^2\overline{\phi}_x\sin^2 2\overline{\phi} \tag{8}$$

directly. From (4)-(6),

$$\begin{aligned}
\overline{\phi}_x \sec^2\overline{\phi} &= \phi_x\sec^2\phi + y_x,\\
y_x &= (2(\phi_x+\eta\sin 2\phi)\tan\phi - 2\eta)y + (\phi_x+\eta\sin 2\phi)y^2,\\
\sec^2\overline{\phi}(\overline{\phi}_x+\eta\sin 2\overline{\phi}) &= (\tan\overline{\phi})_x + 2\eta\tan\overline{\phi},\\
&= (\tan\phi)_x + y_x + 2\eta(\tan\phi + y)\\
&= (\tan\phi)_x + 2\eta\tan\phi + 2\tan\phi(\phi_x+\eta\sin 2\phi)y + (\phi_x+\eta\sin 2\phi)y^2\\
&= (1+\tan^2\phi + 2y\tan\phi + y^2)(\phi_x+\eta\sin 2\phi)\\
&= (1+\tan^2\overline{\phi})(\phi_x+\eta\sin 2\phi)\\
&= \sec^2\overline{\phi}(\phi_x+\eta\sin 2\phi);
\end{aligned}$$

we have

$$\overline{\phi}_x = \phi_x + \eta\sin 2\phi - \eta\sin 2\overline{\phi}, \tag{9}$$

$$\begin{aligned}
\overline{\phi}_{xx} =& \phi_{xx} + 2\eta\phi_x\cos 2\phi - 2\eta\overline{\phi}_x\cos 2\overline{\phi},\\
\overline{\phi}_{xxx} =& \phi_{xxx} + 2\eta\phi_{xx}\cos 2\phi - 4\eta\phi_x^2\sin 2\phi\\
&- 2\eta(\phi_{xx} + 2\eta\phi_x\cos 2\phi - 2\eta\overline{\phi}_x\cos 2\overline{\phi})\cos 2\overline{\phi} + 4\eta^2\overline{\phi}_x^2\sin 2\overline{\phi},
\end{aligned}$$

$$\begin{aligned}
&2(\overline{\phi}_x^3 + 3\eta^2\overline{\phi}_x^2\sin^2 2\overline{\phi})\\
=&2(\phi_x^3 + 3\eta^2\phi_x^2\sin^2 2\phi) + 6\eta\phi_x^2\sin 2\phi + 2\eta^3\sin^3 2\phi - 6\eta\overline{\phi}_x^2\sin 2\overline{\phi} - 2\eta^3\sin^3 2\overline{\phi}.
\end{aligned}$$

Then

$$\overline{\phi}_{xxx} + 2(\overline{\phi}_x^3 + 3\eta^2\overline{\phi}_x\sin^2 2\overline{\phi}) = \phi_t + R,$$

where

$$\begin{aligned}
R =& 2\eta\phi_{xx}(\cos 2\phi - \cos 2\overline{\phi}) + 2\eta(\phi_x^2\sin 2\phi - \overline{\phi}_x^2\sin 2\overline{\phi})\\
&- 4\eta^2(\phi_x\cos 2\phi - \overline{\phi}_x\cos 2\overline{\phi})\cos 2\overline{\phi} + 2\eta^3(\sin^3 2\phi - \sin^3 2\overline{\phi})\\
=& 2\eta\phi_{xx}(\cos 2\phi - \cos 2\overline{\phi}) + 2\eta\sin 2\phi(\phi_x^2 + \eta^2\sin^2 2\phi)\\
&- 2\eta\sin 2\overline{\phi}(\overline{\phi}_x^2 + \eta^2\sin^2 2\overline{\phi}) - 4\eta^2(\phi_x\cos 2\phi - \overline{\phi}_x\cos 2\overline{\phi})\cos 2\overline{\phi}\\
=& 2\eta\phi_{xx}(\cos 2\phi - \cos 2\overline{\phi}) + 2\eta\sin 2\phi(\phi_x^2 + \eta^2\sin^2 2\phi)\\
&- 2\eta\sin 2\overline{\phi}(\phi_x^2 + \eta^2\sin^2 2\phi + 2\eta\phi_x\sin 2\phi - 2\eta\overline{\phi}_x\sin 2\overline{\phi})\\
&- 4\eta^2(\phi_x\cos 2\phi - \overline{\phi}_x\cos 2\overline{\phi})\cos 2\overline{\phi}\\
=& 2\eta\phi_{xx}(\cos 2\phi - \cos 2\overline{\phi}) + 2\eta(\phi_x^2 + \eta^2\sin^2 2\phi)(\sin 2\phi - \sin 2\overline{\phi})\\
&+ 4\eta^2\overline{\phi}_x - 4\eta^2\phi_x(\sin 2\phi\sin 2\overline{\phi} + \cos 2\phi\cos 2\overline{\phi})\\
=& 2\eta(\phi_{xx} + 2\eta\phi_x\cos 2\phi)(\cos 2\phi - \cos 2\overline{\phi})\\
&+ 2\eta(\phi_x^2 + 2\eta\phi_x\sin 2\phi + \eta^2\sin^2 2\phi + 2\eta^2)(\sin 2\phi - \sin 2\overline{\phi}).
\end{aligned}$$

Therefore, we only need to prove

$$\overline{\phi}_t = \phi_t + R. \tag{10}$$

Since

$$\overline{\phi}_t = \phi_t \sec^2 \phi / \sec^2 \overline{\phi} + y_t / \sec^2 \overline{\phi},$$

$$\sec^2 \overline{\phi} = 1 + (\tan \phi + y)^2 = sec^2\phi + 2y \tan \phi + y^2,$$

(10) is established if and only if

$$\phi_t \sec^2 \phi + y_t = \phi_t \sec^2 \overline{\phi} + R \sec^2 \overline{\phi},$$

i.e.,

$$-(2y \tan \phi + y^2)\phi_t + y_t = R \sec^2 \overline{\phi}. \tag{11}$$

From (4)-(6),

$$\begin{aligned} y_t &= 2\phi_t \tan \phi \sec^2 \phi e^{-2\eta \int \cos 2\phi dx} / Q - 2\eta y \Big(\int \cos 2\phi dx \Big)_t - y Q_t / Q \\ &= 2y\phi_t \tan \phi - 2\eta y \Big(\int \cos 2\phi dx \Big)_t - y Q_t / Q, \end{aligned}$$

$$\begin{aligned} Q_t =& \Big(-\phi_t \sec^2 \phi + 2\eta \tan \phi \Big(\int \cos 2\phi dx \Big)_t \Big) e^{-2\eta \int \cos 2\phi dx} \\ &- 2\eta \Big(\int \sin 2\phi e^{-2\eta \int \cos 2\phi dx} dx \Big)_t + \epsilon'; \end{aligned}$$

we have

$$\begin{aligned} y_t =& 2y\phi_t \tan \phi - 2\eta y \Big(\int \cos 2\phi dx \Big)_t + y^2 \phi_t \\ &- 2\eta y \tan \phi \Big(\int \cos 2\phi dx \Big)_t e^{-2\eta \int \cos 2\phi dx} / Q \\ &+ 2\eta y \Big(\int \sin 2\phi e^{-2\eta \cos 2\phi dx} dx \Big)_t \Big/ Q - y\epsilon' / Q \\ =& \phi_t (2y \tan \phi + y^2) - 2\eta y \Big(\int \cos 2\phi dx \Big)_t - \eta y^2 \sin 2\phi \Big(\int cos 2\phi dx \Big)_t \\ &+ 2\eta y \Big(\int \sin 2\phi e^{-2\eta \int \cos 2\phi dx} dx \Big)_t \Big/ Q - y\epsilon' / Q. \end{aligned}$$

Then (11) is reduced to

$$\begin{aligned} &- 2\eta y \Big(\int \cos 2\phi dx \Big)_t - \eta y^2 \sin 2\phi \Big(\int \cos 2\phi dx \Big)_t \\ &+ 2\eta y \Big(\int \sin 2\phi e^{-2\eta \int \cos 2\phi dx} dx \Big)_t \Big/ Q - y\epsilon' / Q = R \sec^2 \overline{\phi}. \end{aligned} \tag{12}$$

Substituting

$$\cos 2\phi - \cos 2\overline{\phi} = 2 \cos^2 \phi - 2 \cos^2 \overline{\phi} = 2y(2 \tan \phi + y) / \sec^2 \phi \sec^2 \overline{\phi},$$

$$\begin{aligned} \sin 2\phi - \sin 2\overline{\phi} &= 2 \tan \phi / \sec^2 \phi - 2(\tan \phi + y) / \sec^2 \overline{\phi} \\ &= 2y(\tan^2 \phi + y \tan \phi - 1) / \sec^2 \phi \sec^2 \overline{\phi} \end{aligned}$$

into R, (12) is reduced to

$$\begin{aligned}
&-2\eta\Big(\int\cos 2\phi dx\Big)_t-\eta y\sin 2\phi\Big(\int\cos 2\phi dx\Big)_t\\
&+2\eta\Big(\int\sin 2\phi e^{-2\eta\int\sin 2\phi dx}dx\Big)_t\Big/Q-\epsilon'/Q\\
=&4\eta\cos^2\phi((\phi_x^2+2\eta\phi_x\sin 2\phi+\eta^2\sin^2 2\phi+2\eta^2)(\tan^2\phi+y\tan\phi-1)\\
&+(\phi_{xx}+2\eta\phi_x\cos 2\phi)(2\tan\phi+y))
\end{aligned}\tag{13}$$

and

$$\begin{aligned}
&-2\eta\Big(\int\cos 2\phi dx\Big)_t-8\eta\cos^2\phi(\phi_{xx}+2\eta\phi_x\cos 2\phi)\tan\phi\\
&-4\eta\cos^2\phi(\phi_x^2+2\eta\phi_x\sin 2\phi+\eta^2\sin^2 2\phi+2\eta^2)(\tan^2\phi-1)\\
=&4\eta(\phi_{xx}\sin 2\phi-\phi_x^2\cos 2\phi-3\eta^2\cos 2\phi+\eta^2\cos^3 2\phi+c(t))\\
&-4\eta\phi_{xx}\sin 2\phi-8\eta^2\phi_x\cos 2\phi\sin 2\phi+4\eta\phi_x^2\cos 2\phi\\
&+8\eta^2\phi_x\sin 2\phi\cos 2\phi+4\eta^3\sin^3 2\phi\cos 2\phi+8\eta^3\cos 2\phi\\
=&c(t),
\end{aligned}$$

$$\begin{aligned}
&-\eta\sin 2\phi\Big(\int\cos 2\phi dx\Big)_t-4\eta\cos^2\phi(\phi_{xx}+2\eta\phi_x\cos 2\phi)\\
&-4\eta\cos^2\phi\tan\phi(\phi_x^2+2\eta\phi_x\sin 2\phi+\eta^2\sin^2 2\phi+2\eta^2)\\
=&2\eta\sin 2\phi(\phi_{xx}\sin 2\phi-\phi_x^2\cos 2\phi-3\eta^2\cos 2\phi+\eta^2\cos^3\cos 2\phi+c(t))\\
&-2\eta\sin 2\phi(\phi_x^2+2\eta\phi_x\sin 2\phi+\eta^2\sin^2 2\phi+2\eta^2)\\
&-4\eta\cos^2\phi(\phi_{xx}+2\eta\phi_x\cos 2\phi)\\
=&\cos^2\phi(-4\eta\phi_{xx}\cos 2\phi-4\eta\phi_x^2\sin 2\phi-8\eta^2\phi_x\\
&-4\eta^3\tan\phi(2+3\cos 2\phi+\sin^2 2\phi-3\cos^2 2\phi)+4\eta c(t)\tan\phi)\\
=&H\cos^2\phi,
\end{aligned}$$

where

$$\begin{aligned}
H\equiv&-4\eta\phi_{xx}\cos 2\phi-4\eta\phi_x^2\sin 2\phi-8\eta^2\phi_x\\
&-4\eta^3(2\sin 2\phi+\sin^3 2\phi)+4\eta c(t)\tan\phi.
\end{aligned}$$

Therefore, (13) is reduced to

$$4\eta c(t)+yH\cos^2\phi+2\eta\Big(\int\sin 2\phi e^{-2\eta\int\cos 2\phi dx}dx\Big)_t\Big/Q-\epsilon'/Q=0,$$

i.e.,

$$4\eta c(t)+He^{-2\eta\int\cos 2\phi dx}/Q+2\eta\Big(\int\sin 2\phi e^{-2\eta\int\cos 2\phi dx}dx\Big)_t\Big/Q-\epsilon'/Q.\tag{14}$$

Substituting (6) into (14), we have

$$\begin{aligned}
&(H-4\eta c(t)\tan\phi)e^{-2\eta\int\cos 2\phi dx}-8\eta c(t)\int\sin 2\phi e^{-2\eta\cos 2\phi dx}dx\\
&+4\eta c(t)\epsilon+2\eta\Big(\int\sin 2\phi e^{-2\eta\cos 2\phi dx}dx\Big)_t-\epsilon'=o,
\end{aligned}$$

i.e., $\epsilon' - 4\eta c(t)\epsilon + h(t) = 0$. The theorem is proved.

According to this theorem, we can obtain the new solution $\overline{\phi}$ from a known solution ϕ by quadrature for the CDF equation.

Example 1. We take the trivial solution $\phi = 0$ and

$$\int \cos 2\phi dx = x, \int \sin 2\phi e^{-2\eta \int \cos 2\phi dx} dx = 0.$$

Then $c = 2\eta^2$, $h = 0$, $\epsilon = \alpha e^{8\eta^3 t} = e^{8\eta^3 t - \delta}$, $y = e^{-(2\eta x + 8\eta^3 t - \delta)}$ (α and δ are arbitrary constants), and we obtain the solution $\overline{\phi} = \arctan e^{-\theta_1}$ ($\theta_1 = 2\eta x + 8\eta^3 t - \delta$) and the corresponding solutions of MKdV equation (2):

$$\overline{q}_1 = \pm(\overline{\phi}_x + \eta \sin 2\overline{\phi}) = 0, \quad \overline{q}_2 = \pm(\overline{\phi}_x - \eta \sin 2\overline{\phi}) = \mp 2\eta \mathrm{sh}\theta_1.$$

Example 2. We take the solution $\phi = \arctan e^{\theta}$, $\theta = 2\eta x + 8\eta^3 t + \delta$ (η, δ are arbitrary constants), and

$$\int \cos 2\phi dx = -\frac{1}{2\eta} \log \mathrm{ch}\theta, \quad \int \sin 2\phi e^{-2\eta \int \cos 2\phi dx} dx = x.$$

Then

$$c = 0, h = -24\eta^3, \quad \epsilon = -24\eta^3 t + \alpha, \quad Q = e^{-\theta}\mathrm{ch}\theta + \omega$$

($\omega = -2\eta x - 24\eta^3 t + \alpha$, α is an arbitrary constant), and we obtain the solution

$$\overline{\phi} = \arctan\Big(\frac{\mathrm{ch}\theta + \omega}{-e^{\theta}\mathrm{ch}\theta + \omega}\Big).$$

Example 3. We take the trivial solution $\phi = \frac{\pi}{4}$ and

$$\int \cos 2\phi dx = 0, \int \sin 2\phi e^{-2\eta \int \cos 2\phi dx} dx = x.$$

Then $c = 0$, $h = -12\eta^3$. We obtain the solution $\phi = \arctan(1 - \frac{1}{\omega})$ ($\omega = \eta x + 6\eta^3 t + \delta$, δ is an arbitrary constant) and the corresponding solutions of MKdV equation

$$\overline{q}_1 = \pm(\overline{\phi}_x + \eta \sin 2\overline{\phi}) = \pm\eta, \quad \overline{q}_2 = \pm(\overline{\phi}_x + \eta \sin 2\overline{\phi}) = \pm(1 - \frac{1}{H})$$

($H = \omega^2 - \omega + \frac{1}{2}$).

Example 4. We take $\phi = \phi_0$ (ϕ_0 ($\neq n\pi \pm \pi/4$) is an arbitrary constant) and

$$\int \cos 2\phi_o dx = x \cos 2\phi_0,$$

$$\int \sin 2\phi_o e^{-2\eta x \cos 2\phi_o} dx = -\frac{1}{2\eta} \tan 2\phi_o e^{-2\eta x \cos 2\phi_0}.$$

Then $c = \eta^2(2\cos 2\phi_0 - \cos^2 2\phi_0)$, $h = 0$ and

$$\begin{aligned} \epsilon =& \alpha e^{4\eta^3(3\cos 2\phi_0 - \cos^3 2\phi_0)t} = e^{4\eta^3(3\cos 2\phi_0 - \cos^3 2\phi_0)t + \delta}, \\ Q =& \frac{1}{2}\sec^2\phi_0 \tan 2\phi_0 e^{-2\eta x \cos 2\phi_0} + \epsilon, \\ y =& 1/\Big(\frac{1}{2}\tan 2\phi_0 + e^{\theta}\Big) \end{aligned}$$

($\theta = 2\eta x \cos 2\phi_0 + 8\eta^3(3\cos 2\phi_0 - \cos^3 2\phi_0)t + \delta$). Therefore, we obtain the solution

$$\overline{\phi} = \arctan\Big(\tan\phi_0 + 1\Big/\Big(\frac{1}{2}\tan 2\phi_0 + e^{\theta}\Big)\Big)$$

and the corresponding solutions of MKdV equation:

$$\begin{aligned}\overline{q}_1 &= \pm(\overline{\phi}_x + \eta\sin 2\overline{\phi}) = \pm\eta\sin 2\phi_0,\\ \overline{q}_2 &= \pm(\overline{\phi}_x - \eta\sin 2\overline{\phi})\\ &= \frac{\mp(\eta\sin 2\phi_0 + 4\eta\cos 2\phi_0 e^{\theta})}{\left(2\tan\phi_0 + \sec 2\phi_0 + \sec^2\phi_0\left(\frac{1}{2}\tan 2\phi_0 + e^{\theta}\right)^2\right)}.\end{aligned}$$

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