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SOME CONTRACTIONS OF THREEFOLD ALGEBRAIC FAMILY**

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Abstract

The objects in this paper are all projective 3-folds over an algebraically closed field of characteristic 0. After simply generalizing the Rationality theorem, a kind of contractions of non-minimal 3-folds is given

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§1. Known Results

Theorem M1. *Let X be a non-singular projective 3-fold with an ample divisor L such that K_X is not nef. If X has an extremal ray $R = \mathbf{R}_+[l]$ which is generated by an extremal rational curve l,then*

(1) *there exists a morphism $\Phi : X \longrightarrow Y$ to a projective variety Y such that $\Phi_* O_X = O_Y$ and for any irreducible curve C in $X, [C] \in R$ iff $\dim\Phi(C) = 0$;*

(2) *there exists an exact sequence*

$$0 \longmapsto Pic\mathbf{Y} \overset{\Phi^*}{\longmapsto} Pic\mathbf{X} \overset{(\ .l)}{\longmapsto} \mathbf{Z},$$

$-K_X$ is Φ-ample;

(3) *if R is not nef, then there exists a divisor D such that $\Phi|_{X-D}$ is an isomorphism and* $\dim\Phi(D) \leq 1$ *and we have five types:*

(b$_1$) *$\Phi(D)$ is a non-singular curve and Y is non-singular, $\Phi|_D : D \longmapsto \Phi(D)$ is a $\mathbf{P}^1-$bundle;*

(b$_2$) *$\Phi(D)$ is a point and Y is non-singular, $D \cong \mathbf{P}^2$ and $\mathbf{O}_D(D) \cong \mathbf{O}_\mathbf{P}(-1)$;*

(b$_3$) *$\Phi(D)$ is a point, $D \cong \mathbf{P}^1 \times \mathbf{P}^1$,$\mathbf{O}_D(D)$ is of bidegree $(-1,-1)$ and $s \times \mathbf{P}^1 \approx \mathbf{P}^1 \times t$ on X,$s,t \in \mathbf{P}^1$;*

(b$_4$) *$\Phi(D)$ is a point, $D \cong$ an irreducible, reduced and quadric surface in $\mathbf{P}^3$,$\mathbf{O}_D(D) \cong \mathbf{O}_D \otimes \mathbf{O}_\mathbf{P}(-1)$;*

(b$_5$) *$\Phi(D)$ is a point, $D \cong \mathbf{P}^2$ and $\mathbf{O}_D(D) \cong \mathbf{O}_\mathbf{P}(-2)$;*

(4) *If R is nef, then Y is non-singular and we have three types:*

(c$_1$) $\dim Y = 2$ *and for any point P of Y, X_P is isomorphic to a conic of $\mathbf{P}^2$(X is called a conic bundle*);

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(c_2) $\dim Y = 1$ *and for any point* P *of* Y, X_P *is an irreducible, reduced surface such that* $\omega_{X_P}^{-1}$ *is ample* (X *is called a del pezzo fibre space*);

(c_3) $\dim Y = 0$, $\rho(X) = 1$, $-K_X$ *is ample* (X *is called a Fano-variety*).

Rationality Theorem. *Suppose that* X *is a projective variety with only canonical singularities on which* K_X *is not nef. Let* H *be an ample divisor on* X. *Then* $\mu(H)$ *is a rational number of the form* $\frac{u}{v}$, *where* $0 < v \leq$ (*index of* X)($\dim X + 1$).

§2. Rationality and Contractions

Let X be a normal projective 3-fold over an algebraically closed field k of characteristic 0. Let K_X be the canonical divisor of X.

$$N(X) = (\{1 - \text{cycles on } X\}/\equiv) \otimes_{\mathbf{Z}} \mathbf{R},$$

where " $\equiv$ " denotes numerical equivalence. Let $NE(X) \subset N(X)$ be the smallest convex cone containing all the effective 1-cycles. Via the intersection pair (.) of 1-cycles and Cartier divisors, $N(X)$ is dual to $NS(X) \otimes_{\mathbf{Z}} \mathbf{R} = N(X)^*$. $\overline{NE}(X)$ is the closure of $NE(X)$ for metrix topology. Let $D(X) \subset N(X)^*$ be the cone generated by all the effective divisors of X, and $\overline{D}(X)$ its closure in $N(X)^*$.

Denifition 2.1. *A divisor* E *is called pseudo-effective if* $E \in \overline{D}(X)$. *For* $D \in N(X)^*$, *we say that* D *is nef if* $D.Z \geq 0$ *for all* $Z \in NE(X)$.

It is easy to show that E is pseudo-effective if E is nef.

Definition 2.2. *For any nef* $\mathbf{Q}$*-divisor* H, *we define*

$$\mu(H) = \sup\{t | t \in \mathbf{R}, H_t = H + tK_X \quad \textit{is nef}\}.$$

Definition 2.3. *A* $\mathbf{Q}$*-divisor* D *on* X *is called ample if some multiple of it is an irreducible, reduced very ample divisor.*

Kieiman's criterion for ampleness is well-known: D is ample iff $D.Z > 0$ for all $Z \in \overline{NE}(X) - 0$.

Definition 2.4. *A divisor* H *on* X *is called Gnef(or good nef) if* $H^{\perp} \cap K_X^{\perp} \cap \overline{NE}(X) = 0$ *and* $\mu(H) > 0$.

Proposition 2.1. *For a nef divisor* H, *if* H *is Gnef, then* $H + qK_X$ *is an ample* $\mathbf{Q}$*-divisor for all rational number* $q \in (0, \mu(H))$. *Conversely if there is a positive rational number* q *such that* $H + qK_X$ *is ample, then* H *is Gnef.*

Proof. If H is Gnef, then for any rational number $q \in (0, \mu(H))$, $H + qK_X$ is nef. If there exists an element $Z \in \overline{NE}(X) - 0$ such that $(H + qK_X) \cdot Z = 0$, then $H \cdot Z = -qK_X \cdot Z > 0$. Thus we have

$$(H + \mu(H)K_X) \cdot Z = (\mu(H) - q)K_X \cdot Z < 0,$$

this is impossible. Hence $H + qK_X$ is ample.

The converse part is obvious.

Theorem 2.1. *Suppose that* X *is a projective variety with only canonical singularities on which* K_X *is not nef. If* X *admits a Gnef Cartier divisor* H, *then* $\mu(H)$ *is a positive* *rational number of the form*

$$\mu(H) = \frac{1}{n_0}\Big[1+\frac{1}{I}\frac{u(n_0)}{v(n_0)}\Big] = \frac{1}{n_0+1}\Big[1+\frac{1}{I}\frac{u(n_0+1)}{v(n_0+1)}\Big]$$
$$= \cdots = \frac{1}{n_0+k}\Big[1+\frac{1}{I}\frac{u(n_0+k)}{v(n_0+k)}\Big] = \cdots,$$

where $\{u(n)\}$ *and* $\{v(n)\}$ *are both sequences of positive integers,* I *=canonical index of* X*,* $0 < v(n) \le I(\dim X+1)$*,* $\lim_{n\to\infty} u(n) = +\infty$.

Proof. Let $\epsilon = \frac{1}{n_0} < \mu$. From Proposition 2.1, we know that $H+\epsilon K_X$ is an ample $\mathbf{Q}$-divisor.

$$H+\mu K_X = (H+\frac{1}{n_0}K_X)+(\mu-\frac{1}{n_0})K_X.$$

Let $H' = n_0 I(H+\epsilon K_X) = In_0H + IK_X$. Then H' is an ample Cartier divisor.

$$n_0 I(H+\mu K_X) = H' + n_0 I(\mu - \frac{1}{n_0})K_X.$$

Thus we get $\mu(H') = n_0 I(\mu-\frac{1}{n_0})$. On the other hand, we know from Rationality Theorem that

$$\mu(H') = \frac{u(n_0)}{v(n_0)},\quad 0 < v(n_0) \le I(\dim X+1).$$

Therefore $u(n_0) = I(n_0\mu - 1)v(n_0)$,

$$I(n_0\mu-1) \le u(n_0) \le I^2(\dim X+1)(n_0\mu-1).$$

For any $n > n_0$, we obtain $u(n)$ and $v(n)$ in a similar way and

$$u(n) = I(n\mu-1)v(n),\quad \mu = \frac{1}{n}\Big[1+\frac{1}{I}\frac{u(n)}{v(n)}\Big].$$

As a simple application of proposition 2.1, Corollary 2.1 is a minor generalization of Rationality Theorem of V. V. Batyrev.

Theorem. *Let* X *be a projective* QFT*-threefold such that* K_X *is not pseudo-effective and* H *is an ample Cartier divisor,*

$$\sigma_X(H) = \sup\{t \in \mathbf{R} | H_t = H + tK_X \in \overline{D}(X)\}.$$

Then $\sigma_X(H)$ *is a rational number.*

Corollary 2.1. *Let* X *be a projective* QFT*-threefold such that* K_X *is not pseudo-effective. If* H *is a Gnef divisor, then* $\sigma_X(H)$ *is a positive rational number.*

Proof. Take a rational number $q \in (0, \mu(H))$. Then $H+qK_X$ is an ample $\mathbf{Q}$-divisor. From the Rationality Theorem of Batyrev, we deduce that $\sigma_X(H+qK_X)$ is rartional. Then $\sigma_X(H) = q + \sigma_X(H+qK_X)$ is rational too.

Remark 2.1. According to the results of Y. Kawamata, we know that under the condition as in Theorem 2.1, there exists a curve $C \subset X$ such that $(H+\mu(H)K_X).C = 0$.

From now on we take a partly view of 3-fold with negative Kodaira dimension.

Remark 2.2. Let X be a projective QFT-threefold. Then $\kappa(X) = -\infty \Longleftrightarrow K_X$ is not pseudo-effective $\Longleftrightarrow X$ is uniruled.

If X is a non-singular projective 3-fold on which K_X is not pseudo-effective, $\rho(X) = 2$, then X admits at least one extremal ray R. Therefore by Theorem M1 we can get an elementary contraction ϕ related to R, $\phi = Cont_R : X \longmapsto Y$, Y is a projective variety. If $\dim Y = 3$, then Y is a Fano 3-fold; if $\dim Y = 2$, then ϕ is a conic bundle (c_1 type); if $\dim Y = 1$, then ϕ is a del pezzo fibre space (c_2 type). In general, X admits a good structure.

Theorem 2.2. *Let X be a non-singular projective 3-fold, $\kappa(X) = -\infty$, $\rho(X) = 3$. If X admits a Gnef divisor H such that $\mu(H)$ is non-integral, then there exists a contraction $\phi : X \longmapsto Y$, ϕ has three types:*

(1) Y is non-singular projective, $\dim Y = 3$, $\rho(Y) = 2$, so the type of Y is clear according to the arguments in above. ϕ is just blowing down a plane $\mathbf{P}^2$;

(2) Y is a non-singular surface and ϕ is a conic bundle;

(3) Y is a non-singular curve and ϕ is a del pezzo fibre space.

From Theorem M1, we obtain the following datum after calculation.

Lemma 2.1. *In the situation (3) in Theorem M1, l is the general extremal curve such that $R = \mathbf{R}_+[l]$, and we have*

(b_1) $K_X = \Phi^*K_Y + D$, $K_X.l = D.l = -1$, $D^3 = 2(1 - g(C))$;

(b_2) $K_X = \Phi^*K_Y + 2D$, $K_X.l = -2$, $D.l = -1$, $D^3 = 1$;

(b_3) $K_X = \Phi^*K_Y + D$, $K_X.l = D.l = -1$, $D^3 = 2$;

(b_4) $K_X = \Phi^*K_Y + D$, $K_X.l = D.l = -1$, $D^3 = 2$;

(b_5) $K_X = \Phi^*K_Y + \frac{1}{2}D$, $K_X.l = -1$, $D.l = -2$, $D^3 = 4$.

Proof of Theorem 2.2. Because H is Gnef, $\mu(H)$ is rational, thus there exists a curve $C \subset X$ such that

$$(H + \mu(H)K_X).C = 0, \quad H.C = -\mu(H)K_X.C > 0, \quad K_X.C < 0.$$

According to Cone theory, we find that there exists at least one extremal rational curve l such that $(H + \mu(H)K_X).l = 0$. Let $R = \mathbf{R}_+[l]$, for any curve C such that $\mathbf{R}_+[C] = R$, we have $(H+\mu(H)K_X).C = 0$. Hence $1 \le -K_X.C = H.C/\mu(H)$, whereas $\mu(H)$ is non-integral, $H.C \ne \mu(H)$, i.e, $-K_X.C > 1$ or $-K_X.C \ge 2$.

Now if $Cont_R$ is birational, then $Y = Cont_R(X)$ is a non-singular projective 3-fold on which $\rho(Y) = 2$. $Cont_R$ is of b_2-type i.e.,

$$D \cong \mathbf{P}^2, O_D(D) \cong O_{\mathbf{P}}(-1).$$

If $Cont_R$ is not birational, then it is either a conic bundle or a del pezzo fibre space.

Theorem 2.3. *Let X be a non-singular Fano 3-fold. If H is an ample divisor on X such that $\mu(H)$ is non-integral and $[\mu(H)] = [\sigma(H)]$, then X has good contractions $f : X \longmapsto Y$ and $g : Y \longmapsto Z$, which satisfy*

(1) Y and Z are both non-singular projective variety, $\dim Y = 3$, $\dim Z \le 2$;

(2) f is just blowing down several planes or trivial. g is a Fano fibration, $\rho(Y) = \rho(Z)+1$, and one of the following is true:

(i) Z is a rational surface, g is a conic bundle and $\rho(Y) \ge 2$;

(ii) $Z \cong \mathbf{P}^1$, g is a del pezzo fibre space and $\rho(Y) = 2$;

(iii) $\dim Z = 0$, Y is a Fano 3-fold, $\rho(Y) = 1$ and Fano index

$$r(Y) \le \Big[\frac{1}{\mu(H) - [\mu(H)]}\Big].$$

I was told by M. Reid that some experts had already known the result. I greatly appreciate his help.

Lemma 2.2.[13,Proposition 1.11 (*iii*)]. *Let $f : X \longmapsto Y$ be a birational morphism between non-singular projective varieties. Then we have $f^*\overline{D}(Y) \subset \overline{D}(X)$ and $f_*\overline{D}(X) \subset \overline{D}(Y)$.*

R. Hartshorne has given the following result in [5]:$-K_X$ on X is ample if and only if $(-K_X.C) > 0$ for every effective 1-cycle C on X.

Lemma 2.3. *Let $\phi : X \longmapsto Y$ be an extremal contraction, X be a non-singular Fano 3-fold, $\phi = Cont_R$, and R be of b_2-type. If H is an ample divisor on X, then ϕ_*H is also ample.*

Proof. Denote by D the exceptional divisor of ϕ. Then

$$D \cong O_{\mathbf{P}}(-1), \quad K_X = \phi^*K_Y + 2D.$$

Let C be any curve on Y, $\widetilde{C}$ the strict transform of C. Then

$$K_Y.C = \phi^*K_Y.\widetilde{C} = (K_X - 2D).\widetilde{C} = K_X.\widetilde{C} - 2D.\widetilde{C} < 0,$$

therefore Y is also Fano.

Now it is sufficient to show that $(\phi_*H.C) > 0$ for any curve $C \subset Y$. We have the exact sequence

$$0 \longmapsto PicY \overset{\phi^*}{\longmapsto} PicX \overset{(.l)}{\longmapsto} \mathbf{Z} \longmapsto 0.$$

There exists a positive number a such that $(H + aK_X).l = 0$. So $H + aK_X \equiv \phi^*\overline{H}$ for a divisor H. Let $H_1 = \Phi_*H$. Then

$$H \equiv H_1 + aK_Y, \quad H + aK_X \equiv \Phi^*(H_1 + aK_X), \quad H + 2aD = \Phi^*H_1.$$

Hence

$$(\Phi_*H.C) = H_1.C = \Phi^*H_1.\widetilde{C} = (H + 2aD).\widetilde{C} > 0.$$

So H_1 is numerically positive and H_1 is ample because Y is Fano.

Proof of Theorem 2.3. We know that $H + \mu(H)K_X$ is nef. Let $\overline{H} = H + [\mu(H)]K_X$. Then $\overline{H}$ is a nef Cartier divisor and $\mu(\overline{H}) = \mu(H) - [\mu(H)]$. Because $\mu(H)$ is non-integral, we have $0 < \mu(\overline{H}) < 1$ and $\overline{H} + K_X$ is not pseudo-effective by the condition. Therefore the problem is reduced to the case when $H + K_X$ is not pseudo-effective. We assume that $H + K_X$ is not pseudo-effective in the next.

Like the situation in the proof of Theorem 2.2, there exists an extremal curve l such that

$$(H + \mu(H)K_X).l = 0, \quad K_X.l = -\frac{1}{\mu}H.l < -1, \quad K_X.l \le -2.$$

Let $R = \mathbf{R}_+[l], \phi_1 = Cont_R$. Then ϕ_1 is of one of the four types: b_2-type, c_1-type, c_2-type and c_3-type.

Let $X_1 = Cont_R(X)$. Then X_1 is non-singular. If ϕ_1 is birational, let $H_1 = \phi_1 H$. Then H_1 is ample by Lemma 2.3, $H_1 + K_{X_1}$ is not pseudo-effective and so $\mu(H) < 1$ by Lemma 2.2 and X_1 is Fano by Hartshorne's result. Thus we can treat X_1 with the same method as to X. Because $\rho(X)$ is finite, this program must terminate at Fano fibration.

Using the classification theorem of extremal ray R of Fano 3-fold in section 2 of [12], we can see that Z is rational, especially if dim $Z = 1$, then $Z \cong \mathbf{P}^1$.

Definition 2.5. *Define $\sigma(X) = \inf \sigma(H)$ for all the ample divisors H on X, $\sigma(X)$ is intrinsic related to X.*

Theorem 2.4. *Let $X \subset \mathbf{P}^n$ be a non-singular Fano 3-fold. $H = O_X(1)$ be the very ample divisor, $d = H^3$ be the degree of X embedded in $\mathbf{P}^n$ $(n \geq 4)$. Then we have $\sigma \leq d(n+1)^2/c_1^3$.*

Lemma 2.4. *Let X be a non-singular projective 3-fold and be embedded in $\mathbf{P}^n$ $(n \geq 4)$. Let $H = O_X(1)$ be the very ample divisor.*

$$q = h^1(O_X) = 0, \quad L = \wedge^{n-3}\mathbf{N}_{X/P^n}.$$

Then $h^0(L) > 0$ i.e., L is linearly equivalent to an effective divisor.

This is a very special case of known results. We can have a cohomological calculation directly.I believe that the conditions here are much sufficient, but I do not think it over.

Proof of Theorem 2.4. We suppose $L = \wedge^{n-3}\mathbf{N}_{X/P^n}$. Then $K_X = L - (n+1)H$. We know that $H + \sigma K_X$ is pseudo-effective. Because X is Fano variety, c_1 is ample. Thus $c_1^2(H + \sigma(H)K_X) \geq 0$.

$$\sigma(H) \leq \frac{c_1^2 H}{c_1^3}, \quad c_1 = (n+1)H - L.$$

$$c_1^2 H = ((n+1)^2 H - L)^2.H = (n+1)^2 H^3 - L.H.(2(n+1)H - L),$$

$$c_1^2 H \leq (n+1)^2 H^3 = d(n+1)^2 \quad \text{and} \quad \sigma \leq \sigma(H) \leq \frac{d(n+1)^2}{c_1^3}.$$

Corollary 2.2. *Under the condition of Theorem 2.4, if $d < \frac{c_1^3}{(n+1)^2}$, then X has good contractions as in Theorem 2.3.*

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