Chin. Ann. of Math.
15B: 3(1994),273-282.

NON-ISOMORPHIC GROUPS WITH ISOMORPHIC SPECTRAL TABLES AND BURNSIDE MATRICES**

W. Kimmerle* K. W. Roggenkamp*

Abstract

It was shown by Formanek and Sibley that the group determinant characterizes a finite group G up to isomorphism. Hoehnke and Johnson (independently the authors — using an argument of Mansfield) showed the corresponding result for k-characters, $k = 1, 2, 3$. The notion of k-characters dates back to Frobenius. They are determined by the group determinant and may be derived from the character table $CT(G)$ provided one knows additionally the functions

$$\Phi_k : G \times \cdots \times G \to C(G), \quad (g_1, \cdots, g_k) \to C_{g_1 \cdot \ldots \cdot g_k},$$

where $C(G) = \{C_g,\ g \in G\}$ denotes the set of conjugacy classes of G.

The object of the paper is to present criteria for finite groups (more precisely for soluble groups G and H which are both semi-direct products of a similar type) when

1. G and H have isomorphic spectral tables (i.e., they form a Brauer pair),
2. G and H have isomorphic table of marks (in particular the Burnside rings are isomorphic),
3. G and H have the same 2-characters.

Using this the authors construct two non-isomorphic soluble groups for which all these three representation-theoretical invariants coincide.

Keywords Finite group, Spectral table, Burnside matrix, Isomorphism.
1991 MR Subject Classification 20C15, 20B25.

§1. Introduction

For a finite group G, we let $\widehat{G}$ be the set of irreducible complex characters, and we denote by

$$CT(G) = (\chi_i^G(K_j^H))_{1 \le i,j \le h(G)}$$

the character table of G, where $\{\chi_i^G\}_{1 \le i \le h(G)}$ are the irreducible characters and $\{K_i^G\}$, $1 \le i \le h(G)$, are the conjugacy classes of G. The character table together with the power map on the conjugacy classes, $K_g \to K_{g^m}$, is called the spectral table of G, denoted by $SP(G)$. The spectral table $SP(G)$ determines the set $\{|x_i|\}_{1 \le i \le h(G)}$, where $x_i \in K_i^G$. We write $B(G)$ for the Burnside matrix of G, which is sometimes called the table of marks.

Manuscript received April 13, 1993.
*Mathematisches Institut B, University of Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart, Germany.
**This research was supported by the Deutsche Forschungsgemeinschaft.
[1]Two such groups are said to form a Brauer pair.
[2]S. K. Sehgal stands here for Surinder K. Sehgal

It is well known that a question of Richard Brauer, whether non-isomorphic groups may have isomorphic spectral tables[1], has a negative answer (cf. E. C. Dade [3] and G. Cliff-S. K. Sehgal [1])[2]. Cliff and Sehgal[1] determine the Brauer pairs among the solvable two fold transitive groups[3]. Recently it was shown by K. Johnson and S. K. Sehgal[4] that these Brauer pairs also have the same 2-characters in the sense of Frobenius. It is a byproduct of the results in this paper that these Brauer pairs also have the same Burnside matrices. These are then the first known examples of Brauer pairs which also have the same table of marks[4].

The aim of this note is to present criteria for a fairly general pair of groups[5] to form a Brauer pair (cf. Proposition 1.1) or to have the same Burnside matrices (cf. Proposition 1.2) or to have the same 2-characters (cf. Proposition 1.3).

As an application we construct some examples of Brauer pairs, the groups of which have the same Burnside matrices and the same 2-characters. These groups are constructed in the same spirit as those in [1], they are given as semi-direct product of a finite group G acting fixed point freely in different ways on a finite vector space V. In the examples of [1] the group G has exactly two orbits on V; in our examples this need not be the case.

We shall next describe the criteria.

Proposition 1.1. *Assume that $G_i,\ 1 \le i \le 2$, are two finite isomorphic groups, which act on the same $\mathbb{F}_p$-vectorspace V fixed point freely, i.e., $Stab_{G_i}(v) = 1$ for every $v \in V \setminus \{0\}$. If each orbit of G_1 on V coincides as a set with an orbit of G_2 on V, then the semi-direct products $H_1 = V \cdot G_1$ and $H_2 = V \cdot G_2$ have isomorphic spectral tables.*

Remark 1.1. The above conditions imply that the groups H_i are Frobenius groups with abelian kernel V and complements G_i; in particular, we have $(p, |G_i|) = 1$.

In order to stable the result on Burnside rings, we have to introduce some more notation:

Definition 1.1. *Assume that G_i and V are as in Proposition 1.1. Then for each $v \in V,\ g_1 \in G_1$, there exists a unique element $\tau_v(g_1) \in G_2$ such that ${}^{g_1}v = {}^{\tau_v(g_1)}v$. Moreover, for each $v \in V,\ g_1, h_1 \in G_1$, there exists a unique $\rho_v(g_1, h_1) \in G_1$ with*

$${}^{\tau_v(g_1)\cdot h_1}v = {}^{\rho_v(g_1,h_1)\cdot h_1}v.$$

Proposition 1.2. *Assume in addition to the hypotheses in Proposition 1.1 that all minimal subgroups in V are conjugate under elements in G_1 and G_2 respectively, and that there exists $v_0 \in V$ such that for every subgroup $H \le G_2$ we have $\rho_v(h, h') \in H$ for all $h, h' \in H$. Then the Burnside matrices of H_1 and H_2 are isomorphic.*

We finally turn to the 2-characters.

Definition 1.2. *Let $\{\chi_i\}_{1\le i\le n}$ be the irreducible characters of the finite group G. With each χ_i there is a 2-character $\chi_i^2 : G \times G \to \mathbb{C}$ associated, which is defined as follows*

$$\chi_i^2(x, y) := \chi_i(x \cdot y) - \chi_i(x) - \chi_i(y).$$

[3]These groups are subgroups of the semidirect product of a finite vectorspace with the group of semi-linear maps on V[5].

[4]H. Pahlings has told the authors that he has by computer search found a Brauer pair of 2-groups, the groups of which also have the same Burnside matrices.

[5]These pairs are constructed by letting a fixed finite group act fixed point freely in two different ways on a finite vectorspace and then consider the semi-direct product.

We say that two groups G, H with irreducible characters $\{\chi_i\}_{1\leq i\leq h}$ of G and $\{\psi_i\}_{1\leq i\leq h}$ of H, where $h = h(G) = h(H)$ denotes the class number, have the same 2-character tables, if there is a bijection $\alpha : G \to H$ and a permutation β on $\{1, \cdots, n\}$, such that G and H have the same character tables via the above maps; and G and H have the same 2-characters via the above maps.

The next result is a very coarse criterion, to see, when certain groups have the same 2-characters.

Proposition 1.3. *Let G and H be isomorphic groups via an isomorphism $\rho : G \to H$. Let M be an $\mathbb{F}_p$-module with fixed point free action for G and H – this implies that p does not divide $|G|$, furthermore, we require that the elements in $M \setminus \{0\}$ form a single orbit under the action of G. Assume that there is a ρ-equivariant bijection $\sigma : M \setminus \{0\} \to M \setminus \{0\}$, i.e., $\sigma(g \cdot m) = \rho(g) \cdot \sigma(m)$ for $g \in G,\ m \in M$, with $\sigma(-m) = -\sigma(m)$. Then $G_1 := M \cdot G$ and $H_1 := M \cdot H$ have the same 2-characters.*

§2. The Proofs of the Propositions 1.1, 1.2, 1.3

We first deal with the spectral tables and prove Proposition 1.1:

Since G_1 and G_2 are isomorphic, there is an isomorphism of spectral tables of G_1 and G_2, i.e., of the characters (and conjugacy classes) of H_1 and H_2, on which V acts trivially (and the classes, which do not meet V respectively). Thus it remains to show that the characters which do not have V in the kernel are in a correspondence for H_1 and H_2 which is compatible with a correspondence of the conjugacy classes. However, the conjugacy classes of H_1 on V and of H_2 on V are in a bijection σ, since by hypothesis the orbits of G_1 and G_2 on V coincide.

We now invoke Clifford's theory for characters[2] (see also [5, 6.13]): Let χ be a non-trivial character of V. Since the action of G_i on V is fixed point free, both the inertia groups of χ in H_1 and H_2 respectively are just V. However, the orders of V and G_i are relatively prime, and so the induced character $\chi_V^{H_i}$ is an irreducible character of H_i. But this induced character has zero value for the elements in H_i outside of V, and for an element $v \in V$ we have

$$\chi_V^{H_i}(v) = \sum_{g\in H_i/\,\mathrm{Stab}_{H_i}(v)} \chi({}^g v),$$

which is the same for both groups, according to our hypotheses. This completes the proof of Proposition 1.1.

We now turn to Burnside matrices and prove Proposition 1.2 in more generality.

Definition 2.1. *Let G and H be groups acting on a set V; we write the actions as $g \cdot m$ and $h \cdot m$ respectively. We shall assume that G and H act fixed point freely on V, and that for every $v \in V$ the orbit of G and H coincide; i.e., $\mathcal{O}_G(v) = \mathcal{O}_H(v)$. Then there exists for each $v \in V$ a unique bijection*

$$\tau_v : G \to H \text{ with } \ g \cdot v = \tau_v \cdot v.$$

Moreover, for every $v \in V$ there is a unique map

$$\rho_v : G \times G \to G \quad \text{with} \quad \tau_v(g) \cdot g' \cdot v = \rho_v(g, g') \cdot g' \cdot v.$$

The question of whether the image of τ_v of a subgroup is again a subgroup, is answered by

Claim 2.1. *Let $U \leq G$. Then $\tau_v(U) \leq H$ if and only if $\rho_v : U \times U \to U$.*

Proof. Note that $\tau_v(g^{-1}) = (\tau_v(g))^{-1}$. Thus we only have to find the necessary and sufficient conditions for $\mathrm{Im}(\tau_v(U))$ to be multiplicatively closed. However,

$$\tau_v(u_1) \cdot \tau_v(u_2) \cdot v = \tau_v(u_1) \cdot u_2 \cdot v = \rho_v(u_1, u_2) \cdot u_2 \cdot v.$$

Hence $\tau_v(u_1) \cdot \tau_v(u_2) \in \tau_v(U)$ if and only if $\rho_v(u_1, u_2) \in U$.

This claim has the following consequence:

Corollary 2.1. *Assume that there exists $v_0 \in V$ such that for every subgroup $U \leq G$ we have $\rho_{v_0} : U \times U \to U$. Then the map τ_{v_0} induces an order preserving map between the subgroups of G and H.*

Proposition 2.1. *In addition to the assumptions in Definition* 1.3*, we suppose now that G and H are isomorphic normal subgroups of a common group $\mathcal{G}$, which also acts on V, such that the action of G and H is induced from that of $\mathcal{G}$, that V is an elementary abelian group of order relatively prime to $|G|$, that there is an element $1 \neq v_0 \in V$ such that for every subgroup $U \leq G$ we have $\rho_{v_0} : U \times U \to U$, and finally that every subgroup W of V has a G-conjugate, which contains v_0. (Note that according to our notation, the conjugation on V with elements in G is written as left multiplication.) Then we conclude, for the normalizer of a subgroup $W \leq V$, that $\tau_{v_0}(N_G(W)) = N_H(W)$.*

For the proof we first note:

Claim 2.2. *Let W be a subgroup of V such that $g^{-1} \cdot W$ contains V_0. Then $g \cdot v_0 \in W$. Moreover, for every subgroup $U \leq G$ we have $\rho_{g \cdot v_0} : U \times U \to U$.*

Proof. The first part is obvious. For the second we shall use multiplication of complexes: $\tau_{g \cdot v_0}(U) \cdot g \cdot v_0 = U \cdot g \cdot v_0$ if and only if

$$g^{-1} \cdot \tau_{g \cdot v_0}(U) \cdot g \cdot v_0 = g^{-1} \cdot U \cdot g \cdot v_0.$$

However, since G and H are normal subgroups of $\mathcal{G}$, we conclude that $g^{-1} \cdot \tau_{g \cdot v_0}(U) \cdot g \leq H$, which is the same as $\tau_{v_0}(g^{-1} \cdot U \cdot g)$ as one sees by looking at the action on v_0. Thus we conclude that $\tau_{g \cdot v_0}(U)$ is a subgroup of H.

We now come to the actual proof of Proposition 2.1. Let W be a subgroup of V. Because of Claim 2.2, we may assume that $v_0 \in W$. Then

$$\tau_{v_0}(n) \cdot n' \cdot v_0 = \rho_{v_0}(n, n') \cdot n' \cdot v_0 \in W$$

for every $n, n' \in N_G(W)$, since by assumption $\rho_{v_0}(n, n') \in N_G(W)$. We now define

$$S(W) = \{w \in W | m \cdot w \in W \text{ for all } m \in \tau_{v_0}(N_G(W)\}.$$

According to the above. $v_0 \in S(W)$, and so $S(W) \neq \{1\}$[6]. $S(W)$ is a subgroup of W, since $m \cdot (w_0 \cdot w_2) = (m \cdot w_1) \cdot (m \cdot w_2)$.

[6] V is written multiplicatively.

We shall now use induction on $|W|$. Let W be a minimal subgroup of V. Then $S(W) = W$, which means that $\tau_{v_0}(N_G(W)) \subseteq N_H(W)$. Note that τ_{v_0} is a bijection, mapping the lattice of subgroups of isomorphically onto the lattice of subgroups of H, and hence $\tau_{v_0}^{-1}$ has the same properties as does τ_{v_0}. Thus we conclude that $\tau_{v_0}(N_G(W)) = N_H(W)$.

Let now W be given. Then — note that V is abelian and has order relatively prime to $|G|$ — W is a module over $N_G(W)$; moreover, V was assumed to be elementary, and so W is completely reducible as $N_G(W)$-module. Assume that W is reducible, i.e., $W = W_1 \oplus W_2$. Then

$$N_G(W_i) \supset N_G(W) \quad \text{and} \quad N_G(W_1) \cap N_G(W_2) = N_G(W).$$

Since τ_{v_0} is compatible with intersections, the result follows from the induction.

We thus may assume that W is a simple module for $N = N_G(W)$. If $S(W) < W$, then $S(W)$ is a $\tau_{v_0}(N)$-module, since $S(W)$ is a union of $\tau_{v_0}(N)$-orbits. But using the inverse map $\tau_{v_0}^{-1}$ and induction on this map, we see that $S(W)$ is an N-module. Hence it follows that $S(W) = W$, and as before we get $\tau_{v_0}(N_G(W)) = N_H(W)$.

We now come to the actual proof of Proposition 1.2:

Claim 2.3. *The lattice of subgroups is the same in both groups. Moreover, corresponding groups have normalizers of the same order.*

Proof of Claim 2.3. Let U be a subgroup of H_1, and put $U_0 = U \cap V$. Then U_0 has a complement in U, say U_1. However $(|V|, |G_1|) = 1$ and so all complements to V in H_1 are conjugate. We therefore may assume that $U = U_0 \cdot U_1$ with $U_o \leq V$ and $U_1 \leq {}^{v_U}G_1$, where $v_U \in V/U_0$ is uniquely determined, since H_1 is a Frobenius group, and so all complements are conjugate. We note that then $U_1 \leq N_{({}^{v_U})G_1}(U_0)$. We now define $\Phi(U) = U_0 \cdot \tau_{v_0}(U_1)$. According to the results proved above, this is a subgroup of H_2, and Φ surely gives an order preserving bijection on the subgroups, presevering the order of $N_{H_1}(U)/U$ (Note that $N_{H_1}(U_0 \cdot U_1) = N_{{}^{v_U}G_1}(U_1) \cdot U_0$).

We note that Claim 2.3 now shows that we have an isomorphism of the Burnside matrices.

We now turn to the 2-characters and prove Proposition 1.3. We shall use the hypotheses and notation introduced there; in addition we define $\sigma(0) = 0$. The groups G and H are isomorphic via a map, say ρ, and thus they have isomorphic 2-characters. Since the groups G and H operate transitively and fixed point freely on $M \setminus \{0\}$, there is only one faithful irreducible character χ_{G_1} and χ_{H_1} of $G_1 = M \cdot G$ and $H_1 = M \cdot H$ respectively. All other characters are just the characters of G and H, since by assumption all complements are conjugate $-$ $H^1(G, M) = 0$ because $(p, |G|) = 1$. Similarly as in the paper of Johnson and Sehgal[4], we construct a map $\alpha : M \cdot G \to M \cdot H$ by means of

$$\alpha : (x, g) \to (\sigma(x), \rho(g)) \text{ provided } 0 \neq x \in M,\ g \in G$$

and

$$\alpha : (0, g) \to (0, \rho(g)).$$

For simplicity we write χ for χ_{G_1} and ψ for χ_{H_1}.

We thus only have to verify for $a, b \in G_1$ the relation

$$\chi(a \cdot b) - \chi(a) - \chi(b) = \psi(\alpha(a) \cdot \alpha(b)) - \psi(\alpha(a)) - \psi(\alpha(b)).$$

We have to treat some cases separately: Since χ and ψ are induced from faithful linear characters of M, we have $\chi(g) = 0$ for $g \in G$ and $\psi(h) = 0$ for $h \in H$. The above equality thus holds

1. if $a, b \in M$,
2. if $a \in M,\ b \in G$ or conversely,
3. if $a = (x, g)$ and $b = (x', g')$ with $g^{-1} \neq g'$ and $x, x' \neq 0$.

We thus have to verify the above equation only in case $a = (x, g)$ and $b = (x', g^{-1})$ with $x, x' \neq 0$; i.e., we then have to show: $\chi(a \cdot b) = \psi(\alpha(a) \cdot \alpha(b))$; i.e.,

$$\chi(x + g \cdot x') = \psi(\sigma(x) + \rho(g) \cdot \sigma(x')).$$

These values are different if and only if $(x + g \cdot x') \neq 0$ but $\sigma(x) + \rho(g) \cdot \sigma(x') = 0$ or conversely. Now

$$\sigma(x) + \rho(g) \cdot \sigma(x') = \sigma(x) + \sigma(g \cdot x').$$

However, σ is bijective and so it is impossible that we have $-x \neq g \cdot x'$ but $\rho(-x) = \rho(g \cdot x')$, since $\sigma(-m) = -\sigma(m)$ by assumption.

§3. The Construction of the Groups

We shall now describe a family of non-isomorphic groups, which contain those of Cliff and Sehgal[1], and which can be collected in families in such a way that groups in the same family have isomorphic character tables and isomorphic Burnside matrices. However, these groups can be distinguished by their p-adic group rings for the prime p, since they have no normal p'-subgroup[8] (This result also implies that integral group rings determine Frobenius groups[6]).

These groups can be considered as subgroups of the semi-direct product of the group of semi-linear maps acting on a fimite vectorspace.

Definition 3.1. *Let G_n be a group with a cyclic normal subgroup $D_n = \langle d_n \rangle$ of order n with complement $C_{\mu(n)} = \langle c \rangle$, where $\mu(n)$ is the Euler function, i.e., the number of primitive n-th roots of unity. The group D_n should be interpreted as the group of n-th roots of unity, on which $C_{\mu(n)}$ acts as Galois group.*

By $G_{n,m}$ we denote the subgroup of G_n, where the complement $C_m = \langle c_m \rangle$ is generated by an element of order n dividing $\mu(n)$. In addition, we require that m^2 divides n, and that m and n/m^2 are relatively prime.

A further assumption is that p is a rational prime, such that $\mathbb{F}_{p^m}$ is the smallest field of characteristic p containing all n-th roots of unity. We note that this is a consequence of the requirements which we specify now: We postulate that either $n = p^m - 1$ — this is the case, which is considered in Cliff-Sehgal[1], where $V \setminus \{0\}$ consists of one orbit under the action of G_n — or

$$\Big(\sum_{i=0}^{m-1} p^i\Big) \cdot n_1 = ((p^m - 1)/(p-1)) \cdot n_1 = n, \tag{3.1}$$

where n_1 then is a factor of $p-1$. We require in addition that m is odd and for $y := (p-1)/n_1$

we have both of the following conditions for the greatest common divisor: $(y, n_1) = 1$ *and* $(y, m) = 1$.

Then $M = \mathbb{F}_{p^m}$ *is an irreducible* $G_{n,m}$*-module, on which* d_n *acts by multiplication with a primitive n-th root of unity, and* c_m *acts as Frobenius automorphism, i.e., raising to the p-th power.* $\mathrm{End}_{G_{n,m}}(M) = \mathbb{F}_p$ *is the prime field.*

By $G_{n,m,p}$ *we denote the semi-direct product of* M *with* $G_{n,m}$.

Let $b_m = d_n^{n/m^2}$ *be an element of order* m^2 *in* D_n. *There is no loss of generality, if we assume that* ${}^{c_m}b_m = b_m^{m+1}$. *We note that* b_m^m *lies in the centre of* $G_{n,m}$.

Claim 3.1. *Let* $1 \le i \le m-1$ *be relatively prime to* m. *Then*

$$(c_m \cdot b_m^i)^m = b_m^{i \cdot m} \tag{3.2}$$

is central.

Proof. We have

$$\begin{aligned}(c_m \cdot b_m^i)^m &= \prod_{j=1}^{m} {}^{c_m^j}(b_m^i) \\ &= b_m^{\,i \cdot \left(\sum_{j=1}^{m}(m+1)^j\right)},\end{aligned}$$

and so it remains to show that

$$i \cdot \Big(\sum_{j=1}^{m}(m+1)^j\Big) \equiv i \cdot m \bmod m^2.$$

However, $(m+1)^j \equiv 1 + j \cdot m \bmod m^2$, and so

$$i \cdot \Big(\sum_{j=1}^{m}(m+1)^j\Big) \equiv i \cdot (m + m^2 \cdot (m+1) \cdot 1/2) \bmod m^2,$$

which is congruent to $i \cdot m$ as claimed, since m is odd.

We are now finally in the position to define our groups as subgroups of $G_{n,m,p}$ (cf. Definition 3.1).

Definition 3.2. *Let* $1 \le i \le m-1$ *be relatively prime to* m*, and define the group* $H_{n,m,i}$ *as a subgroup of* $G_{n,m}$ *by*

$$H_{n,m,i} = \langle d_n^{m^2}, b_m^m, c_m \cdot b_m^i \rangle,$$

and put $H_{n,m,i,p} = M \cdot H_{n,m,i}$*, the semi-direct product with the module* M *(cf. Definition 3.1).*

All of the above conditions are satisfied, for the following subgroups of the semi-linear group $\mathbb{F}_{7^3} \cdot \mathbb{F}_{7^3}^* \cdot C_3$ where $d^9,\ b \in \mathbb{F}_{7^3}$ have order 19 and 9 respectively, c generates C_3, and $H_i\langle d^9, b^3, c \cdot b^i\rangle,\ i = 1, 2$, then $G_i = \mathbb{F}_{7^3} \cdot H$ is a pair of such groups.

Proposition 3.1. *Let* $1 \le i \le m-1$ *be relatively prime to* m. *Then the groups* $H_{n,m,p,i}$ *have isomorphic spectral tables.*

Proof. The groups $H_{n,m,i}$ are isomorphic for all i with $1 \le i \le m-1$ relatively prime to m. In fact, since m and n/m^2 are relatively prime, all these groups are isomorphic to a semi-direct product of a cyclic group of order n/m^2, generated by a, with a cyclic group of order m^2, generated by e, and e acts on a in the same way as c_m acts on $d_n^{m^2}$.

Consequently we may apply Proposition 1.1 and the result will follow, if we can show that the orbits of $H_{n,m,i}$ on $M=\mathbb{F}_{p^m}$ are the same for all admissible i. To do so, let $\zeta\in M=\mathbb{F}_{p^m}$ be a primitive n-th root of unity with respect to the multiplication.

Claim 3.2. $\mathrm{Stab}_{H_{n,m,i}}(\zeta)=1$.

Proof. Recall from Definition 3.2 that

$$H_{n,m,i}=\langle d_n^{m^2}, b_m^m, c_m\cdot b_m^i\rangle.$$

To simplify the notation, we shall write $a=d_n^{m^2},\ b=b_m,\ c=c_m$. Then $H_i=\langle a, b^m, b^i\cdot c\rangle$ has an abelian normal subgroup $A=\langle a, b^m\rangle$, which has trivial stabilizer on ζ. We recall from above that $H_i=\langle a, c\cdot b^i\rangle$ and that $(c\cdot b^i)^m=b^{i\cdot m}$, which has order m. Since $(|a|,m)=1$, all complements to $\langle a\rangle$ in H_i are conjugate, and we may replace ζ by another primitive n-th root of unity, to conclude that

$$\mathrm{Stab}_{H_i}(\zeta)\subset\langle c\cdot b^i\rangle\supset\langle b^m\rangle.$$

However, no non-trivial subgroup of $\langle b^m\rangle$ can stabilize ζ.

We now note that our group H_i has order n, and from Claim 3.2 it follows that the orbit $\omega_{H_i}(\zeta)$ has length n. It follows from the construction that then this orbit must be the subgroup of $\mathbb{F}_{p^m}$ generated by ζ, i.e., $\omega_{H_i}(\zeta)$ consists precisely of the n-th roots of unity in $\mathbb{F}_{p^m}$. However, this set is independent of the index i in H_i.

Recall from Definition 3.1 that we have two cases: First, $n=p^m-1$; then the elements $\mathbb{F}_{p^m}\setminus\{0\}$ form a single orbit, and the statement follows.

In the second case (cf. Equation 3.1) we write

$$n=n_0\cdot n_1,\quad\text{where}\quad n_0=(p^m-1)/(p-1). \tag{3.3}$$

Claim 3.3. *The greatest common divisor of $y=(p-1)/n_1$ and n is* 1, *and the group $\langle\zeta\rangle$ of order n as a subgroup of $\mathbb{F}_{p^m}$ has coset representatives, which can be chosen to lie in the units of the prime field of $\mathbb{F}_{p^m}$.*

Proof. The case $p=2$ is of no interest, and so we may assume that p is an odd prime. Let

$$\nu=\Big(p-1,\sum_{j=0}^{m-1}p^j\Big)$$

be the greatest common divisor. Then ν divides m — this follows by induction on m. In order to see that $(y,n)=1$, we note that by assumption $(y,m)=1$ and $(y,n_1)=1$, the second statement now follows easily.

This completes the proof of Proposition 3.1.

Proposition 3.2. *Let $1\le i\ne j\le m-1$ be relatively prime to m. Then the groups $H_{n,m,i,p}$ and $H_{n,m,j,p}$ (cf. Definition* 3.2) *are not isomorphic.*

Proof. Assume to the contrary that $H:=H_{n,m,i,p}$ and $H':=H_{n,m,j,p}$ are isomorphic, via an isomorphism ϕ. We denote by S the common subgroup of H and H' generated by $\mathbb{F}_{p^m}\cup\{d_n^{m^2}, b_m^m\}$, and by $C_{n/m}$ the subgroup generated by $\{d_n^{m^2}, b_m^m\}$.

Claim 3.4. $M:=\mathbb{F}_{p^m}$ *is an irreducible module for* $C_{n/m}$.

Proof. According to our construction (cf. Definition 3.1) M is an irreducible module for C_n. Now $C_{n/m}$ is a subgroup of index m in C_n. If N is an irreducible constituent of $M\downarrow_{C_{n/m}}$, then $N = \mathbb{F}_{p^\nu}$ is a field, since $C_{n/m}$ is cyclic. But then $m = \mu \cdot \nu$. On the other hand,

$$|M : N| = p^{\mu\cdot(\nu-1)} \leq m.$$

The last inequality follows from orbit considerations, using Claim 3.3: Recall that $p^m - 1 = n \cdot y$ with $y|(p-1)$. Since C_n acts fixed point freely on M, there are y orbits of length n of C_n on M, which are transformed into each other by elements in the prime field (cf. Claim 3.3). Since each of these splits for $C_{n/m}$ into m orbits, the inequality follows.

However, $m^2|(p^{\mu\cdot\nu} - 1)$, which is only possible, if $\nu = 1$, since $2 \cdot \mu \cdot (\nu - 1) \geq \mu \cdot \nu$. Hence M remains irreducible, when restricted to $C_{n/m}$.

We now continue with the proof of Proposition 3.2. The isomorphism ϕ induces an automorphism ϕ' of S. Moreover, according to the Claim 3.4, $M = \mathbb{F}_{p^m}$ is an irreducible module for $C_{n/m}$ with $\mathrm{End}_{C_{n/m}}(M) = \mathbb{F}_{p^m}$.

Hence — if necessary after a conjugation — we may assume

$$\begin{aligned} \phi' : \quad S &\to \quad S, \\ (x,\alpha) &\to (\sigma(x), \alpha^i), \end{aligned}$$

where α is a generator of $C_{n/m}$ and σ is an $\mathbb{F}_p$-automorphism of M with $\sigma(\alpha \cdot x) = \alpha^i \cdot \sigma(x)$ and i is relatively prime to n/m. It should be noted that all complements to M in S are conjugate. However, not all indices i are possible. In fact, only those values can be attained, such that the twisted module ${}^iM$ under the automorphism $\alpha \to \alpha^i$ is isomorphic to M.

Alltogether, $C_{n/m}$ has $\mu(n/m)$ automorphisms, and there are $\mu(n/m)/m$ non-isomorphic faithful irreducible $\mathbb{F}_pC_{n/m}$-modules, since $\mathbb{F}_{p^m}$ is a splitting field for $C_{n/m}$ and so each of them is isomorphic to $\mathbb{F}_{p^m}$. On each of these modules the Frobenius automorphism acts $\mathbb{F}_p$-linearly. Moreover, the unique subgroup of order m in $\mathrm{Aut}(C_{n/m})$ stabilizes M. But this is exactly the group generated by the Frobenius automorphism of $\mathbb{F}_{p^m}$, i.e., the group generated by c_m (cf. Definition 3.1), and so the twisted module ${}^iM$ is the module M twisted be a power of the Frobenius automorphism. Since the conjugation action of $c_m \cdot b_m^i$ on S is the same as that of c_m — in both cases it is the Frobenius automorphism — we may apply a conjugation in H and H' respectively to arrange that

$$\begin{aligned} \phi' : \quad S &\to \quad S, \\ (x,\alpha) &\to (\sigma(x), \alpha), \end{aligned} \tag{3.4}$$

But then σ is an $\mathbb{F}_pC_{n/m}$ linear map, which is given by multiplication with $x_0 \in \mathbb{F}^*_{p^m}$.

Since all groups of order m^2 in H and H' respectively are conjugate by the theorem of Schur-Zassenhaus, we may assume that ϕ maps $c_m \cdot b_m^i$ to $(c_m \cdot b_m^j)^k$. We recall from Claim 3.1 that

$$(c_m \cdot b_m^i)^m = b_m^{i\cdot m} \quad \text{and} \quad (c_m \cdot b_m^j)^{k\cdot m} = b_m^{j\cdot k\cdot m}.$$

Applying ϕ and Equation (3.4), we get $i = k \cdot j \bmod m$. On the other hand, we have for

$x \in \mathbb{F}_{p^m}$ the equality

$$\phi({}^{c_m \cdot b_m^i}x) = (b_m^i \cdot x)^p \cdot x_0.$$

Evaluating this equation on both sides and solving with respect to x we conclude that for every $x \in \mathbb{F}_{p^m}$ the expression $x^{p\cdot(1-p^k)}$ is a constant. Since raising to the p-th power is an automorphism of $\mathbb{F}_{p^m}$, we have that for every $x \in \mathbb{F}_{p^m}$ the expression x^{1-p^k} is constant. But that is a contradiction, since $1 < k < m$.

We now show that our groups have isomorphic Burnside matrices, by proving

Claim 3.5. *The groups $H_{n,m,i,p}$ and $H_{n,m,j,p}$, $1 \le i \ne j \le m-1$, satisfy the hypotheses of Proposition* 1.2 *in Section* 1.

Proof. It only remains to show that we can find an element v_0 in $\mathbb{F}_{p^m}$ such that τ_{v_0} maps the subgroups of $H_{n,m,i}$ to those of $H_{n,m,j}$. We choose $v_0 \in \mathbb{F}_p$, the prime field in $\mathbb{F}_{p^m}$. Then τ_{v_0} acts trivially on $\langle d_n^{m^2}, b_m^m \rangle$ and τ_{v_0} maps $\langle c_m \cdot b_m^i \rangle$ to $\langle c_m \cdot b_m^j \rangle$ as one sees by inspection of the action of these groups on v_0 (It is worthwile to note that

$$\tau_{v_0} : \langle c_m \cdot b_m^i \rangle \to \langle c_m \cdot b_m^j \rangle$$

is not a group homomorphism).

Finally, it is clear that the groups in case $n = p^m - 1$ surely satisfy the hypotheses of Proposition 1.3, and hence have the same 2-character tables. Note the codition $\sigma(-m) = -\sigma(m)$ is satisfied, since in case p is odd, 2 divides n. (A similar statement was in great detail proved by K. W. Johnson and S. K. Sehgal[4].)

References

[1] Cliff, G. & Sehgal, S. K., On groups having the same character tables, *Com. Algebra,* **9** (1981), 627-640.

[2] Curtis, C. W. & Reiner, I., Methods of representation theory II, John Wiley & Sons, 1987.

[3] Dade, E. C., Answer to a question of R. Brauer, *J. Algebra,* **1** (1964), 1-4.

[4] Johnson, K. & Sehgal, S. K., The 2-character table does not determine a group (manuscript, 1991).

[5] Huppert, B., Endliche Gruppen I, Springer Verlag 1967.

[6] Kimmerle, W., Beiträge zur ganzzahligen Darstellungstheorie endlicher Gruppen, *Bayreuther Mathematische Schriften,* **36** (1991), 1-139.

[7] Pahlings, H., Private communication, 1992.

[8] Scott, L. L., Recent progress on the isomorphism problem, *Proc. Symposia in pure Math.,* **47** (1987), 259-273.