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WEAK TRAVELLING WAVE FRONT SOLUTIONS OF GENERALIZED DIFFUSION EQUATIONS WITH REACTION

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Abstract

The author demonstrate that the two-point boundary value problem

$$\begin{cases} p'(s) = f'(s) - \lambda p^{\beta}(s) & \text{for } s \in (0,1); \beta \in (0,1), \\ p(0) = p(1) = 0, p(s) > 0 & \text{if } s \in (0,1), \end{cases}$$

has a solution $(\bar{\lambda}, \bar{p}(s))$, where $|\bar{\lambda}|$ is the smallest parameter, under the minimal stringent restrictions on $f(s)$, by applying the shooting and regularization methods. In a classic paper, Kolmogorov et. al. studied in 1937 a problem which can be converted into a special case of the above problem.

The author also use the solution $(\bar{\lambda}, \bar{p}(s))$ to construct a weak travelling wave front solution $u(x,t) = y(\xi)$, $\xi = x - Ct$, $C = \bar{\lambda}N/(N+1)$, of the generalized diffusion equation with reaction

$$\frac{\partial}{\partial x}\left(k(u)\left|\frac{\partial u}{\partial x}\right|^{N-1}\frac{\partial u}{\partial x}\right) - \frac{\partial u}{\partial t} = g(u),$$

where $N > 0$, $k(s) > 0$ a.e. on $[0,1]$, and $f(s) := \frac{N+1}{N}\int_0^s g(t)k^{1/N}(t)dt$ is absolutely continuous on $[0,1]$, while $y(\xi)$ is increasing and absolutely continuous on $(-\infty, +\infty)$ and

$$(k(y(\xi))|y'(\xi)|^N)' = g(y(\xi)) - Cy'(\xi) \quad \text{a.e. on } (-\infty, +\infty),$$

$$y(-\infty) = 0, \qquad y(+\infty) = 1.$$

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§1. Introduction

The second-order quasilinear parabolic equation

$$\frac{\partial}{\partial x}\left(k(u)\left|\frac{\partial u}{\partial x}\right|^{N-1}\frac{\partial u}{\partial x}\right) - \frac{\partial u}{\partial t} = 0 \tag{1.1}$$

has been suggested as a model for certain generalized diffusion processes by Philip[1] and some similarity solutions of (1.1) have been given by Atkinson and Bouillet[2], Bouillet and

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Gomes[3], and Wang[4], where N is a positive constant and $k(u)$ is assumed to be positive a.e. on $(-\infty,+\infty)$ so that $k^{1/N}(u)$ is locally Lebesgue integrable in $(-\infty,+\infty)$.

As the title suggests, this paper is concerned with the existence of weak travelling wave front solutions of the equation (1.1) with reaction, namely

$$\frac{\partial}{\partial x}\left(k(u)\left|\frac{\partial u}{\partial x}\right|^{N-1}\frac{\partial u}{\partial x}\right)-\frac{\partial u}{\partial t}=g(u). \tag{1.2}$$

Throughout this paper the following hypotheses are adopted:

(H_1) N is a positive constant.

(H_2) $k(s)$ is a measurable function which is defined and positive a.e. on $(-\infty,+\infty)$ such that $k^{1/N}(s)$ is Lebesgue integrable on $[0,1]$.

(H_3) $g(s)$ is a bounded measurable function defined on $(-\infty,+\infty)$, and

$$g(s)=0 \qquad \text{for } s\le 0 \text{ and } s\ge 1.$$

(H_4) $f(s):=\frac{N+1}{N}\int_0^s g(t)k^{1/N}(t)dt$ is an absolutely continuous function defined on $[0,1]$ such that one of the following five conditions holds:

(H_{41}) There exists a point $s=A\in(0,1)$ such that $f(s)$ is positive in $(0,A)$ and negative in $(A,1]$. Moreover, $D_+f(s)$, the right hand lower derivative of $f(s)$, is negative in $(A,1)$.

(H_{42}) $f(s)$ is negative in $(0,1]$ and $D_+f(s)$ is negative and bounded in $(0,1)$.

(H_{43}) There is a point $s=B\in(0,1)$ such that $f(s)-f(1)$ is negative in $[0,B)$ and positive in $(B,1)$. Moreover, $D^-f(s)$, the left hand upper derivative of $f(s)$, is positive in $(0,B)$.

(H_{44}) $f(s)$ is positive in $(0,1]$ and $D^-f(s)$ is positive and bounded in $(0,1)$.

(H_{45}) $f(s)$ is positive in $(0,1)$ and $f(1)=0$.

It goes without saying that for almost all $s\in(0,1)$, $f'(s)$, a derivative of $f(s)$, exists and

$$D_+f(s)=D^-f(s)=f'(s)=\frac{N+1}{N}g(s)k^{1/N}(s).$$

The following hypotheses (H_2') and (H_3') can replace (H_2) and (H_3), respectively, because they also ensure that the function $f(s)$ is absolutely continuous on $[0,1]$.

(H_2') $k(s)$ is a bounded measurable function defined on $(-\infty,+\infty)$. Moreover, $k(s)$ is positive a.e. on $(-\infty,+\infty)$.

(H_3') $g(s)$ is a Lebesgue integrable function defined on $(-\infty,+\infty)$ and

$$g(s)=0 \qquad \text{for } s\le 0 \text{ and } s\ge 1.$$

In their classic paper[5], Kolmogorov et. al. discussed the existence and stability of travelling wave solutions for the simple parabolic equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=g(u), \tag{1.3}$$

where $g(u)$ has the same properties as the function $u(u-1)$. Subsequently, many authors have considered the existence and stability of travelling wave solutions of (1.3) under certain restrictions on $g(u)$. For details, see [6-11].

Aronson[12], Hosono[13], Grindrod and Sleeman[14], de Pablo and Vazquez[15], and Wang[16] have studied some special cases of (1.2) where $N=1$, $k(s)$ is a continuously differentiable function defined on $[0,+\infty)$ with $k(0)=k'(0)=0$ and $k'(s)>0$ for $s>0$, and $k(s)g(s)$ has the same properties as the function $-s^\alpha$ with $\alpha>0$ or $s^\alpha(s^\beta-1)$ with $\alpha,\beta>0$ or $s^\alpha(s^\beta-1)(s^\gamma-a)$ with $\alpha,\beta,\gamma>0$ and $a\in(0,1)$.

All of the above authors, including the authors of [6-11], applied the phase plane method in proving the existence of (weak) travelling wave front solutions.

In the present paper, we demonstrate that under the hypothesis (H_4) a two-point boundary value problem of the form

$$\begin{cases} p'(s)=f'(s)-\lambda p^\beta(s) & \text{for } s\in(0,1);\beta\in(0,1),\\ p(0)=p(1)=0, p(s)>0 & \text{if } s\in(0,1),\end{cases} \tag{1.4}$$

has a solution $(\bar\lambda,\bar p(s))$, by applying the shooting and regularization methods, where $|\bar\lambda|$ is the smallest parameter. Subsequently, we construct a weak travelling wave front solution of the equation (1.2) utilizing the solution $(\bar\lambda,\bar p(s))$.

§2. Formal Reduction

In this section we convert the problem of finding travelling wave front solutions of the equation (1.2) into the two-point boundary value problem (1.4).

Definition 2.1. *We call a function of the form*

$$u(x,t)=y(\xi),\quad \xi=x-Ct,$$

a weak travelling wave front solution of (1.2), *if the following conditions hold:*

(a) $y(\xi)$ *is an increasing, absolutely continuous function defined on* $(-\infty,+\infty)$.

(b) $y(-\infty)=0$ *and* $y(+\infty)=1$. (2.1)

(c) $z(\xi):=k(y(\xi))|y'(\xi)|^N$ *is* (*equivalent to*) *an absolutely continuous function defined on* $(-\infty,+\infty)$.

(d) $z(-\infty)=0$ *and* $z(+\infty)=0$. (2.2)

(e) *There exists a finite real number* C *such that*

$$g(y(\xi))-Cy'(\xi)=(k(y(\xi))|y'(\xi)|^N)':=z'(\xi)\quad \textit{a.e. on } (\infty,+\infty). \tag{2.3}$$

Let $u(x,t)=y(x-Ct)$ be a weak travelling wave front solution of (1.2). If $y(\xi)$ is strictly increasing, then the function $\xi=v(s)$, inverse to $s=y(\xi)$, exists, and hence $s=y(v(s))$ for all $s\in(0,1)$ and $y'(v(s))=1/v'(s)>0$ a.e. in $(0,1)$. Inserting $\xi=v(s)$ into (2.3) and then putting

$$w(s):=k(s)|v'(s)|^{-N},\qquad 0<s<1, \tag{2.4}$$

i.e.,

$$v'(s)=k^{1/N}(s)w^{-1/N}(s),\qquad 0<s<1, \tag{2.5}$$

we arrive at a two-point boundary value problem of the form

$$\begin{cases} w'(s)=g(s)k^{1/N}(s)w^{-1/N}(s)-C, & 0<s<1,\\ w(0)=0,\qquad w(1)=0.\end{cases} \tag{2.6}$$

Let us denote

$$p(s):=w^{(N+1)/N}(s),\quad \lambda:=C(N+1)/N,\quad \beta:=1/(N+1). \tag{2.7}$$

Then the problem (2.6) is transformed into (1.4).

It must be pointed out that the two endpoints $s=0$ and $s=1$ are singular in (2.6) (resp. (1.4)) and the parameter C (resp. λ) is unknown a priori and must be determined as part of the solution.

There is another formal reduction. In fact, (2.1)-(2.3) can be written as

$$\begin{cases} \frac{dy}{d\xi} = \left(\frac{z}{k(y)}\right)^{1/N}, \quad \frac{dz}{d\xi} = g(y) - C\left(\frac{z}{k(y)}\right)^{1/N}, \quad \xi \in (-\infty, +\infty), \\ (y,z)|_{\xi=-\infty} = (0,0), \qquad (y,z)|_{\xi=+\infty} = (1,0). \end{cases} \tag{2.8}$$

Eliminating ξ from (2.8), we obtain

$$\begin{cases} \frac{dz}{dy} = g(y)k^{1/N}(y)z^{-1/N} - C, \qquad 0<y<1, \\ z|_{y=0} = 0, \quad z|_{y=1} = 0, \end{cases}$$

which is identical to (2.6).

Some particular cases of (2.8) have been investigated by many authors[5–16] in the (y,z) phase plane. However, to our knowledge, the singular two-point boundary value problem (1.4) has not been directly studied before.

§3. Two-Point Boundary Value Problem

The present section is the core of this paper. In this section we demonstrate that the two-point boundary value problem (1.4) has at least one solution.

When $f(1) \neq 0$, we need to consider the two-point boundary value problem only involving one singular endpoint

$$\begin{cases} p'(s) = f'(s) - \lambda p^\beta(s), \quad 0<s<1 \\ p(0) = h \in (0, -f(1)), \quad p(1) = 0 \end{cases} \tag{$(1.4)_h^-$}$$

if $f(1)<0$ or

$$\begin{cases} p'(s) = f'(s) - \lambda p^\beta(s), \quad 0<s<1, \\ p(0) = 0, \qquad\qquad p(1) = h \in (0, f(1)) \end{cases} \tag{$(1.4)_h^+$}$$

if $f(1)>0$.

Definition 3.1. *A pair $(\lambda, p(s))$ is called a solution of $(1.4)_h^-$ (resp. $(1.4)_h^+$), if*

(a) *λ is a finite real number,*

(b) *$p(s)$ is a nonnegative, absolutely continuous function defined on $[0,1]$,*

(c) *$p(0)=h$ and $p(1)=0$ (resp. $p(0)=0$ and $p(1)=h$), and*

(d) *$p'(s) = f'(s) - \lambda p^\beta(s)$ a.e. on $[0,1]$.* (3.1)

In this definition, the parameter h is allowed to be zero. When $h=0$, $(1.4)_h^-$ (resp. $(1.4)_h^+$) is identical to (1.4).

Lemma 3.1. *Let $(\lambda, p(s))$ be a solution of $(1.4)_h^-$ (resp. $(1.4)_h^+$). Then*

$$\lambda = \frac{f(1)+h}{\int_0^1 p^\beta(s)ds} < 0 \left(resp.\ \lambda = \frac{f(1)-h}{\int_0^1 p^\beta(s)ds} > 0 \right). \tag{3.2}$$

Proof. Integrating (3.1) over $[0,1]$, we obtain (3.2).

Clearly, when $f(1)=0$, the pair $(0, f(s))$ is a unique solution of (1.4).

Lemma 3.2. *Let $(\lambda, p(s))$ be a solution of (1.4). Then $p(s)>0$ in $(0,1)$.*

Proof. When $f(1)=0$, $\lambda=0$, $p(s)=f(s)$. The lemma is obviously true.

First assume $f(1) < 0$. If $p(s_0) = 0$, where $s_0 \in (0,1)$, then Lemma 3.1 implies that $s_0 \in (A,1)$ when $f(s)$ satisfies (H_{41}). Integrating (3.1) over $[s_0, s_0+\delta]$, a subinterval of $[s_0,1)$, dividing the result by δ and then letting $\delta \downarrow 0$, we get $0 \le D_+p(s_0) = D_+f(s_0) < 0$, which is absurd.

Next, assume $f(1) > 0$. If $p(s_0) = 0$, where $s_0 \in (0,1)$, then Lemma 3.1 implies that $s_0 \in (0,B)$ when $f(s)$ satisfies (H_{43}). In the same way as above, we get

$$0 \ge D^-p(s_0) = D^-f(s_0) > 0,$$

which is also absurd. This completes the proof of the lemma.

Lemma 3.3. *Let* $(\lambda_1, p_1(s))$ *and* $(\lambda_2, p_2(s))$ *be solutions of* $(1.4)^{\pm}_h$ *with* $h \ge 0$. *If*

$$p_1(a) = p_2(a), p_1(b) = p_2(b), \quad 0 \le a < b \le 1, \text{ and } p_1(a) + p_1(b) > 0,$$

then $\lambda_1 = \lambda_2$ *and* $p_1(s) \equiv p_2(s)$ *on* $[a,b]$.

Proof. If this is not the case, by Lemma 3.1 and the continuity of $p_1(s)$ and $p_2(s)$, we may without loss of generality assume that $\lambda_1 < 0$, $\lambda_2 < 0$, and $p_1(s) < p_2(s)$ in (a,b). Note that

$$p_1'(s) - p_2'(s) = \lambda_2 p_2^\beta(s) - \lambda_1 p_1^\beta(s) \quad \text{a.e. in } (0,1). \tag{3.3}$$

When $\lambda_2 \le \lambda_1 < 0$, we integrate (3.3) over $[a,b]$ to give

$$0 = \lambda_2 \int_a^b [p_2^\beta(s) - p_1^\beta(s)]ds + (\lambda_2 - \lambda_1)\int_a^b p_1^\beta(s)ds < 0,$$

which is a contradiction.

When $\lambda_1 < \lambda_2 < 0$ and (say) $p_1(a) > 0$, there exists a subinterval $[a, s_0]$ of $[a,b)$, where $\lambda_2 p_2^\beta(s) - \lambda_1 p_1^\beta(s) > 0$. Integrating (3.3) over $[a, s_0]$, we get

$$p_1(s_0) - p_2(s_0) > 0, \qquad s_0 \in (a,b),$$

which contradicts the assumption that $p_1(s) - p_2(s) < 0$ in (a,b). The lemma is proved.

Corollary 3.1. $(1.4)^{\pm}_h$ *with* $h > 0$ *has at most one solution.*

Corollary 3.2. *Let both* $(\lambda_1, p_1(s))$ *and* $(\lambda_2, p_2(s))$ *be solutions of* (1.4). *Then either* $p_1(s) \equiv p_2(s)$ *on* $[0,1]$, $\lambda_1 = \lambda_2$ *or* $p_1(s) > p_2(s)$ *in* $(0,1)$, $|\lambda_1| < |\lambda_2|$.

Proof. From Lemma 3.2, we know that $p_1(s) > 0$ and $p_2(s) > 0$ in $(0,1)$. There are two possibilities. If there exists a point $c \in (0,1)$ such that $p_1(c) = p_2(c)$, then $p_1(s) \equiv p_2(s)$ on $[0,1]$ and $\lambda_1 = \lambda_2$, by Lemma 3.3. If there is no interior point of $(0,1)$ at which $p_1(s) = p_2(s)$, then (say) $p_1(s) > p_2(s)$ in $(0,1)$ and hence

$$|\lambda_1| = |f(1)| / \int_0^1 p_1^\beta(s)ds < |f(1)| / \int_0^1 p_2^\beta(s)ds = |\lambda_2|$$

by Lemma 3.1. The proof concludes.

In what follows we demonstrate that under the hypothesis (H_{41}) or (H_{42}) the problem $(1.4)^-_h$ has a solution by applying the shooting and regularization methods.

Lemma 3.4. *For each pair of fixed* $\lambda \le 0$ *and* $h \in (0, -f(1))$, *the initial value problem*

$$\begin{cases} p'(s) = f'(s) - \lambda p^\beta(s), & s > 0, \\ p(0) = h \end{cases} \tag{3.4$^\lambda_h$}$$

has a unique solution $p(s;\lambda,h)$ on $[0,r)$, its maximal interval of existence. If $r<1$, then $p(r-0;\lambda,h)=0$; if $r=1$, then $p(1-0;\lambda,h)\geq 0$. Moreover, $p(s;\lambda,h)$ is continuous and strictly increasing in h and $-\lambda$ in the following sense:

(a) *If $h_1>h_2>0$, then $p(s;\lambda,h_1)>p(s;\lambda,h_2)$ in the maximal interval of existence for $p(s;\lambda,h_2)$.*

(b) *If $-\lambda_1>-\lambda_2\geq 0$, then $p(s;\lambda_1,h)>p(s;\lambda_2,h)$ in the maximal interval of existence for $p(s;\lambda_2,h)$.*

Although the proof of Lemma 3.4 (for the case when $f'(s)$ is continuous on $[0,1]$) can be found in [17, pp.8-15], we shall reproduce it here, in consideration of not only the completeness but also the fact that a detailed knowledge of the properties of p will be important in our discussion of existence.

Proof of Lemma 3.4. Let $[0,\delta]$ be a (maximal) subinterval of $[0,1]$ on which

$$q(s):=\frac{h}{2}+f(s)-\lambda s\left(\frac{h}{2}\right)^{\beta}\geq 0$$

and put

$$M^{1/\beta}:=\max_{s\in(0,\delta)}\{q(s)\},\qquad L:=\max\left\{M,\beta\left(\frac{h}{2}\right)^{\beta-1}\right\},$$

$$p_0(s):=\frac{h}{2},\qquad p_{n+1}(s):=(\Phi p_n)(s),\qquad n=0,1,2,\cdots,\tag{3.5}$$

where

$$(\Phi p)(s):=h+f(s)-\lambda\int_0^s p^{\beta}(t)dt.\tag{3.6}$$

It is clear that

$$0\leq p_1(s)-p_0(s)=q(s)\leq M^{1/\beta},\qquad s\in[0,\delta],$$

and

$$0\leq p_1^{\beta}(s)-p_0^{\beta}(s)\leq[p_1(s)-p_0(s)]^{\beta}\leq L,\qquad s\in[0,\delta].$$

Here we have used the inequality

$$(a+b)^{\beta}\leq a^{\beta}+b^{\beta}\text{ for }a,b\geq 0,0<\beta<1.$$

The definition of Φ implies that

$$\frac{h}{2}=p_0(s)\leq p_1(s)\leq\cdots\leq p_n(s)\leq p_{n+1}(s)\leq\cdots,\qquad s\in[0,\delta],$$

and hence

$$0\leq p_{n+1}^{\beta}(s)-p_n^{\beta}(s)\leq L[p_{n+1}(s)-p_n(s)],\qquad s\in[0,\delta],n=1,2,\cdots.$$

Thus, it is readily verified by induction that

$$0\leq p_{n+1}(s)-p_n(s)\leq\frac{1}{n!}(-\lambda Ls)^n,\qquad s\in[0,\delta],n=1,2,\cdots.$$

It follows that the series

$$p_1(s)+\sum_{n=1}^{\infty}[p_{n+1}(s)-p_n(s)]$$

is uniformly convergent on $[0,\delta]$, that is,

$$p(s;\lambda,h) := \lim_{n\to\infty} p_n(s) \text{ exists uniformly on } [0,\delta]. \tag{3.7}$$

Thus, term-by-term integration is applicable to the integrals in (3.5) and gives

$$p(s;\lambda,h) = h + f(s) - \lambda \int_0^s p^\beta(t;\lambda,h)dt. \tag{3.8}$$

Therefore, (3.7) is a solution of $(3.4)_h^\lambda$ on $[0,\delta]$.

When $\delta < 1$, we consider the initial value problem

$$\begin{cases} p'(s) = f'(s) - \lambda p^\beta(s), & s \geq \delta, \\ p(\delta) = h_1 := p(\delta;\lambda,h). \end{cases}$$

Repeating the above argument, we can conclude that there is a subinterval $[0,\delta+\delta_1]$ of $[0,1]$ on which a solution $p(s;\lambda,h)$ of $(3.4)_h^\lambda$ is defined. Continuing this procedure, we can obtain a solution $p(s;\lambda,h)$ of $(3.4)_h^\lambda$ on $[0,r)$, where $r = \delta+\delta_1+\delta_2+\cdots$.

The local Lipschitz continuity of the nonlinear function p^β with respect to $p>0$ implies the uniqueness of $p(s;\lambda,h)$. The remainders of Lemma 3.4 follows from Lemma 3.3 and the part of Lemma 3.4 which has already been proved.

Lemma 3.5. *Let $E_h := \{\lambda \leq 0; p(1-0;\lambda,h) > 0\}$. Then there is a $\lambda_0 < 0$ such that $(-\infty,\lambda_0] \subset E_h$.*

Proof. If $f(s)$ satisfies (H_{41}), we choose a $\lambda_0 < 0$ such that

$$\frac{h}{2} + f(1) - \lambda_0 A\left(\frac{h}{2}\right)^\beta = 0;$$

if $f(s)$ satisfies (H_{42}), we pick out a $\lambda_0 < 0$ such that

$$\frac{h}{2} + \inf_{0<s<1} D_+ f(s) - \lambda_0 \left(\frac{h}{2}\right)^\beta = 0.$$

Therefore, for all $\lambda \leq \lambda_0$

$$q(s) := \frac{h}{2} + f(s) - \lambda s\left(\frac{h}{2}\right)^\beta \geq 0, \qquad s\in[0,1]$$

and

$$p(s;\lambda,h) := \lim_{n\to\infty} p_n(s) \geq \frac{h}{2}, \qquad s\in[0,1].$$

This means that $(-\infty,\lambda_0]\subset E_h$.

Lemma 3.6. *E_h is an open set.*

Proof. The lemma follows from Lemma 3.4 and the fact that $\lambda = 0$ is not in E_h.

Lemma 3.7. *$(1.4)_h^-$ with $h>0$ has a solution $(\lambda(h), p(s,h))$. Moreover, $p(s,h)$ and $\lambda(h)$ are continuous and strictly increasing in $h>0$.*

Proof. Let us define $\lambda(h) := \sup E_h(<0)$ and pick out a sequence $\{\lambda_j\in E_h; j=1,2,\cdots\}$ which is strictly increasing and converges to $\lambda(h)$. Then the sequence $\{p(s;\lambda_j,h); j = 1,2,\cdots\}$ is strictly decreasing and converges to a limit $p(s,h)$ uniformly on $[0,1]$. Inserting the pair $(\lambda_j, p(s;\lambda_j,h))$ into the equation (2.8) and then letting $j\to\infty$, we get

$$p(s;h) = h + f(s) - \lambda(h)\int_0^x p^\beta(t,h)dt(\geq 0), \qquad x\in[0,1]. \tag{3.9}$$

We claim that the pair $(\lambda(h), p(s,h))$ is a solution of $(1.4)_h^-$. It is enough to show that $p(1,h)=0$. If $p(1,h)>0$, then $\lambda(h)\in E_h$ by Lemma 3.4. This contradicts the fact that E_h is an open set.

If $h_1>h_2>0$, then $p(s,h_1)>p(s,h_2)>0$ in $[0,1)$ by Lemmas 3.3 and 3.2. From this it follows by (3.9) that

$$h_1-h_2=\lambda(h_1)\int_0^1 p^\beta(t,h_1)dt-\lambda(h_2)\int_0^1 p^\beta(t,h_2)dt>0,$$

which implies that $0>\lambda(h_1)>\lambda(h_2)$. This completes the proof.

Lemma 3.8. *Let* (H_{41}) *or* (H_{42}) *hold. Then the problem* (1.4) *has a smallest parameter solution* $(\bar\lambda,\bar p(s))$ *in the following sense:*

If $(\lambda,p(s))$ *is any solution of* (1.4)*, then* $|\lambda|\ge|\bar\lambda|$ *and* $p(s)\le\bar p(s)$ *in* $(0,1)$.

Proof. Let $(\lambda(h),p(s,h))$ be a solution of $(1.4)_h^-$. Then $p(s,h)$ and $\lambda(h)$ are continuous and strictly increasing in $h>0$. Consequently,

$$\bar p(s):=\lim_{h\downarrow 0}p(s;h)(\ge 0)\text{ exists uniformly on }[0,1],$$

$$\bar\lambda:=\lim_{h\downarrow 0}\lambda(h)=f(1)/\int_0^1\bar p^\beta(s)ds<0.$$

Inserting $(\lambda(h),p(s,h))$ into (3.9) and then letting $h\downarrow 0$, we get

$$\bar p(s)=f(s)-\bar\lambda\int_0^s\bar p^\beta(t)dt\ge 0\qquad\text{on }[0,1],$$

which shows that $(\bar\lambda,\bar p(s))$ is a solution of (1.4) and $\bar p(s)>0$ in $(0,1)$ by Lemma 3.2.

We now prove that $|\bar\lambda|$ is the smallest parameter. If $(\bar\lambda,\bar p(s))$ is not the smallest parameter solution of (1.4), then there exists a solution $(\lambda,p(s))$ such that $|\lambda|<|\bar\lambda|$ and $p(s)>\bar p(s)$ in $(0,1)$ by Corollary 3.2. Since $\bar p(s)=\lim_{h\downarrow 0}p(s,h)$, there are two numbers $h>0$ and $a\in(0,1)$ such that $p(a)=p(a,h)$. According to Lemma 3.3, $p(s,h)\equiv p(s)$ on $[a,1]$. This is not possible. The proof of Lemma 3.8 concludes.

In very much the same way, we can demonstrate that under the hypothesis (H_{43}) or (H_{44}) the problem (1.4) has a solution $(\bar\lambda,\bar p(s))$, where $|\bar\lambda|$ is the smallest parameter.

We summarize the results above in the following statement.

Theorem 3.1. *Suppose that* (H_4) *holds. Then the smallest parameter solution* $(\bar\lambda,\bar p(s))$ *of the problem* (1.4) *exists. Moreover,* $\bar p(s)$ *is positive in* $(0,1)$.

§4. Weak Travelling Wave Front Solutions

In this section we construct a weak travelling wave front solution $u(x,t)=y(\xi)$, $\xi=x-Ct$, of (1.2) for some constant wave speed C, utilizing the solution $(\bar\lambda,\bar p(s))$ of (1.4).

We first introduce four propositions that are used in the ensuing paragraphs.

Proposition 4.1 (Corollary 4 in [18]). *If* $y(\xi)$ *is increasing on* $[a,b]$ *and if* $w(s)$ *is absolutely continuous on* $[y(a),y(b)]$*, then* $w(y(\xi))$ *has a finite derivative a.e. on* $[a,b]$ *and the chain rule*

$$\frac{d}{d\xi}w(y(\xi))=w'(y(\xi))y'(\xi)$$

holds.

Proposition 4.2 (Corollary 6 in [18]). *Suppose that $y(\xi)$ is increasing and absolutely continuous on $[a,b]$ and $\phi(s)$ is Lebesgue integrable on $[y(a),y(b)]$. Then $\phi(y(\xi))y'(\xi)$ is integrable on $[y(a),y(b)]$ and the change of variables formula*

$$\int_a^b \phi(y(\xi))y'(\xi)d\xi = \Phi(y(b)) - \Phi(y(a))$$

holds, where $\Phi(s)$ is an indefinite integral of $\phi(s)$.

Proposition 4.3. *If $v(s)$ is strictly increasing and locally absolutely continuous in $(0,1)$, then the function $s = y(\xi)$, inverse to $\xi = v(s)$, is strictly increasing and absolutely continuous on (ξ_0,ξ_1), where $\xi_0 := v(0+0)$ and $\xi_1 := v(1-0)$. Moreover,*

$$y(\xi_0) := \lim_{\xi\downarrow\xi_0} y(\xi) = 0 \text{ and } y(\xi_1) := \lim_{\xi\uparrow\xi_1} y(\xi) = 1.$$

Proof. Clearly, it is enough to show that $y(\xi)$ is absolutely continuous on (ξ_0,ξ_1).

Since $y(\xi)$ is strictly increasing and absolutely continuous in (ξ_0,ξ_1), $y'(\xi)$ exists a.e. and is locally integrable in (ξ_0,ξ_1). Let $[a,b]$ be a subinterval of $(0,1)$. Then an application of the change of variables formula gives

$$\int_{v(a)}^{v(b)} y'(\xi)d\xi = \int_\alpha^\beta y'(v(s))v'(s)ds = b-a = y(v(b)) - y(v(a)).$$

Here we have used the fact that $s = y(v(s))$ in $(0,1)$. Letting $a\downarrow 0$ and $b\uparrow 1$ yields

$$\int_{\xi_0}^{\xi_1} y'(\xi) = 1,$$

which shows that $y(\xi)$ is absolutely continuous on (ξ_0,ξ_1).

Proposition 4.4. *Let $y(\xi)$ be an increasing, absolutely continuous function defined on $(-\infty,+\infty)$ with $y(-\infty)=0$ and $y(+\infty)=1$ and let $w(s)$ be an absolutely continuous function defined on $[0,1]$. The $w(y(\xi))$ is absolutely continuous on $(-\infty,+\infty)$.*

Proof. The lemma is an immediate consequence of Propositions 4.1 and 4.2.

Let $(\bar\lambda,\bar p(s))$ be the smallest parameter solution of (1.4) and let

$$w(s) := \bar p^{N/(N+1)}(s), \quad C := \frac{\bar\lambda N}{N+1}, \quad N := \frac{1-\beta}{\beta}, \quad v(s) := \int_{\frac12}^s \left(\frac{k(t)}{w(t)}\right)^{1/N} dt.$$

Then

$$\begin{aligned} &w(0)=w(1)=0, w(s)>0 \text{ in } (0,1),\\ &v'(s) = k^{1/N}(s)w^{-1/N}(s) \text{ a.e. in } (0,1), &(4.1)\\ &w'(s) = g(s)v'(s) - C \text{ a.e. in } (0,1), &(4.2)\end{aligned}$$

and

$$w(s) = k(s)|v'(s)|^{-N} \text{ a.e. in } (0,1). \tag{4.3}$$

It follows from (4.1) that $v(s)$ is strictly increasing and locally absolutely continuous in $(0,1)$ and hence the function $s=y(\xi)$, inverse to $\xi=v(s)$, exists in (ξ_0,ξ_1), where

$$\xi_0 := v(0+0), \quad \xi_1 := v(1-0), \qquad \text{i.e., } y(\xi_0)=0, y(\xi_1)=1.$$

Proposition 4.3 asserts that $y(\xi)$ is absolutely continuous and strictly increasing on (ξ_0, ξ_1). When ξ_0 (resp. ξ_1) is finite, we define

$$y(\xi) = 0 \text{ for all } \xi \le \xi_0 \qquad (\text{resp. } y(\xi) = 1 \text{ for all } \xi \ge \xi_1).$$

Clearly, $y(\xi)$ is increasing and absolutely continuous on $(-\infty, +\infty)$ no matter whether ξ_0 (or ξ_1) is finite or not.

Inserting $s = y(\xi)$ into (4.2) and (4.3), we obtain

$$w'(y(\xi))y'(\xi) = g(y(\xi)) - Cy'(\xi) \text{ a.e. in } (\xi_0, \xi_1), \tag{4.4}$$

$$w(y(\xi)) = k(y(\xi))|y'(\xi)|^{N-1}y'(\xi) \text{ a.e. in } (\xi_0, \xi_1), \tag{4.5}$$

and

$$\lim_{\xi \downarrow \xi_0} w(y(\xi)) = \lim_{\xi \uparrow \xi_1} w(y(\xi)) = 0,$$

and hence (by the chain rule)

$$\begin{aligned}(k(y(\xi))|y'(\xi)|^{N-1}y'(\xi))' &= w'(y(\xi))y'(\xi) \\ &= g(y(\xi)) - Cy'(\xi) \text{ a.e. in } (\xi_0, \xi_1).\end{aligned} \tag{4.6}$$

Here we have used the fact that $y'(\xi) = 1/v'(y(\xi)) > 0$ a.e. in (ξ_0, ξ_1). When ξ_0 (or ξ_1) is finite, the equalities (4.4), (4.5) and (4.6) read $0 = 0$ on $(-\infty, \xi_0]$ (or on $[\xi_1, +\infty)$).

Proposition 4.4 claims that the function $w(y(\xi))$ is absolutely continuous on $(-\infty, +\infty)$ and hence $u(x,t) = y(x - Ct)$ is a weak travelling wave front solution of the equation (1.2), where $|C| = |\bar{\lambda}|N/(N+1)$ is the smallest wave speed.

We can summarize the preceding discussion in the following statement.

Theorem 4.1. *Let* (H_1)-(H_4) *hold. Then the equation* (1.2) *possesses a weak travelling wave front solution* $u(x,t) = y(x - Ct)$ *with* $|C| = |\bar{\lambda}|N/(N+1)$ *being the smallest wave speed, where* $\bar{\lambda}$ *is given by Theorem* 3.1.

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