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A COMPLETE METRIC OF POSITIVE CURVATURE ON R^n AND EXISTENCE OF CLOSED GEODESICS**

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Abstract

An example of complete Riemannian metric of positive (or nonnagative) curvature on R^n such as $ds^2 = a(x)dx^2$ is obtained by direct caculations. Furthermore, by using a geodesic convex condition and a theorem for complete noncompact Riemannian manifold, an existence result of periodic solution of prescribed energy for a singular Hamiltonian system is also obtained.

Keywords Positive curvature, Complete metric, Geodesic, Riemannian manifold, Hamiltonian system.
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§1. Introduction

This paper is initiated by looking for a periodic solution of

$$q''(t) + V'(q(t)) = 0 \tag{1.1}$$

such that

$$\frac{1}{2}|q'(t)|^2 + V(q(t)) = h, \ \forall t \in R. \tag{1.2}$$

Here $n \geq 2$, $h \in R$ is a given number, $q \in C^2(R, R^n \backslash \{0\})$, $q''(t)$ is the second derivative of $q(t)$, $V \in C^2(R^n \backslash \{0\}, R)$, and $V'(x)$ is the gradient of the function V at x.

Recently, many papers concern the study of prescribed energy problem (1.1)-(1.2) (see, for example, [2-4, 12, 20]). In [6] Benci-Giannoni studied the existence of periodic solution (1.1)-(1.2) confined in an annulus $\{x \in R^n; r_0 \leq |x| \leq r\}$ and in [23] we gave a geometric explanation for their result and proved the following

Theorem 1.1. *Let $h \in R$, and $D(y;t) = \{x \in R^n; |x - y| < t\}$ for $y \in R^n$ and $t > 0$. Assume that there exist $r > 0$, $r_i \in (0,r)$ and x^0, $x^i \in R^n$, $i = 1,\dots,m$, such that $\bigcup D(x^i;r_i) \subset D(x^0;r)$ and $D(x^i;r_i) \bigcap D(x^j;r_j) = \phi$ for $i \neq j$. Set $G = \overline{D}(x^0;r) \backslash \bigcup D(x^i;r_i)$. If $V \in C^2(G,R)$ satisfies*

(i) $h - V > 0$ *on* G;

(ii) $h - V(x) - \frac{1}{2}\langle V'(x), x - x^i\rangle < 0$, $\forall x \in \partial D(x^i;r_i)$, $1 \leq i \leq m$;

(iii) $h - V(x) - \frac{1}{2}\langle V'(x), x - x^0\rangle > 0$, $\forall x \in \partial D(x^0;r)$;

then we can enlarge the Jacobi metric tensor $(h - V)\delta_{ij}$ in a neighborhood of G so that G is geodesic convex with respect to the enlarged Jacobi metric.

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This paper is largely motivated by the above result and the following result (see, for example, [13, Proposition 3] or [22, Theorem 4.3]).

Theorem 1.2. *Let M be a complete Riemannian manifold with nonnegative sectional curvature outside some compact set. Then for any $y \in M$ there is a family $\{G(t); t > 0\}$ of compact totally convex sets with $G(t) \subset G(s)$ for $t < s$ and the balls $B_t(y) := \{x \in M; d(x,y) < t\}$ are contained in $G(t)$ for $t > 0$.*

The main results of the paper can be stated as the following

Theorem 1.3. *Let $h \in R$ and let $D(y;t) = \{x \in R^n; |x-y| < t\}$ for $y \in R^n$ and $t > 0$. Assume that there exist $r_i > 0$ and $x^i \in R^n, i = 1, \dots, m$, such that $D(x^i;r_i) \bigcap D(x^i;r_j) = \phi$ for $i \neq j$. Let $G_0 = R^n \backslash \bigcup D(x^i;r_i)$. If $V \in C^2(G_0, R)$ satisfies*

(i) *$h - V > 0$ on G_0 and there exists a constant $C > 0$ such that $h - V(x) \geq C|x|^{-2}$ for $|x|$ big enough;*

(ii) *$h - V(x) - \frac{1}{2}\langle V'(x), x - x^i \rangle \leq 0$, $\forall x \in \partial D(x^i;r_i)$, $1 \leq i \leq m$;*

(iii) *the Hessian matrix $V''(x)$ of $V(x)$ is semi-positive definite for $|x|$ big enough;*

then (1.1)-(1.2) *possesses a nonconstant periodic solution in G_0.*

Theorem 1.4. *Let $a \in C^2(R^n, R)$ satisfy*

(i) *$a > 0$ on R^n and there exists a constat $C > 0$ such that $a(x) \geq C|x|^{-2}$ for $|x|$ big enough;*

(ii) *The Hessian matrix $a''(x)$ of $a(x)$ is semi-negative (or negative) definite for all $x \in R^n$.*

Then the metric $ds^2 = a(x)\sum dx_i^2$ is a complete one on R^n with nonnegative (or resp. positive) sectional curvature.

Remark 1.1. It is well-known that the metric $ds^2 = (1 + \frac{1}{4}K|x|^2)^{-2}\sum dx_i^2$ where K =constant> 0, defined on R^n, has curvature K. But unfortunately, this metric is not complete. In fact, letting $x^m = (m, 0, \dots, 0)$ for $m = 1, 2, \dots$, we see that $d(x^m;0) \leq \int_0^m (1+Kt^2)^{-2}dt \leq \int_0^{+\infty}(1+Kt^2)^{-2}dt < \infty$ for all m and $|x^m| \longrightarrow \infty$ as $m \longrightarrow \infty$. It seems that, as we know (cf. [9]), Theorem 1.4 is the simplest example of complete metrics with positive curvature on R^n.

Remark 1.2. Usually, there does not exist any closed geodesic in a complete, noncontractible and noncompact Riemannian manifold if we do not add other geometric conditions on the manifold. For example, by Gordon [11, Theorem 2], the metric $ds^2 = (|x|^{-2} + |x|^{-1})\sum dx_i^2$ defined on $R^n \backslash \{0\}$ is complete, but no closed geodesic exists. In fact, let $a(x) = |x|^{-2} + |x|^{-1}$ and suppose that p is a closed geodesic, then by Maupertuis-Jacobi principle (see [1] or [5]), there exists correspondingly a nonconstant T-period function $q \in C^2(R, R^n \backslash \{0\})$ such that

$$q''(t) - a'(q(t)) = 0 \quad \text{and} \quad \frac{1}{2}|q'(t)|^2 - a(q(t)) = 0.$$

Thus we have

$$\int_0^T (\langle q'', q \rangle - \langle a'(q), q \rangle)dt = 0$$

and hence

$$\int_0^T \left(a(q) + \frac{1}{2}\langle a'(q), q \rangle\right)dt = 0.$$

But this is impossible because $a(x) + \frac{1}{2}\langle a'(x), x\rangle > 0$ for all $x \in R^n\backslash\{0\}$.

The organization of the paper is as follows. We first give some preliminaries in Section 2, and then we show Theorems 1.3-1.4.

Notations. Throughout the paper we let $\langle x, y\rangle = \sum x_i y_i, \forall x, y \in R^n$ and $|x| = (\langle x, x\rangle)^{\frac{1}{2}}$.

Given $y \in R^n, r > r_0 > 0$, we denote $D(y;r) = \{x \in R^n; |x-y| < r\}$, $D(y; r_0, r) = \{x \in R^n; r_0 < |x-y| < r\}$, and $D(r) = \{x \in R^n; |x| < r\}$.

Given $G \subset R^n$, we denote by $\overline{G}$ its closure and by ∂G its boundary.

§2. Preliminaries

We begin with the following defination (cf. [16] or [22]).

Definition 2.1. *Let M be a Riemannian manifold.*

(i) *A non-empty set G in M is called strongly convex, if for any $x, y \in G$ there is a unique minimal geodesic joining x and y with image in G. A non-empty set G is called convex if for any $x \in G$, there exists an $r = r(x)$ so that $B_r(x) \cap G$ is strongly convex. Here $B_r(x) := \{y \in M; d(x,y) < r\}$.*

(ii) *A non-empty set G in M is called totally convex, if for any $x, y \in G$ and any geodesic segment p joining x and y, the image of p lies in G.*

Remark 2.1. From this definition, one can see that if G is a non-empty compact set in M, then G is convex if and only if there is an $\eta > 0$ such that for any $x, y \in G$, with $d(x,y) < \eta$, there exists a unique minimal geodesic joining x and y with image in G.

Lemma 2.1. *If $a \in C^2(R^n, R)$ satisfies*

(i) *$a > 0$ on R^n;*

(ii) *there exists a proper C^3 funtion U on R^n such that $a(x) \geq |U'(x)|^2$ for all $x \in R^n$; then the Riemannian metric $ds^2 = a(x)\sum dx_i^2$ on R^n is complete.*

Proof. See [11, Theorem 2].

From Lemma 2.1, we have the following

Lemma 2.2. *If $a \in C^2(R^n, R)$ satisfies $a > 0$ on R^n and there exists a constant C such that $a(x) \geq C|x|^{-2}$ for $|x|$ big enough, then the metric $ds^2 = a(x)\sum dx_i^2$ on R^n is complete.*

Proof. From Lemma 2.1, we need only to prove that there exists a proper smooth function U on R^n such that $a(x) \geq |U'(x)|^2$ for all $x \in R^n$. We first choose a smooth function U_1 on R^n so that $U_1(x) = \log|x|$ for $|x|$ big enough. Then it is easy to see that there exists a constant $C_1 > 0$ small enough so that $U := C_1 U_1$ satifies $a(x) \geq |U'(x)|^2$ for all $x \in R^n$. Furthermore, U is a proper function on R^n because $U(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$.

§3. Proof of Theorems 1.3-1.4

In this section, we will first prove Theorem 1.4 and then prove Theorem 1.3. Now we first prove the following lemma.

Lemma 3.1. *Let $a \in C^2(R^n, R)$ be a positive function. Then the sectional curvature K of the Riemannian metric $ds^2 = a(x)\sum dx_i^2$ defined on R^2 can be calculated by*

$$K(T_{\widetilde{x}}R^2) = -\frac{1}{2a(\widetilde{x})}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{a(x)}\frac{\partial a(x)}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{a(x)}\frac{\partial a(x)}{\partial x_2}\right)\right]_{x=\widetilde{x}}$$

where $x=(x_1,x_2)$ is the standard coordinate system of R^2 and $T_{\tilde{x}}R^2$ is the tangent space of R^2 at $\tilde{x}$

Proof. Let $X=X^i\frac{\partial}{\partial x_i}=\frac{\partial}{\partial x_1}$ and $Y=Y^i\frac{\partial}{\partial x_i}=\frac{\partial}{\partial x_2}$, where $X^1=Y^2=1$ and $X^2=Y^1=0$, be the tangent vectors correspounding to the coordinate $x=(x_1,x_2)$ at $\tilde{x}\in R^2$. Then $T_{\tilde{x}}R^2=\text{span}\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right\}$, $\|X\|^2=a(\tilde{x})=\|Y\|^2$ and $XY=0$. Thus we have

$$K(T_{\tilde{x}}R^2)=-\frac{R(X,Y,X,Y)}{\|X\|^2\|Y\|^2XY}=-(a(x))^{-2}R(X,Y,X,Y).$$

Since

$$R(X,Y,X,Y)=R_{ijkl}X^iY^jX^kY^l=R_{1212}=R^m_{112}g_{m2}=a(x)R^2_{112},$$

$$R^j_{ikl}=\frac{\partial T_{il}}{\partial x_k}-\frac{\partial T_{ik}}{\partial x_l}+T^m_{il}T^j_{mk}-T^m_{ik}T^j_{ml},$$

$$T_{ij}=\frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x_j}+\frac{\partial g_{jl}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_l}\right)=\frac{1}{2a(x)}\left(\frac{\partial g_{ik}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_i}-\frac{\partial g_{ij}}{\partial x_l}\right),$$

where $g_{ij}=a(x)\delta_{ij}$ and $g^{ij}=\delta_{ij}/a(x)$, it follows that

$$\begin{aligned}R^2_{112}&=\frac{\partial T_{12}}{\partial x_1}-\frac{\partial T_{11}}{\partial x_2}+T^1_{12}T^2_{11}+T^2_{12}T^2_{21}-T^1_{11}T^2_{12}-T^2_{11}T^2_{22}\\&=\frac{\partial T_{12}}{\partial x_1}-\frac{\partial T_{11}}{\partial x_2}+T^2_{11}(T^1_{12}-T^2_{22})+T^2_{21}(T^2_{21}-T^1_{11}).\end{aligned}$$

But we have by direct calculations

$$T^2_{12}=\frac{1}{2a(x)}\frac{\partial a(x)}{\partial x_1}=T^1_{21}=T^1_{11},\quad T^2_{11}=-\frac{1}{2a(x)}\frac{\partial a(x)}{\partial x_2}=T^1_{12}=T^2_{22}.$$

Thus

$$R^2_{112}=-\frac{\partial}{\partial x_1}\left(\frac{1}{2a(x)}\frac{\partial a(x)}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{1}{2a(x)}\frac{\partial a(x)}{\partial x_2}\right),$$

and hence the claim of the lemma follows.

Proof of Theorem 1.4. By Lemma 2.2, we need only to show that the sectional curvature K of the metric ds^2 is nonnegative (or positive) if $a''(x)$ is semi-negative (resp. negative) definite for any $x\in R^n$. For convienience, we only consider the case that a'' is semi-negative definite.

If $n=2$, then by Lemma 3.1 it is easy to see that $K(T_xR^2)\geq 0$ for any $x\in R^2$.

Now we assume that $n>2$. $\forall\tilde{x}\in R$, let $E\subset T_{\tilde{x}}(R^n)$ be an arbitrary two dimensional tangent subspace. We can assume that there exist $X,Y\in E$ such that $\|X\|=\|Y\|=1$ and $XY=0$. Then we can expand X, Y to an orthonormal base $X_1=X$, $X_2=Y$, $X_3,\dots,X_n$ in $T_{\tilde{x}}(R^n)$. Let

$$X_i=d_{ij}\frac{\partial}{\partial x_j},i=1,\dots,n,\text{ and }D:=(d_{ij}).$$

Then $a(\tilde{x})DD^T=(\delta_{ij})$, where D^T is the transposed matrix of D. If we choose $u_i=d_{ij}x_j,i=1,\dots,n$, as a new coordinate system of R^n and let $\tilde{u}=D\tilde{x}$, then we have

$$X_1=\frac{\partial}{\partial u_1},\ X_2=\frac{\partial}{\partial u_2},\dots,\ X_n=\frac{\partial}{\partial u_n}$$

and

$$ds^2=a(x)dx_i^2=\frac{a(D^{-1}u)}{a(\tilde{x})}du_i^2=\tilde{a}(u)du_i^2,$$

where $\widetilde{a}(u) := \frac{a(D^{-1}u)}{a(\bar{x})}$ and $\widetilde{a}(\bar{u}) = 1$.

Now according to the proof of Lemma 2.1 we have

$$K(E) = -R(X_1, X_2, X_1, X_2) = -R_{1212} = -R^2_{112}$$
$$= -\frac{1}{2}\left[\frac{\partial}{\partial u_1}\left(\frac{1}{\widetilde{a}(u)}\frac{\partial \widetilde{a}(u)}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{\widetilde{a}(u)}\frac{\partial \widetilde{a}(u)}{\partial u_2}\right)\right]_{u=\bar{u}}.$$

It follows that $K(E) \geq 0$ if and only if

$$\left[\frac{\partial^2 \widetilde{a}(u)}{\partial u_1^2} + \frac{\partial^2 \widetilde{a}(u)}{\partial u_2^2}\right]_{u=\bar{u}} \leq \left[\left(\frac{\partial \widetilde{a}(u)}{\partial u_1}\right)^2 + \left(\frac{\partial \widetilde{a}(u)}{\partial u_2}\right)^2\right]_{u=\bar{u}}.$$

But the assumption that $a''(x)$ is semi-negative definite implies

$$\frac{\partial^2 \widetilde{a}(\bar{u})}{\partial u_1^2} + \frac{\partial^2 \widetilde{a}(\bar{u})}{\partial u_2^2} = \frac{1}{a(\bar{x})}\left[\frac{\partial^2 a(x)}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial u_1}\frac{\partial x_j}{\partial u_1} + \frac{\partial^2 a(x)}{\partial x_i \partial x_j}\frac{\partial x_i}{\partial u_2}\frac{\partial x_j}{\partial u_2}\right]_{x=\bar{x}} \leq 0.$$

Thus we have $K(E) \geq 0$.

Remark 3.1. From the proof of Theorem 1.4 and because of a theorem of [8], one can see that we have actually proved the following

Proposition 3.1. *Let M be a complete noncompact Riemannian manifold which is locally conformal to R^n with its usual Euclidean metric, and the locally conformal functions have their positive definite Hessians. Then M is a complete manifold with positive curvature and hence M is diffeomorphic to R^n.*

Proof of Theorem 1.3. First we can choose a C^2 function $\widetilde{V}$ on R^n such that $\widetilde{V}(x) = V(x)$ for any $x \in G_0 = R^n \backslash \bigcup D(x^i; r_i)$ and $h - \widetilde{V} > 0$ on R^n. Then by the assumptions (i), (ii) and the proof of Theorem 1.4, we see that, with respect to the enlarged Jacobi metric $ds^2 = (h - \widetilde{V})\sum dx_i^2$, R^n is a complete Riemannian manifold with nonnegative sectional curvature outside some compact set. Thus, from Theorem 1.4, it follows that for any fixed $y \in R^n$ there is a family $\{G(t); t > 0\}$ of compact totally convex set with $G(t) \subset G(s)$ for $t < s$ and the balls $B_t(y) = \{x \in R^n; d(x, y) < t\}$ are contained in $G(t)$ for $t > 0$. Since the $(n-1)$-th homotopy group $\prod_{n-1}(G_0) \neq 0$ (see [7, Proposition 17.11]), there is a homotopically non-trivial map $f : S^{n-1} \longrightarrow G_0$. Thus there exist a $t_0 > 0$ big enough and an $r > \max\{r_1, \ldots, r_m\}$ such that the image of f is contained in $G(t_0)$ and $\bigcup D(x^i; r_i, r) \subset G(t_0)$. Let $G = G(t_0) \bigcap G_0$. Then $\prod_{n-1}(G) \neq 0$ because f is of course homotopically trivial in G_0 if it is in G (see [19, p.405, Corollary 24]).

Now we first assume that V satisfies the following strict inequalities other than the inequalities (ii) of Theorem 1.3.

$$h - V(x) - \frac{1}{2}\langle V'(x), x - x^i\rangle < 0, \quad \forall x \in \partial D(x^i; r_i),\ 1 \leq i \leq m. \tag{3.1}$$

Then according to the proof of Theorem 1.1 (see [23, Proposition 1.4]), by (3.1) and the convexity of $G(t_0)$, one can see that G is actually geodesic convex. Now by [22, Theorem 2] or by an argument similar to the proof of [15, Appendix, Theorem A.1.5], there exists a closed geodesic p in G. Thus p coorespounds to a nonconstant periodic solution of (1.1)-(1.2) in G because $\widetilde{V} = V$ on G_0 and $G \subset G_0$.

Now assume that V satisfies all hypotheses of Theorem 1.3. Then we can modify V only in a sufficiently small neighborhood N of $\bigcup \partial D(x^i; r_i)$ to get a sequence of approximate potentials $V_k \in C^2(G_0, R), k = 1, 2, \dots$ such that (3.1) holds with V replaced by each $V_k, V_k(x) = V(x)$ for $x \in G_0 \backslash N$, and

$$\lim_{k\to\infty} \sup_{x\in G_0} [|V_k(x) - V(x)| + |V_k'(x) - V'(x)|] = 0.$$

But according to the construction of $G(t_0)$, there exists k_0 big enough so that for each $k > k_0$, $G = G(t_0) \cap G_0$ is also geodesic convex with respect to the metric $(h - V_k)\delta_{ij}$. Hence, by the conclusion of the first part, for each $k > k_0$ there is a closed geodesic p^k in G with respect to the metric $(h - V_k)\delta_{ij}$. Finally by an argument similar to the proof of [23, Theorem 1.2], we can also deduce that there is a closed geodesic in G with respect to the metric $(h - V)\delta_{ij}$.

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