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NEUMANN PROBLEM OF ELLIPTIC EQUATIONS WITH LIMIT NONLINIEARITY IN BOUNDARY CONDITION***

DENG YINBING* WANG XUJIA** WU SHAOPING**

Abstract

This paper deals with a problem proposed by H. Brezis on the existence of positive solutions to the equation $\Delta u + u^{(n+2)/(n-2)} + f(x,u) = 0$ under the Neumann boundary condition $D_\gamma u = u^{n/(n-2)}$, where $f(x,u)$ is a lower order perturbation of $u^{(n+2)/(n-2)}$ at infinity.

Keywords Neumann problem, Semilinear elliptic equation, Positive solution.
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§1. Introduction

Let Ω be a bounded domain in R^n with C^1 boundary, $n \geq 3$. In this paper we are concerned with the existence of positive solutions to the nonlinear elliptic equation

$$-\Delta u = u^p + f(x,u) \text{ in } \Omega \tag{1.1}$$

with the boundary condition

$$D_\gamma u = u^q \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $p = (n+2)/(n-2)$, $q = n/(n-2)$, γ denotes the unit outward normal to $\partial\Omega$, and $f(x,u)$ is a lower order perturbation of u^p at infinity.

We say $u \in H^1(\Omega)$ is a weak solution of (1.1), (1.2) if $u \geq 0$, $u \not\equiv 0$, and

$$\int_\Omega [D_i u D_i v - u^p v - f(x,u)v]\,dx - \int_{\partial\Omega} u^q v d\sigma = 0, \ \forall v \in H^1(\Omega).$$

Hence u is a critical point of the functional

$$J(u) = \int_\Omega \left[\frac{1}{2}|Du|^2 - \frac{1}{p+1}u_+^{p+1} - F(x,u)\right] dx - \frac{1}{q+1}\int_{\partial\Omega} u_+^{q+1} d\sigma, \tag{1.3}$$

where $F(x,u) = \int_0^u f(x,t)dt$, $u_+ = \max(u,0)$. Note that both $p+1$ and $q+1$ are critical Sobolev exponents for the embeddings $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{q+1}(\partial\Omega)$, which causes new difficulties in treating the problem (1.1), (1.2).

The Dirichlet counterpart of (1.1), (1.2) was studied by Brezis and Nirenberg[3], and later by many other authors. In 1985 Brezis[1] proposed several open problems in this aspect,

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*Department of Mathematics, Huazhong Normal University, Wuhan 430070, China.
**Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
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including the problem of finding a positive solution of (1.1) satisfying the homogeneous Neumann condition

$$D_\gamma u = 0, \tag{1.4}$$

and the problem (1.1), (1.2).

Problem (1.1), (1.4) in the subcritical case $1 < p < (n+2)/(n-2)$ was studied in [8], [6]. But in the critical case $p = (n+2)/(n-2)$ it was studied by Wang[9] in which some delicate integral computation was made in order to estimate the critical value of the associated functional. Both arguments in [3] and in [9] are based on the fact that the best constant S in the Sobolev inequality

$$\|u\|_{p+1,R^n} \le S\|Du\|_{2,R^n} \tag{1.5}$$

is achieved by the function $u(x) = (1+|x|^2)^{(2-n)/2}$. But this fact can not be applied to problem (1.1), (1.2) since we also have critical Sobolev exponents in the boundary condition. Recently Escobar[5] considered the best constant $S_{a,b}$ in the Sobolev inequality

$$a\|u\|_{p+1,R^n_+} + b\|u\|_{q+1,\partial R^n_+} \le S_{a,b}\|Du\|_{2,R^n_+} \tag{1.6}$$

and proved that $S_{a,b}$ is achieved by the function $\psi(x) = (1+|x'|^2+|x_n+x_n^0|^2)^{(2-n)/2}$, where a, b are nonnegative constants with $a+b>0$, x_n^0 is a constant depending only on a, b, n. Escobar's result enables us to deal with the problem (1.1),(1.2). The function $\psi(x)$ will play a crucial role in our argument.

We will prove for a class of $f(x,u)$, for instance $f(x,u) = -\lambda u$, the existence of a positive solution to (1.1), (1.2). In Section 2, we present a general existence theorem which is based on a variant of the Mountain Pass Lemma. In Section 3, by a way similar to the one in [9], we verify the conditions of the above theorem to obtain solutions of (1.1), (1.2).

We will always denote $x' = (x_1, \cdots, x_{n-1})$, $R^n_+ = R^n \cap \{x_n > 0\}$. For simplicity we will write $\|u\|_{L^\alpha(\Omega)} = \|u\|_{\alpha,\Omega}$ and $\|u\|_{L^\alpha(\partial\Omega)} = |u|_{\alpha,\partial\Omega}$.

§2. An Existence Theorem

Let Ω be a bounded domain in R^n with C^1 boundary, $n \ge 3$. Assume that $f(x,u)$ is measurable in x, continuous in u and that $\sup\{f(x,u); x \in \Omega,\ 0 \le u \le M\} < \infty$ for every $M > 0$. Consider the problem

$$\begin{cases} -\Delta u = u^p + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma} = u^q & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $p = (n+2)/(n-2)$, $q = n/(n-2)$, γ stands for the unit outward normal of $\partial\Omega$. Suppose that there exists $a(x) \in L^\infty(\Omega)$ such that

$$\lim_{u\to 0} f(x,u)/u = a(x) \quad \text{uniformly for } x \in \Omega, \tag{2.2}$$

$$\lim_{u\to+\infty} f(x,u)/u^p = 0 \text{ uniformly for } x \in \Omega. \tag{2.3}$$

Moreover, we suppose that the elliptic operator $-\Delta + a(x)$ with the Neumann boundary condition $\frac{\partial u}{\partial \gamma} = 0$ has its least eigenvalue l_1 positive, i.e.,

$$l_1 = \inf\left\{\int_\Omega (|Du|^2 - a(x)u^2)dx; \quad \int_\Omega u^2 dx = 1\right\} > 0. \tag{2.4}$$

The values of $f(x,u)$ for $u<0$ are irrelevant and we may define

$$f(x,u)=a(x)u \quad \text{for } x\in\Omega, u\le 0.$$

Set $F(x,u)=\int_0^u f(x,t)dt$, and

$$J(u)=\int_\Omega\left[\frac{1}{2}|Du|^2-\frac{1}{p+1}u_+^{p+1}-F(x,u)\right]dx-\frac{1}{q+1}\int_{\partial\Omega}u_+^{q+1}d\sigma. \tag{2.5}$$

If $u\in H^1(\Omega)$ is a critical point of $J(u)$, let $u_-=\max(-u,0)$, then

$$\int_\Omega(|Du_-|^2-a(x)u_-^2)dx=\langle J'(u),u_-\rangle=0,$$

which implies by (2.4) that $u_-\equiv 0$. Hence in order to obtain a solution of (2.1) it suffices to find a nonzero critical point of $J(u)$.

Set

$$c=\inf_{\psi\in\Psi}\sup_{t\in(0,1)}J(\psi(t)), \tag{2.6}$$

where $\Psi=\{\psi\in C([0,1],H^1(\Omega));\ \psi(0)=\psi_0\equiv t_0\}$, t_0 being a constant large enough so that $J(t\psi_0)\le 0$ for all $t\ge 1$. By (2.4) we have

$$\begin{aligned}J(u)&\ge C\|u\|_{H^1}^2-\int_\Omega\left[F(x,u)-\frac{1}{2}a(x)u^2+\frac{1}{p+1}u_+^{p+1}\right]dx-\frac{1}{q+1}\int_{\partial\Omega}u_+^{q+1}d\sigma\\&\ge(C-\varepsilon)\|u\|_{H^1}^2-C_\varepsilon\int_\Omega u_+^{p+1}dx-\frac{1}{q+1}\int_{\partial\Omega}u_+^{q+1}d\sigma,\end{aligned}$$

and hence

$$c>0 \tag{2.7}$$

Before stating the main theorem we first introduce a few lemmas.

Lemma 2.1. *For any constants $a>0$ and $b\ge 0$, the infimum*

$$S(a,b)=\inf_{u\ne 0}\left\{\|Du\|_{2,R_+^n}/(a\|u\|_{p+1,R_+^n}+b|u|_{q+1,\partial R_+^n})\right\} \tag{2.8}$$

is achieved by the function $(1+|x'|^2+|x_n+x_n^0|^2)^{-(n-2)/2}$ for some constant x_n^0 depending only on a,b.

This lemma was proved by Escobar (see Theorem 3.3 in [5]). By the same argument as that of Escobar we have

Lemma 2.2. *For any $\theta\in(0,1]$, the infimum*

$$S_\theta=\inf_{u\ne 0}\{\|Du\|_{2,R_+^n}^2/(\theta\|u\|_{p+1,R_+^n}^2+(1-\theta)|u|_{q+1,\partial R_+^n}^2)\} \tag{2.9}$$

is achieved by the function $u(x)=(1+|x'|^2+|x_n+x_n^0|^2)^{(2-n)/2}$, or after rescaling by any of the functions

$$u_\varepsilon(x)=\left(\frac{\varepsilon}{\varepsilon^2+|x'|^2+|x_n+\varepsilon x_n^0|^2}\right)^{(n-2)/2}, \tag{2.10}$$

where x_n^0 is a constant depending only on n,θ.

Since $u_\varepsilon(x)$ reaches the infimum S_θ, it verifies the Eular-Langrange equation

$$\begin{cases}-\Delta u=\theta S_\theta\|u\|_{p+1,R_+^n}^{1-p}u^p & \text{in } R_+^n,\\ -\dfrac{\partial u}{\partial x_n}=(1-\theta)S_\theta|u|_{q+1,\partial R_+^n}^{1-q}u^q & \text{on } \partial R_+^n.\end{cases} \tag{2.11}$$

From the Neumann condition it follows that

$$x_n^0 = \frac{1-\theta}{n-2} S_\theta |u_1|_{q+1,\partial R_+^n}^{1-q}, \quad u_1 = u_\varepsilon \big|_{\varepsilon=1}.$$

The value of S_θ can be solved in the following way.

For $\tau \geq 0$, let

$$\psi_{\varepsilon,\tau}(x) = \left(\frac{\varepsilon\sqrt{n(n-2)}}{\varepsilon^2 + |x'|^2 + |x_n + \varepsilon\tau x_n^0|^2} \right)^{(n-2)/2}, \quad x_n^0 = \sqrt{\frac{n}{n-2}}. \tag{2.12}$$

Simple computation shows that $\psi_{\varepsilon,\tau}$ satisfies

$$\begin{cases} -\Delta u = u^p & \text{in } R_+^n, \\ -\dfrac{\partial u}{\partial x_n} = \tau u^q & \text{on } \partial R_+^n. \end{cases} \tag{2.13}$$

Let $\theta = \|\psi_{\varepsilon,\tau}\|_{p+1,R_+^n}^{p-1} / (\|\psi_{\varepsilon,\tau}\|_{p+1,R_+^n}^{p-1} + \tau|\psi_{\varepsilon,\tau}|_{q+1,\partial R_+^n}^{q-1})$, which is independent of ε. Then $\psi_{\varepsilon,\tau}$ reaches the infimum S_θ with

$$S_\theta = \|\psi_{\varepsilon,\tau}\|_{p+1,R_+^n}^{p-1} + \tau|\psi_{\varepsilon,\tau}|_{q+1,\partial R_+^n}^{q-1},$$

and (2.13) is congruent to (2.11).

Denote

$$\Phi(u) = \int_{R_+^n} \left[\frac{1}{2}|Du|^2 - \frac{1}{p+1} u_+^{p+1} \right] dx - \int_{\partial R_+^n} \frac{1}{q+1} u_+^{q+1} d\sigma \tag{2.14}$$

and set

$$A = \inf_{u \neq 0} \sup_{t>0} \Phi(tu). \tag{2.15}$$

Lemma 2.3. *The infimum A is achieved by $\psi_\varepsilon =: \psi_{\varepsilon,1}$.*

Proof. Set

$$\widetilde{A} = \inf_{\tau\geq 0} \sup_{t>0} \Phi(t\psi_{\varepsilon,\tau}),$$

which is independent of $\varepsilon > 0$, then $A \leq \widetilde{A}$. We claim that $A = \widetilde{A}$. Indeed, if $A < \widetilde{A}$, then there exists $u \in H^1(R_+^n)$ with $\|Du\|_{2,R_+^n} = \|D\psi_{\varepsilon,0}\|_{2,R_+^n}$, such that $A(u) = \sup_{t>0} \Phi(tu) < \widetilde{A}$. If

$$\|u\|_{p+1,R_+^n}^2 / |u|_{q+1,\partial R_+^n}^2 > \|\psi_{\varepsilon,0}\|_{p+1,R_+^n}^2 / |\psi_{\varepsilon,0}|_{q+1,\partial R_+^n}^2, \tag{2.16}$$

since $\psi_{\varepsilon,0}$ reaches the infimum S_1 in (2.9) we have

$$\|Du\|_{2,R_+^n}^2 / \|u\|_{p+1,R_+^n}^2 \geq S_1.$$

This, together with (2.16), implies

$$A(u) \geq \sup_{t>0} \Phi(t\psi_{\varepsilon,0}),$$

contradicting $A(u) < \widetilde{A}$. If (2.16) is not true, there must be some $\tau \geq 0$ such that

$$\|u\|_{p+1,R_+^n}^2 / |u|_{q+1,\partial R_+^n}^2 = \|\psi_{\varepsilon,\tau}\|_{p+1,R_+^n}^2 / |\psi_{\varepsilon,\tau}|_{q+1,\partial R_+^n}^2. \tag{2.17}$$

In this case since $\psi_{\varepsilon,\tau}$ reaches the infimum S_θ for some θ, we have

$$S_\theta \leq \|Du\|_{2,R_+^n}^2 / (\theta\|u\|_{p+1,R_+^n}^2 + (1-\theta)|u|_{q+1,\partial R_+^n}^2),$$

which also contradicts $A(u) < \widetilde{A}$.

Consequently $A = \widetilde{A}$. It remains to verify $\widetilde{A} = \sup_{t>0} \Phi(t\psi_{\varepsilon,1})$. To show this, first note that $\psi_{\varepsilon,1}$ is the only function in $\{\psi_{\varepsilon,\tau}; \tau \geq 0\}$ which satisfies the Eular equation of Φ. Next let $\widetilde{\psi}_\tau(x) = \tau^{n-2}\psi_{\varepsilon,\tau}(\tau x)$. Then

$$\widetilde{\psi}_\tau(x) \to \widetilde{\psi}_0(x) =: \left(\frac{\varepsilon\sqrt{n(n-2)}}{|x'|^2 + |x_n + x_n^0|^2}\right)^{(n-2)/2}$$

as $\tau \to \infty$. Moreover

$$A(\psi_{\varepsilon,\tau}) = A(\widetilde{\psi}_\tau) \to A(\widetilde{\psi}_0) > A$$

since $\widetilde{\psi}_0$ does not satisfy the Eular equation of Φ. It therefore follows that $\widetilde{A} = \sup_{t>0}\Phi(t\psi_\varepsilon)$.

Let B_R be the ball $\{x \in R^n; |x| < R\}$, and $\widetilde{B} = B_1 \cap \{x_n > h(x')\}$, where $h(x')$ is a given C^1 function defined on $\{x' \in R^{n-1}; |x'| < 1\}$ with h, Dh vanishing at $x' = 0$.

Lemma 2.4. *$\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ depending only on ε such that if $|Dh| < \delta$, we have*

$$\widetilde{S}_\theta = \inf_{u \in H_0^1(B_1)} \{\|Du\|_{2,B}^2 / (\theta\|u\|_{p+1,B}^2 + (1-\theta)|u|_{q+1,\partial B}^2) \geq S_\theta - \varepsilon.$$

Proof. By making the transformation $y' = x', y_n = x_n - h(x')$, this lemma follows from Lemma 2.2 immediately.

The main theorem of this section is

Theorem 2.1. *Suppose that* (2.2)-(2.4) *hold, and*

$$c < A. \tag{2.18}$$

Then there exists a solution u of (2.1) *with $J(u) \leq c$.*

Proof. By Theorem 2 in [3], there exists a sequence $(u_j) \subset H^1(\Omega)$ such that $J(u_j) \to c$ and $J'(u_j) \to 0$ in $H^{-1}(\Omega)$ as $j \to \infty$, that is,

$$\int_\Omega \left[\frac{1}{2}|Du_j|^2 - \frac{1}{p+1}(u_j)_+^{p+1} - F(x, u_j)\right] dx - \frac{1}{q+1}\int_{\partial\Omega}(u_j)_+^{q+1} d\sigma = c + o(1), \tag{2.19}$$

$$\int_\Omega [Du_j D\varphi - (u_j)_+^p \varphi - f(x, u_j)\varphi] dx - \int_{\partial\Omega} (u_j)_+^q \varphi d\sigma = o(\|\varphi\|_{H^1(\Omega)}). \tag{2.20}$$

Let $\varphi = u_j$. We obtain

$$\begin{aligned} &\frac{1}{n}\int_\Omega (u_j)_+^{p+1} dx + \frac{1}{2(n-1)}\int_{\partial\Omega}(u_j)_+^{q+1} d\sigma \\ &= \int_\Omega [F(x, u_j) - \frac{1}{2}u_j f(x, u_j)] dx + c + o(1 + \|u_j\|_{H^1(\Omega)}). \end{aligned} \tag{2.21}$$

Since $f(x, u) = a(x)u$ for $u < 0$, we have

$$F(x, u) - \frac{1}{2}uf(x, u) = 0 \quad \text{for } u < 0.$$

From (2.21) it therefore follows that

$$\int_\Omega (u_j)_+^{p+1} dx + \int_{\partial\Omega} (u_j)_+^{q+1} dx \leq C(1 + \|u_j\|_{H^1(\Omega)}),$$

and hence by (2.19), $\|u_j\|_{H^1(\Omega)} \leq C$.

Extract a subsequence, still denoted by (u_j), so that

$$u_j \longrightarrow u \text{ weakly in } H^1(\Omega) \text{ and in } (L^{p+1}(\Omega))^*,$$
$$u_j \longrightarrow u \text{ weakly in } (L^{q+1}(\partial\Omega))^*,$$

$$u_j \longrightarrow u \text{ strongly in } L^t(\Omega) \text{ for any } t < p+1.$$

Passing to the limit in (2.20) we see that u is a critical point of J.

To show $u \neq 0$, we prove it by contradiction. If $u \equiv 0$, we have (see [3])

$$\int_\Omega F(x,u_j)dx \to 0, \int_\Omega u_j f(x,u_j)dx \to 0 \quad \text{as } j \to \infty. \tag{2.22}$$

Let ε be a small positive constant to be determined, and let $(\varphi_\alpha)_{\alpha=1}^N$ be a unit partition on $\overline{\Omega}$ with diam $(\text{supp}\varphi_\alpha) \leq \delta$ for each α, where diam(D) stands for the diameter of the set D. Since $\partial\Omega \in C^1$, from Lemma 2.4, it follows that

$$\|D(u\varphi_\alpha)\|_{2,\Omega}^2 \geq (S_\theta - \varepsilon)(\theta\|u\varphi_\alpha\|_{p+1,\Omega}^2 + (1-\theta)|u\varphi_\alpha|_{q+1,\partial\Omega}^2), \quad \forall\, 1 \leq \alpha \leq N,\ u \in H^1(\Omega)$$

provided δ is sufficiently small. For any $\theta \in (0,1]$ we thus have

$$\begin{aligned}
&\theta\|u_j\|_{p+1,\Omega}^2 + (1-\theta)|u_j|_{q+1,\partial\Omega}^2 \\
&= \theta\|\sum_{\alpha=1}^N \varphi_\alpha u_j^2\|_{(p+1)/2,\Omega} + (1-\theta)|\sum_{\alpha=1}^N \varphi_\alpha u_j^2|_{(q+1)/2,\partial\Omega} \\
&\leq \theta\sum_{\alpha=1}^N \|\varphi_\alpha u_j^2\|_{(p+1)/2,\Omega} + (1-\theta)\sum_{\alpha=1}^N |\varphi_\alpha u_j^2|_{(q+1)/2,\partial\Omega} \\
&\leq (S_\theta - \varepsilon)^{-1}\sum_{\alpha=1}^N \|D(u_j\varphi_\alpha^{1/2})\|_{2,\Omega}^2 \\
&\leq (S_\theta - \varepsilon)^{-1}[(1+\varepsilon)\|Du_j\|_{2,\Omega}^2 + C_\varepsilon\|u_j\|_{2,\Omega}^2] \\
&= (S_\theta - \varepsilon)^{-1}(1+\varepsilon)\|Du_j\|_{2,\Omega}^2 + o(1) \qquad \text{as } j \to \infty \\
&\leq (S_\theta - \varepsilon)^{-1}(1+2\varepsilon)\|Du_j\|_{2,\Omega}^2.
\end{aligned}$$

The last inequality holds provided j is large enough, say, $j \geq j_0$. Similarly to the proof of Lemma 2.3 we have

$$\sup_{t>0} \widetilde{\Phi}(tu_j) \geq A - K\varepsilon \tag{2.23}$$

for some constant K independent of $j \geq j_0$, where

$$\widetilde{\Phi}(u) = \int_\Omega \left[\frac{1}{2}|Du|^2 - \frac{1}{p+1}u_+^{p+1}\right]dx - \frac{1}{q+1}\int_{\partial\Omega} u_+^{q+1}d\sigma.$$

Let $\varphi = u_j$ in (2.20). By (2.22) we find

$$\int_\Omega [|Du_j|^2 - (u_j)_+^{p+1}]dx - \int_{\partial\Omega} (u_j)_+^{q+1}d\sigma = o(1) \quad \text{as } j \to \infty,$$

which implies $\widetilde{\Phi}(u_j) = \sup_{t>0}\widetilde{\Phi}(tu_j) + o(1)$. Again by (2.22), and from (2.19), we conclude $\widetilde{\Phi}(u_j) = J(u_j) + o(1) = c + o(1)$. This, combined with (2.23), leads to $c \geq A - (K+1)\varepsilon$ provided j is large enough, which contradicts (2.18) if ε is small enough. Hence $u \not\equiv 0$.

Finally we show that $J(u) \leq c$. Since $u_j \to u$ weakly in $H^1(\Omega)$, we have

$$\int_\Omega F(x,u_j)dx \to \int_\Omega F(x,u)dx, \int_\Omega u_j f(x,u_j)dx \to \int_\Omega uf(x,u)dx.$$

Set $v_j = u_j - u$. From [2] we have

$$\int_\Omega (u_j)_+^{p+1}dx = \int_\Omega (v_j)_+^{p+1}dx + \int_\Omega u^{p+1}dx + o(1),$$

$$\int_{\partial\Omega}(u_j)_+^{q+1}dx = \int_{\partial\Omega}(v_j)_+^{q+1}dx + \int_{\partial\Omega}u^{q+1}dx + o(1).$$

Obviously

$$\int_\Omega |Du_j|^2dx = \int_\Omega |Dv_j|^2dx + \int_\Omega |Du|^2dx + o(1).$$

Hence (2.19) and (2.20) reduce to

$$J(u) + \int_\Omega \left[\frac{1}{2}|Dv_j|^2 - \frac{1}{p+1}(v_j)_+^{p+1}\right]dx - \frac{1}{q+1}\int_{\partial\Omega}(v_j)_+^{q+1}d\sigma = c + o(1)$$

and

$$\int_\Omega \left[|Dv_j|^2 - (v_j)_+^{p+1}\right]dx - \int_{\partial\Omega}(v_j)_+^{q+1}dx = o(1)$$

respectively. Consequently

$$J(u) = c + o(1) - \frac{1}{n}\int_\Omega (v_j)_+^{p+1}dx - \frac{1}{2(n-1)}\int_{\partial\Omega}(v_j)_+^{q+1}d\sigma$$

and hence $J(u) \le c$.

§3. Verification of the Condition (2.18)

Set $c^* = \inf\{\sup_{t>0} J(tu);\ u \ge$ and $u \ne 0\}$. Then $c \le c^*$ (see, e.g. [7]). Hence the condition (2.18) in Theorem 2.1 can be replaced by

$$c^* < A = \sup_{t>0}\Phi(t\psi_\varepsilon). \tag{H}$$

We first consider the problem

$$\begin{cases} -\Delta u = u^p - \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \gamma} = u^q & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Theorem 3.1. *If* $\partial\Omega \in C^2$, *then problem* (3.1) *admits a solution for any* $\lambda > 0$.

Proof. The functional associated with (3.1) is

$$J(u) = \int_\Omega \left[\frac{1}{2}|Du|^2 + \frac{1}{2}\lambda u^2 - \frac{1}{p+1}u_+^{p+1}\right]dx - \frac{1}{q+1}\int_{\partial\Omega}u_+^{q+1}dx. \tag{3.2}$$

Let $v \equiv 1$. Then $\sup_{t>0} J(tv) < A$ for $\lambda > 0$ small enough, which implies by Theorem 2.1 the existence of a solution to (3.1).

But to show (3.1) has a solution for $\lambda > 0$ large is much harder, we follow the outline of [9] and proceed as follows.

Let $B(\bar{x}, R)$ be a ball containing Ω so that $\partial B(\bar{x}, R) \cap \partial\Omega \ne \emptyset$. Let $x_0 \in \partial B(\bar{x}, R) \cap \partial\Omega$, and $\alpha_1, \cdots, \alpha_{n-1}$ denote the principal curvatures of $\partial\Omega$ at x_0 (relative to the inner normal). Then $\alpha_i \ge R^{-1}$ for each $1 \le i \le n-1$. Without loss of generality we may suppose that x_0 is the origin and $\Omega \subset \{x_n > 0\}$. Hence the boundary $\partial\Omega$ near the origin can be represented by (rotating the $x_1, \cdots, x_{n-1}$ axes if needed)

$$x_n = h(x') = \frac{1}{2}\sum_{i=1}^{n-1}\alpha_i x_i^2 + o(|x'|^2),$$

for $x' = (x_1, \cdots, x_n) \in D(0,\delta) =: \{x' \in R^{n-1};\ |x'| < \delta\}$ for some $\delta > 0$. Set

$$\psi_\varepsilon(x) = \left(\frac{\varepsilon\sqrt{n(n-2)}}{\varepsilon^2 + |x'|^2 + |x_n + \varepsilon x_n^0|^2}\right)^{(n-2)/2}, \quad x_n^0 = \sqrt{\frac{n}{n-2}}. \tag{3.3}$$

We claim that

$$Y_\varepsilon = \sup_{t>0} J(t\psi_\varepsilon) < A \tag{3.4}$$

for $\varepsilon > 0$ sufficiently small, and hence (H) holds.

Denote

$$K_1(\varepsilon) = \int_\Omega |D\psi_\varepsilon|^2 dx,\ K_2(\varepsilon) = \int_\Omega \psi_\varepsilon^{p+1} dx,\ K_3(\varepsilon) = \int_{\partial\Omega} \psi_\varepsilon^{q+1} d\sigma.$$

Let $g(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2$. The proof of (3.4) is divided into two cases.

Case 1, $n \geq 4$. We have

$$\begin{aligned} K_1(\varepsilon) &= \int_{R_+^n} |D\psi_\varepsilon|^2 dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |D\psi_\varepsilon|^2 dx_n + O(\varepsilon^{n-2}) \\ &= \int_{R_+^n} |D\psi_\varepsilon|^2 dx - \left[\int_{R^{n-1}} dx' \int_0^{g(x')} + \int_{D(0,\delta)} \int_{g(x')}^{h(x')}\right] |D\psi_\varepsilon|^2 dx_n + O(\varepsilon^{n-2}). \end{aligned}$$

Observing that

$$\begin{aligned} I(\varepsilon) &=: \int_{R^{n-1}} dx' \int_0^{g(x')} |D\psi_\varepsilon|^2 dx_n \\ &= (n-2)^2 C_n \varepsilon^{n-2} \int_{R^{n-1}} dx' \int_0^{g(x')} \frac{|x'|^2 + |x_n + \varepsilon x_n^0|^2}{(\varepsilon^2 + |x'|^2 + |x_n + \varepsilon x_n^0|^2)^n} dx_n \\ &= (n-2)^2 C_n \int_{R^{n-1}} dy' \int_0^{\varepsilon g(y')} \frac{|y'|^2 + |y_n + x_n^0|^2}{(1 + |y'|^2 + |y_n + x_n^0|^2)^n} dy_n, \end{aligned}$$

where $C_n = [n(n-2)]^{(n-2)/2}$, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon^{-1} I(\varepsilon) &= (n-2)^2 C_n \int_{R^{n-1}} \frac{(|y'|^2 + |x_n^0|^2) g(y')}{(1 + |x_n^0|^2 + |y'|^2)^n} dy' \\ &= \frac{(n-2)^2}{2(n-1)} C_n \sum_{i=1}^{n-1} \alpha_i \int_{R^{n-1}} \frac{(|y'|^2 + |x_n^0|^2)|y'|^2}{(1 + |x_n^0|^2 + |y'|^2)^n} dy' =: I \end{aligned} \tag{3.5}$$

Next by

$$\begin{aligned} I_1(\varepsilon) &=: \left|\int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} |D\psi_\varepsilon|^2 dx_n\right| \\ &= (n-2)^2 C_n \varepsilon^{n-2} \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \frac{|x'|^2 + |x_n + \varepsilon x_n^0|^2}{(\varepsilon^2 + |x'|^2 + |x_n + \varepsilon x_n^0|^2)^n} dx_n \\ &\leq \varepsilon^{n-2} C \int_{D(0,\delta)} \frac{|h(x') - g(x')|}{(\varepsilon^2 + |x'|^2 + |x_n + \varepsilon x_n^0|^2)^{n-1}} dx', \end{aligned}$$

and noting that $|h(x') - g(x')| = o(|x'|^2)$, we obtain $I_1(\varepsilon) = o(\varepsilon)$. Hence

$$K_1(\varepsilon) = K_{1,0} - I(\varepsilon) + o(\varepsilon), \tag{3.6}$$

where $K_{1,0} = \int_{R^n_+} |D\psi_\varepsilon|^2 dx$, which is independent of ε. Similarly we have

$$K_2(\varepsilon) = \int_{R^n_+} \psi_\varepsilon^{p+1} dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} \psi_\varepsilon^{p+1} dx_n + O(\varepsilon^n)$$
$$= \int_{R^n_+} \psi_\varepsilon^{p+1} dx - \left[\int_{R^{n-1}} dx' \int_0^{g(x')} + \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')}\right] \psi_\varepsilon^{p+1} dx_n + O(\varepsilon^n).$$

Since

$$\mathit{II}(\varepsilon) = \int_{R^{n-1}} dx' \int_0^{g(x')} \psi_\varepsilon^{p+1} dx_n$$
$$= C'_n \varepsilon^n \int_{R^{n-1}} dx' \int_0^{g(x')} \frac{1}{(\varepsilon^2 + |x'|^2 + |x_n + \varepsilon x_n^0|^2)^n} dx_n$$
$$= C'_n \int_{R^{n-1}} dy' \int_0^{\varepsilon g(y')} \frac{1}{(1 + |y'|^2 + |y_n + x_n^0|^2)^n} dy_n,$$

where $C'_n = [n(n-2)]^{n/2}$, we obtain

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathit{II}(\varepsilon) = C'_n \int_{R^{n-1}} \frac{g(y')}{(1 + |x_n^0|^2 + |y'|^2)^n} dy' =: \mathit{II}. \tag{3.7}$$

Similarly we have

$$\mathit{II}_1(\varepsilon) = \left| \int_{D(0,\delta)} dx' \int_{g(x')}^{h(x')} \psi_\varepsilon^{p+1} dx_n \right| = o(\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Therefore

$$K_2(\varepsilon) = K_{2,0} - \mathit{II}(\varepsilon) + o(\varepsilon), \tag{3.8}$$

where $K_{2,0} = \int_{R^n_+} \psi_\varepsilon^{p+1} dx$. To estimate $K_3(\varepsilon)$, we extend $h(x')$ to R^{n-1} so that $|h(x')| + |Dh(x')|^2 = O|x'|^2)$ as $|x'| \to \infty$. We have

$$K_3(\varepsilon) = \varepsilon^{n-1} C''_n \int_{D(0,\delta)} \frac{\sqrt{1 + |Dh|^2}}{(\varepsilon^2 + |x'|^2 + |h(x') + \varepsilon x_n^0|^2)^{n-1}} dx' + O(\varepsilon^{n-1})$$
$$= C''_n \int_{R^{n-1}} \frac{\sqrt{1 + |Dh|^2(\varepsilon y')}}{(1 + |y'|^2 + |\frac{1}{\varepsilon} h(\varepsilon y') + x_n^0|^2)^{n-1}} dy' + O(\varepsilon^{n-1}),$$

where $C''_n = [n(n-2)]^{(n-1)/2}$. Since $h(x') = g(x') + o(|x'|^2)$, we obtain

$$\frac{d}{d\varepsilon} K_3(\varepsilon)|_{\varepsilon=0} = -(n-1) C''_n \int_{R^{n-1}} \frac{2x_n^0 g(y')}{(1 + |x_n^0|^2 + |y'|^2)^n} dy' =: -\mathit{III}. \tag{3.9}$$

Hence

$$K_2(\varepsilon) = K_{3,0} - \mathit{III}(\varepsilon) + o(\varepsilon), \tag{3.10}$$

where $K_{3,0} = \int_{R^{n-1}} \psi_\varepsilon^{q+1} d\sigma$. Moreover, we have (see [3])

$$K_4(\varepsilon) = \int_\Omega \psi_\varepsilon^2 dx = \begin{cases} O(\varepsilon), & n = 3, \\ O(\varepsilon^2 \log \varepsilon), & n = 4, \\ O(\varepsilon^2), & n \geq 5. \end{cases} \tag{3.11}$$

Let $t_\varepsilon > 0$ be a constant such that

$$J(t_\varepsilon \psi_\varepsilon) = Y_\varepsilon = \sup_{t>0} J(t\psi_\varepsilon)$$
$$= \sup_{t>0} \left[\frac{1}{2}(K_1(\varepsilon) + \lambda K_4(\varepsilon)) t^2 - \frac{K_2(\varepsilon)}{p+1} t^{p+1} - \frac{K_3(\varepsilon)}{q+1} t^{q+1}\right].$$

Then $\frac{d}{dt}J(t\psi_\varepsilon)=0$ at $t=t_\varepsilon$, that is, t_ε is the positive root of

$$K_1(\varepsilon)+\lambda K_4(\varepsilon)-K_2(\varepsilon)t^{p-1}-K_3(\varepsilon)t^{q-1}=0. \tag{3.12}$$

Noting that $p-1=2(q-1)=4/(n-2)$, we have

$$t_\varepsilon^{q-1}=\left[-K_3(\varepsilon)+\sqrt{K_3^2(\varepsilon)+4K_2(\varepsilon)(K_1(\varepsilon)+\lambda K_4(\varepsilon))}\right]/2K_2(\varepsilon). \tag{3.13}$$

From (3.5)-(3.11) we thus conclude $\Delta t_\varepsilon =: t_\varepsilon - 1 = O(\varepsilon) \qquad \text{as } \varepsilon\to 0.$

By Lemma 2.3 it follows that

$$A=\frac{1}{2}K_{1,0}-\frac{1}{p+1}K_{2,0}-\frac{1}{q+1}K_{3,0}. \tag{3.14}$$

and $K_{1,0}-K_{2,0}-K_{3,0}=0$. Hence

$$\begin{aligned} J(t_\varepsilon\psi_\varepsilon)&=\frac{1}{2}(K_1(\varepsilon)+K_4(\varepsilon))t_\varepsilon^2-\frac{K_2(\varepsilon)}{p+1}t_\varepsilon^{p+1}-\frac{K_3(\varepsilon)}{q+1}t_\varepsilon^{q+1}\\ &=\frac{1}{2}(K_{1,0}-I(\varepsilon))t_\varepsilon^2-\frac{K_{2,0}-I\!I(\varepsilon)}{p+1}t_\varepsilon^{p+1}-\frac{K_{3,0}-I\!I\!I(\varepsilon)}{q+1}t_\varepsilon^{q+1}+o(\varepsilon)\\ &=\frac{1}{2}(K_{1,0}-I(\varepsilon))-\frac{1}{p+1}(K_{2,0}-I\!I(\varepsilon))-\frac{1}{q+1}(K_{3,0}-I\!I\!I(\varepsilon))\\ &\quad+(K_{1,0}-K_{2,0}-K_{3,0})\Delta t_\varepsilon+o(\varepsilon)\\ &=A-(\frac{1}{2}I(\varepsilon)-\frac{1}{p+1}I\!I(\varepsilon)-\frac{1}{q+1}I\!I\!I(\varepsilon))+o(\varepsilon)\\ &=A-(\frac{1}{2}I-\frac{1}{p+1}I\!I-\frac{1}{q+1}I\!I\!I)\varepsilon+o(\varepsilon). \end{aligned} \tag{3.15}$$

To verify (3.4) it suffices to show

$$I>\frac{n-2}{n}I\!I+\frac{n-2}{n-1}I\!I\!I. \tag{3.16}$$

Set $\alpha_0=\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\alpha_i$. Noting that $C_n'=n(n-2)C_n, C_n''=\sqrt{n(n-2)}C_n$, and recalling that $x_n^0=\sqrt{\frac{n}{n-2}}$, by (3.7) and (3.9) we deduce

$$\begin{aligned}\frac{1}{\alpha_0 C_n}\left(\frac{n-2}{n}I\!I+\frac{n-2}{n-1}I\!I\!I\right)&=\int_{R^{n-1}}\left[\frac{(n-2)^2|x'|^2}{(1+|x_n^0|^2+|x'|^2)^n}+\frac{2n(n-2)|x'|^2}{(1+|x_n^0|^2+|x'|^2)^n}\right]dx'\\ &=(n-2)(3n-2)\int_{R^{n-1}}\frac{|x'|^2}{(1+|x_n^0|^2+|x'|^2)^n}dx'.\end{aligned}$$

On the other hand, by (3.5),

$$I/\alpha_0 C_n=(n-2)^2\int_{R^{n-1}}\frac{|x'|^2\left(\frac{n}{n-2}+|x'|^2\right)}{(1+|x_n^0|^2+|x'|^2)^n}dx'.$$

Hence we need only to check

$$\int_{R^{n-1}}\frac{|x'|^4}{(1+|x_n^0|^2+|x'|^2)^n}dx'>\frac{2(n-1)}{n-2}\int_{R^{n-1}}\frac{|x'|^2}{(1+|x_n^0|^2+|x'|^2)^n}dx'. \tag{3.17}$$

To show (3.17), observe that $\forall\,\beta<2n-1$, integrating by parts we have

$$\int_0^\infty\frac{r^{\beta-2}}{(a^2+r^2)^{n-1}}dr=\frac{2(n-1)}{\beta-1}\int_0^\infty\frac{r^\beta}{(a^2+r^2)^n}dr.$$

Next by

$$\int_0^\infty \frac{r^\beta}{(a^2+r^2)^n}dr = \int_0^\infty \frac{r^{\beta-2}}{(a^2+r^2)^{n-1}}dr - a^2\int_0^\infty \frac{r^{\beta-2}}{(a^2+r^2)^n}dr,$$

we obtain

$$\int_0^\infty \frac{r^\beta}{(a^2+r^2)^n}dr = \frac{(\beta-1)a^2}{2n-\beta-1}\int_0^\infty \frac{r^{\beta-2}}{(a^2+r^2)^n}dr.$$

Letting $\beta = n+2$, we conclude

$$\begin{aligned}\int_{R^{n-1}} \frac{|x'|^4}{(1+|x_n^0|^2+|x'|^2)^n}dx' &= \frac{(n+1)(1+|x_n^0|^2)}{n-3}\int_{R^{n-1}} \frac{|x'|^2}{(1+|x_n^0|^2)^n}dx' \\ &= \frac{2(n+1)(n-1)}{(n-3)(n-2)}\int_{R^{n-1}} \frac{|x'|^2}{(1+|x_n^0|^2+|x'|^2)n}dx',\end{aligned}$$

which implies (3.17) and hence (3.4) holds.

Case 2, $n=3$. Let a, a^* be two positive constants such that $a|x'|^2 \le h(x') \le a^*|x'|^2$ for $x' \in D(0,\delta)$. We have

$$\begin{aligned}K_1(\varepsilon) &= \int_{R_+^n} |D\psi_\varepsilon|^2dx - \int_{D(0,\delta)} dx' \int_0^{h(x')} |D\psi_\varepsilon|^2 dx_n + O(\varepsilon^{n-2}) \\ &\le K_{1,0} - \int_{D(0,\delta)} dx' \int_0^{a|x'|^2} |D\psi_\varepsilon|^2 dx_n + O(\varepsilon^{n-2}).\end{aligned}$$

The second term of the right hand side

$$\ge C\int_{D(0,\delta/\varepsilon)} dy' \int_0^{a\varepsilon|y'|^2} \frac{|y'|^2+|y_n+x_n^0|^2}{(1+|y'|^2+|y_n+x_n^0|^2)^n}dy_n \ge C_0\varepsilon|\log\varepsilon|.$$

We conclude

$$K_1(\varepsilon) \le K_{1,0} - C_0\varepsilon|\log\varepsilon| + O(\varepsilon). \tag{3.18}$$

In the same way we have

$$K_1(\varepsilon) \ge K_{1,0} - C_1\varepsilon|\log\varepsilon| + O(\varepsilon). \tag{3.19}$$

Similarly to (3.8), (3.10) we have $K_2(\varepsilon) = K_{2,0}+O(\varepsilon)$, $K_3(\varepsilon) = K_{3,0}+O(\varepsilon)$. Let $t_\varepsilon > 0$ so that $J(t_\varepsilon\psi_\varepsilon) = Y_\varepsilon = \sup\limits_{t>0} J(t\psi_\varepsilon)$. From (3.12), (3.13) and by (3.18), (3.19) we infer that

$$\Delta t_\varepsilon = 1 - t_\varepsilon = O(\varepsilon|\log\varepsilon|).$$

Hence by (3.11)

$$\begin{aligned}J(t_\varepsilon\psi_\varepsilon) &= \frac{1}{2}K_1(\varepsilon)t_\varepsilon^2 - \frac{1}{p+1}K_{2,0}t_\varepsilon^{p+1} - \frac{1}{q+1}K_{3,0}t_\varepsilon^{q+1} + O(\varepsilon) \\ &\le \frac{1}{2}(K_{1,0} - C_0\varepsilon|\log\varepsilon|) - \frac{1}{p+1}K_{2,0} - \frac{1}{q+1}K_{3,0} \\ &\quad + (K_{1,0}-K_{2,0}-K_{3,0})\Delta t_\varepsilon + O(\varepsilon) \\ &= A - \frac{1}{2}C_0\varepsilon\log\varepsilon| + O(\varepsilon) < A\end{aligned} \tag{3.20}$$

provided $\varepsilon > 0$ is small enough. This completes the proof.

We now turn to the general problem (2.1).

Theorem 3.2. *Suppose that $\partial\Omega \in C^2$, (2.2)-(2.4) holds. If*

$$f(x,u) \ge -C(u+u^\alpha), \quad \forall\, x \in \Omega, u \ge 0 \tag{3.21}$$

for some $C \geq 0$, and $\alpha \in (1, n/(n-2))$, then there exists a solution of (2.1).

Proof. Let $x_0 \in \partial\Omega$ so that the principal curvatures $\alpha_1, \cdots, \alpha_{n-1}$ of $\partial\Omega$ at x_0 (relative to the inner normal) are positive. We may suppose that x_0 is the origin and $\Omega \subset \{x_n > 0\}$. Let ψ_ε and $K_1(\varepsilon), K_2(\varepsilon), K_3(\varepsilon)$ be as in the proof of Theorem 3.1. Set

$$K_4(\varepsilon) = K_4(\varepsilon, t) = \int_\Omega F(x, t\psi_\varepsilon)dx.$$

From (3.21) we have

$$K_4(\varepsilon) \geq \begin{cases} O(\varepsilon), & n = 3, \\ o(\varepsilon), & n \geq 4. \end{cases} \tag{3.22}$$

Let $t_\varepsilon > 0$ so that $J(t_\varepsilon\psi_\varepsilon) = \sup\limits_{t>0} J(t\psi_\varepsilon)$, where

$$J(u) = \int_\Omega \left[\frac{1}{2}|Du|^2 - \frac{1}{p+1}u_+^{p+1} - F(x, u)\right] dx - \frac{1}{q+1}\int_{\partial\Omega} u_+^{q+1} d\sigma$$

is the functional associated with the problem (2.1). Similarly to the proof of Theorem 3.1 we conclude that $t_\varepsilon \to 1$ and hence

$$J(t_\varepsilon\psi_\varepsilon) \leq \sup_{t>0} \widetilde{\Phi}(t\psi_\varepsilon) + \begin{cases} O(\varepsilon), & n = 3, \\ o(\varepsilon), & n \geq 4, \end{cases}$$

where $\widetilde{\Phi}(u) = J(u) + \int_\Omega F(x, u)dx$. We have verified in the proof of Theorem 3.1 that

$$\sup_{t>0} \widetilde{\Phi}(t\psi_\varepsilon) \leq \begin{cases} A - C\varepsilon|\log\varepsilon|, & n = 3, \\ A - C\varepsilon, & n \geq 4, \end{cases}$$

for some $C > 0$ (see (3.15) and (3.20)). Hence $J(t_\varepsilon\psi_\varepsilon) < A$ for $\varepsilon > 0$ small enough. This completes the proof.

Remark. More delicate result has been recently obtained by J. Escobar[10].

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