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A NOTE ON RECURRENT SETS**

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Abstract

The authors determine the Hausdorff and Bouligand dimensions of a class of recurrent sets by using elementary methods and, as a corollary, give a new proof of a conjecture by Dekking which has been proved by Bedford by using the ergodic techniques.

Keywords Recurrent set, Hausdorff dimension, Bouligand dimension.
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For studying the formalism of fractals[4], Dekking has introduced the recurrent sets[2,3], accounting for many well-known constructions of fractals. In [2], Dekking obtained the best estimate of the upper bound of Hausdorff dimension, and Bedford determined a lower bound of the dimension under some conditions by using the techniques of dynamic systems[1]. In this note, we use an elementary method to study the Hausdorff and Bouligand dimensions which we denote respectively by $\dim_H$ and $\dim_B$. For the definitions of these dimensions, see [5].

Let us first recall some notations and basic results on recurrent sets. Let S be a finite alphabet of letters, S^* the free semigroup generated by S, and $\sigma : S^* \to S^*$ a semigroup endomorphism. Let f: $S^* \to \mathbf{R}^d$ be a homomorphism, that is, for all words $v, w \in S^*$, we have

$$f(vw) = f(v) + f(w).$$

Denote by $\mathcal{C}(\mathbf{R}^d)$ the space of compact subset of $\mathbf{R}^d$, let $K[.] : S \to \mathcal{C}(\mathbf{R}^d)$ be a map with the property that

$$K[vw] = K[v] \cup \{K[w] + f(v)\}$$

for all $v, w \in S^*$.

A letter $s \in S$ is said to be virtual if $K[s] = \emptyset$. The set of virtual letter is denoted by Q. By [3], we may assume that $\sigma Q^* \subset Q^*$ and that $\sigma(s) \notin Q^*$, if $s \notin Q$. We say that s is an essential letter if $s \in E = S \backslash Q$. Suppose that $L : \mathbf{R}^d \to \mathbf{R}^d$ is a linear map so that $f\sigma(s) = Lf(s)$ for all $s \in S$, and suppose that L is expansive, that is, all the eigenvalues of L have a modulus larger than one. Moreover, in this note, we suppose that L is a similitude and we denote by λ the eigenvalue of L. With these notations, Dekking proved[2] that if w is a word containing at least an essential letter, then there exists a compact set $\overline{K}[w]$ such

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that $L^{-n}K[\sigma^n(w)]$ converges to $\overline{K}[w]$ in the Hausdorff metric. We call $\overline{K}[w]$ a recurrent set. Moreover, $\overline{K}[w]$ is independent of the choice of $K[.]$ and in this note we take

$$\overline{K}[s]=\{\alpha f(s);0\le\alpha\le 1\}.$$

Now we define a non-negative $|E|\times|E|$ matrix $A=(a_{ij})$ as follows: suppose that $|E|=r$ and $s_1,s_2,\cdots,s_r$ are the elements of E, then a_{ij} is defined as the number of s_j in the word $\sigma(s_i)$. Let λ_E be the eigenvalue of A with the greatest modulus. We shall assume that σ is essential mixing, that is, there exists a positive integer n such that for all s and t in $E, s\in\sigma^n(t)$. This ensures that

$$\dim_H\overline{K}[s]=\dim_H\overline{K}[t]$$

for all $s,t\in E$ and that

$$\lambda_E^n\sim|\sigma^n(s)|_E, \tag{1}$$

where $|w|_E$ is the number of the letter belonging to E (see [1]).

Let $s\in E$. Set

$$\begin{aligned}\alpha_0&=\sup\Big\{\alpha;\exists\,\varepsilon>0\text{ such that }\liminf_{n\to\infty}\frac{m((K[\sigma^n(s)])^\varepsilon)}{|\sigma^n(s)|_E^\alpha}=\infty\Big\}\\&=\inf\Big\{\alpha;\exists\,\varepsilon>0\text{ such that }\liminf_{n\to\infty}\frac{m((K[\sigma^n(s)])^\varepsilon)}{|\sigma^n(s)|_E^\alpha}=0\Big\},\end{aligned} \tag{2}$$

where m is the Lebesgue measure of $\mathbf{R}^d$, and if $A\subset\mathbf{R}^n$, then

$$A^\varepsilon=\{x\in\mathbf{R}^d;|x-y|\le\varepsilon,y\in A\}.$$

We define β_0 by replacing lim inf by lim sup in the formula (2).

It is easy to establish the following lemma by the definitions of the recurrent sets.

Lemma 1.

(i) *If* $s\in E$, *then* $\overline{K}[\sigma(s)]=L\overline{K}[s]$;

(ii) *If* $u,v\in S^*$, *then*

$$\overline{K}[uv]=\overline{K}[u]\cup\{\overline{K}[v]+f(u)\}.$$

Remark. It is readily checked by calculations and Lemma 1 (ii) that if $w\in S^*$, then

$$\begin{aligned}r_d2^dm(K[w])^\varepsilon&\ge m(K[w])^{2\varepsilon}\ge m(K[w])^\varepsilon,\\m(K[w])^{r+\varepsilon}&\ge m(\overline{K}[w])^\varepsilon,\\m(\overline{K}[w])^{r+\varepsilon}&\ge m(K[w])^\varepsilon,\end{aligned}$$

where r_d is a constant which is independent of w, $2r:=\max_{s\in E}\text{diam }\overline{K}[s]$, and diam $\overline{K}[s]$ is the diameter of s.

Hence if there is an $\varepsilon_0\ge 0$ such that the formula (2) holds, then for any $\varepsilon\ge 0$, the formula (2) holds. Moreover, we can replace K by $\overline{K}$ in the definition of the formulas (2).

Proposition 1. $\dim_H\overline{K}[s]\ge\alpha_0\log\lambda_E/\log|\lambda|, s\in E$.

Proof. Let $2r=\max_{s\in E}\text{diam }\overline{K}[s]$ as above.

Let $\sigma^n(s)=x_1x_2\cdots x_{|\sigma^n(s)|}$. Then from Lemma 1 (ii), we have

$$\overline{K}[\sigma^n(s)]=\bigcup_{j=1}^{|\sigma^n(s)|}\{\overline{K}[x_j]+f(x_1x_2\cdots x_{j-1})\}.$$

Choose a ball containing the recurrent set $\overline{K}[x_j] + f(x_1x_2\cdots x_{j-1})$ with radius r, which we denote by $B(\tilde{x}_j, r)$. Notice that, by convention, if $x_j \in Q$, then $\overline{K}[x_j] = \emptyset$, so $\overline{K}[\sigma^n(s)]$ is in fact the union of $|\sigma^n(s)|_E$ recurrent sets. Since the end points of the segment $K[s]$ belong to the recurrent set $\overline{K}[s]$, the segment $K[x_j]+f(x_1x_2\cdots x_{j-1})$ is contained in $B(\tilde{x}_j, r)$. Thus

$$\bigcup_{j=1}^{|\sigma^n(s)|} B(\tilde{x}_j, r+\varepsilon) \supset (K[\sigma^n(s)])^{\varepsilon}. \tag{3}$$

Now we choose $J_n(s)$ balls $\{B(\tilde{x}_{j_k}, r+\varepsilon)\}_{1\le j_k\le J_n(s)}$ from the above $|\sigma^n(s)|$ balls such that

a) if $k \neq l$, then $d(\tilde{x}_{j_k}, \tilde{x}_{j_l}) > 3(r+\varepsilon)$;

b) if $x_m \notin Q$, and $m \notin \{j_k\}_{1\le k\le J_n(s)}$, then there is an index j_{k_0} such that

$$d(\tilde{x}_m, \tilde{x}_{j_{k_0}}) \le 3(r+\varepsilon).$$

By b), we have

$$\bigcup_{k=1}^{J_n(s)} B(\tilde{x}_{j_k}, 4(r+\varepsilon)) \supset \bigcup_{j=1}^{|\sigma^n(s)|} B(\tilde{x}_j, r+\varepsilon).$$

Then by (3), we have

$$\begin{aligned} J_n(s)m(B(0,4(r+\varepsilon))) &\ge m\Big(\bigcup_{k=1}^{J_n(s)} B(\tilde{x}_{j_k}, 4(r+\varepsilon))\Big) \\ &\ge m\Big(\bigcup_{j=1}^{|\sigma^n(s)|} B(\tilde{x}_j, r+\varepsilon)\Big) \\ &\ge m((K[\sigma^n(s)])^{\varepsilon}). \end{aligned} \tag{4}$$

Let $0<\alpha<\alpha_0$ and note $\tau := \alpha\log\lambda_E/\log|\lambda|$. Then $\lambda_E^{\alpha} = |\lambda|^{\tau}$. By (2), we have

$$m((K[\sigma^n(s)])^{\varepsilon})/|\sigma^n(s)|_E^{\alpha} \to \infty, n\to\infty. \tag{5}$$

Let $0<p<\tau$. Then by (3), (4) and (5), there is a positive integer $N(s)$ such that for $n \ge N(s)$, $J_n(s)/|\lambda|^{np} \ge 1$. Set $N_0 := \max_{s\in E} N(s)$. Then for any $s\in E$, if $n\ge N_0$, we have

$$J_n(s)/|\lambda|^{np} \ge 1. \tag{6}$$

Suppose that $\{U_i\}$ is any $(r+\varepsilon)/|\lambda|^{N_0}$ covering family of open balls of $\overline{K}[s]$. Since $\overline{K}[s]$ is a compact set, we can choose a finite number of balls that cover $\overline{K}[s]$. This finite covering family is still denoted by $\{U_i\}$.

Since L is a similitude, we have

$$\sum_{i\ge1} |U_i|^p = \sum_{i\ge1} \frac{|L^{N_0}(U_i)|^p}{|\lambda|^{N_0p}}, \tag{7}$$

where $|A|$ is the diameter of set A.

By Lemma 1 (i), $\{L^{N_0}(U_i)\}$ is a $(r+\varepsilon)$-covering of $\overline{K}[\sigma^{N_0}(s)]$.

Notice that $\overline{K}[\sigma^{N_0}(s)]$ is the union of the $|\sigma^{N_0}(s)|$ recurrent sets. As above, we choose J_{N_0} recurrent sets

$$\overline{K}[x_{j_k}] + f(x_1\cdots x_{j_k-1}),\ 1\le k\le J_{N_0},$$

which is contained in the ball $B(\tilde{x}_{j_k}, r+\varepsilon)$ respectively. We denote by V^k the corresponding covering family of

$$\overline{K}[x_{j_k}] + f(x_1 \cdots x_{j_k-1}),\ 1 \leq k \leq J_{N_0}$$

obtained by keeping those elements of $\{L^{N_0}(U_I)\}$ which intersect

$$\overline{K}[x_{j_k}] + f(x_1 \cdots x_{j_k-1}),\ 1 \leq k \leq J_{N_0}.$$

Notice that the diameter of $L^{N_0}(U_i)$ is at most $r+\varepsilon$. Then by condition a), these convering families V^k are disjoint (that is, any element of V^i and any element of V^j are disjoint). Thus, by (6) and (7),

$$\sum_{i\geq 1} | U_i |^p \geq \frac{J_{N_0}}{\lambda^{N_0 p}}\Big(\sum_{1\leq k\leq J_{N_0}} \Big(\sum_{U_i\in V^k} | L^{N_0}(U_i) |^p \Big)\Big)$$

$$\geq \sum_{1\leq k\leq J_{N_0}} \Big(\sum_{U_i\in V^k} | L^{N_0}(U_i) |^p \Big). \tag{8}$$

Choose k_0 such that $\sum\limits_{U_i\in V^{k_0}} | L^{N_0}(U_i) |^p$ is minimum among the J_{N_0} sum. Then

$$\sum_{i\geq 1} | U_i |^p \geq \sum_{U_i\in V^{k_0}} | L^{N_0}(U_i) |^p . \tag{9}$$

If

$$\max_{U_i\in V^{k_0}} | L^{N_0}(U_i) | \geq \frac{r+\varepsilon}{| \lambda |^{N_0}},$$

then by (8),

$$\sum_{i\geq 1} | U_i |^p \geq \Big(\frac{r+\varepsilon}{| \lambda |^{N_0}}\Big)^p > 0.$$

Otherwise, $\{L^{N_0}(U_i)\}_{i\in V^{k_0}}$ is also a $\frac{r+\varepsilon}{|\lambda|^{N_0}}$-covering family of some recurrent set $\overline{K}[t], t \in E$ and we may repeat the preceding procedure. Notice that $\{U_i\}$ is a finite family. Therefore after a finite number of steps, we shall obtain finally

$$\sum_{i\geq 1} | U_i |^p \geq \Big(\frac{r+\varepsilon}{| \lambda |^{N_0}}\Big)^p > 0.$$

Since $p \leq \tau$ is chosen arbitrarily, we have

$$\dim_H \overline{K}[s] \geq \alpha \log \lambda_E / \log | \lambda |;$$

but $\alpha \leq \alpha_0$ is also chosen arbitrarily, we obtain

$$\dim_H \overline{K}[s] \geq \alpha_0 \log \lambda_E / \log | \lambda | .$$

Proposition 2. *For any $s \in E$, we have*

$$\dim_B \overline{K}[s] \leq \beta_0 \log \lambda_E / \log | \lambda | .$$

Proof. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive numbers with

$$\log \varepsilon_{n+1} / \log \varepsilon_n \to 1, n \to \infty.$$

Then by (7), if $A \subset \mathbf{R}^d$, we have

$$\dim_B(A) = \lim_{n\to\infty} (d - \log m(A^{\varepsilon_n}) / \log \varepsilon_n). \tag{10}$$

Let $\beta \geq \beta_0$. By (2) and the remark,

$$\limsup_{n\to\infty} m((\overline{K}[\sigma^n(s)])^\varepsilon)/\mid \sigma^n(s) \mid_E^\beta = 0. \tag{11}$$

Since L is a similitude, we have

$$L(A^\varepsilon) = (L(A))^{|\lambda|\varepsilon}. \tag{12}$$

Now taking $\varepsilon_n = \mid \lambda \mid^{-n} \varepsilon \downarrow 0$, from (1), (10), (11), (12) and Lemma 1 (i) we obtain

$$\begin{aligned} m((\overline{K}[s])^{\varepsilon_n}) &= m(L^{-n}(\overline{K}[\sigma^n(s)])^{\varepsilon_n}) \\ &= m(L^{-n}(\overline{K}[\sigma^n(s)])^{\varepsilon}) \\ &= \mid \lambda \mid^{-nd} m((\overline{K}[\sigma^n(s)])^{\varepsilon}) \\ &\leq \mid \lambda \mid^{-nd} \mid \lambda \mid^{n\beta \log \lambda_E/\log|\lambda|}, \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty}\left(d - \frac{\log m((\overline{K}[s])^{\varepsilon_n})}{\log \varepsilon_n}\right) \\ &\leq d - \lim_{n\to\infty} \frac{-nd + n\beta(\log \lambda_E/\log \mid \lambda \mid)}{-n} \\ &= \beta \log \lambda_E/\log \mid \lambda \mid, \end{aligned}$$

that yields

$$\dim_B \overline{K}[s] \leq \beta \log \lambda_E/\log \mid \lambda \mid.$$

Since $\beta \geq \beta_0$ is chosen arbitrarily, we obtain Proposition 2. Combining Proposition 1 and Proposition 2, we have

Theorem.

$$\begin{aligned} \alpha_0 \log \lambda_E/\log \mid \lambda \mid &\leq \dim_H \overline{K}[s] \\ &\leq \dim_B \overline{K}[s] \\ &\leq \beta_0 \log \lambda_E/\log \mid \lambda \mid. \end{aligned}$$

Corollary. *If there is $\varepsilon \geq 0$ such that*

$$\begin{aligned} 0 &< \liminf_{n\to\infty} \frac{m((K[\sigma^n(s)])^\varepsilon)}{\mid \sigma^n(s) \mid_E^\alpha} \\ &= \limsup_{n\to\infty} \frac{m((K[\sigma^n(s)])^\varepsilon)}{\mid \sigma^n(s) \mid_E^\alpha} < \infty, \end{aligned} \tag{13}$$

then

$$\begin{aligned} \dim_H \overline{K}[s] &= \dim_B \overline{K}[s] \\ &= \alpha \log \lambda_E/\log \mid \lambda \mid. \end{aligned}$$

In particular, if

$$\liminf_{n\to\infty} \frac{m((K[\sigma^n(s)])^\varepsilon)}{\mid \sigma^n(s) \mid_E^\alpha} > 0$$

(this condition is called resolubility by Dekking), we have

$$\begin{aligned} \dim_H \overline{K}[s] &= \dim_B \overline{K}[s] \\ &= \log \lambda_E/\log \mid \lambda \mid \end{aligned}$$

(in this case, the inequality of the right hand side of (12) is automatically satisfied), and we prove again the conjecture of Dekking (see (1)).

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