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ON THE ESSENTIAL SPECTRUM OF PSEUDODIFFERENTIAL OPERATORS (II)

Fan Qihong*

Abstract

This paper continues the studies of the essential spectrum of nonsemi-bounded pseudodifferential operators. The author improves the results in [5] in some sense. For the relativistic Schrödinger operator, $\sqrt{-\Delta+m^2}+v(x)$, complete results are obtained.

Keywords Essential spectrum, Pseudodifferential operator, Schrödinger operator.
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§1. Introduction

The purpose of this paper is to continue the study of the essential spectrum of pseudodifferential operators. In [4], [5], we studied the self-adjointness and the essential spectrum of nonsemi-bounded pseudodifferential operators with symbol $p(\xi)+q(x)+Q(x,\xi)$. They include the Schödinger operator, $-\Delta+v(x)$, and its relativistic corrections, $\sqrt{-\Delta+m^2}+v(x)$, with potential $v(x)$ tending to negative infinity, as $|x|$ tends to infinity. There exists an important literature on the studies of the spectral properties of the relativistic Hamiltonians $\sqrt{-\Delta+m^2}+v(x)$, (see [2] and the references there). Most of them are under the assumption that the negative part of the potential $v(x)$ is small in some sense at the infinity (see [3, 7, 8, 11]). In [4], [5], we studied them with $v(x)$ large at infinity, especially, $v(x)\to-\infty$, as $x\to\infty$ in some directions. For the spectral properties of global elliptic pseudodifferential operators on $\mathbb{R}^n$, see [6]. Here, we consider the spectrum in Hilbert space $L^2(\mathbb{R}^n)$ and the operator under consideration is not global elliptic. For the spectral properties of pseudodifferential operators on Banach space (see [12]).

Our studies on the essential spectrum of pseudodifferential operators are motivated by Titchmarsh's work on spectrum for one dimensional differential operator $-d^2/dx^2+q(x)$ on $L^2(\mathbb{R}^+)$, with $q(x)\to-\infty$, $q'(x)<0$ and $q'(x)=O(|q(x)|^c)$, $(0<c<3/2)$. He proved that if $\int^\infty |q(x)|^{-1/2}dx$ is divergent, then there is a continuous spectrum over $(-\infty,+\infty)$ and if $\int^\infty |q(x)|^{-1/2}dx$ is convergent, then the spectrum is discrete over $(-\infty,+\infty)$ (see [10]).

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*Institute & Department of Mathematics, Beijing Unversity, Beijing 100071, China.
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§2. Preliminaries and the Statements of Results

For a symbol $\sigma(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, the corresponding operator $\sigma(x,D)$ is defined by

$$\sigma(x,D)f(x)=\int e^{2\pi ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi. \tag{2.1}$$

The motivation for this definition is that if σ is a polynomial in ξ, say $\sigma(x,\xi)=\sum a_\alpha(x)\xi^\alpha$, then $\sigma(x,D)=\sum a_\alpha(x)D^\alpha$. So one obtains differential operators written in the usual way, with the differentiation on the right. For the relativistic Schrödinger operator, the corresponding symbol is $\sqrt{\xi^2+m^2}+v(x)$.

In order to perform effective calculations, one needs to restrict attention to some class of symbols and operators. For our purpose, we assume that $\sigma(x,\xi)$ is polynomial bounded, that is, there is a positive polynomial $m(x,\xi)$ such that

$$|D_x^\alpha D_\xi^\beta\sigma(x,\xi)|\le C_{\alpha\beta}m(x,\xi)\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}. \tag{2.2}$$

We use $S(m)$ to denote the set of all functions which satisfy (2.2), use $S^{m,k}$ to denote $S(\langle x\rangle^k\langle\xi\rangle^m)$, where

$$\langle x\rangle=(1+|x|^2)^{1/2},\quad \langle\xi\rangle=(1+|\xi|^2)^{1/2}.$$

Another symbol class which we frequently use is the Hörmander class $S^m_{\rho,\delta}$ defined by

$$S^m_{\rho,\delta}=\{\sigma(x,\xi)\in C^\infty(\mathbb{R}^{2n}),\ |D_x^\alpha D_\xi^\beta\sigma(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|}\},$$

where $0\le\delta\le\rho\le1$; usually one uses S^m to denote $S^m_{1,0}$. In [5], we get the following result about the essential spectrum of pseudodifferential operators.

Theorem. *Let $p(\xi)\in S^m$, $m>0$, and*

$$p(\xi)\ge C|\xi|^m, \tag{2.3}$$

for $|\xi|$ sufficiently large in some directions. Assume that $q(x)$ is a real value function and, for some $k>0$, satisfies

$$|D^\alpha q(x)|\le C_\alpha|x|^{k-|\alpha|},\quad q(x)\le -C|x|^k, \tag{2.4}$$

for $|x|$ sufficiently large, where $c>0$ is a constant.

If $\frac{1}{m}+\frac{1}{k}>2$, then $\sigma_e(p(D)+q(x))=\mathbb{R}$.

We note that the condition $\frac{1}{m}+\frac{1}{k}>2$ in this theorem is not necessary. By Titchmarsh's results, we think the best possible condition is $\frac{1}{m}+\frac{1}{k}\ge1$. In this paper, our purpose is to prove the following theorem, which in some sense improves the results in [5].

Theorem 2.1. *Let $p(x,\xi)\in C^\infty(\mathbb{R}^{2n})$ be polynomial bounded, and for some $N>0$, $C_0>0$ sufficiently large,*

$$|D_\xi^\beta p(x,\xi)|\le C_\beta\langle\xi\rangle^{N-|\beta|}$$

for any β with $|\beta|\ge C_0$ where C_β is a constant. For $\lambda\in\mathbb{C}$, when $|x|$ is sufficiently large, there is a C^∞ function $\Phi(x)$ with the following properties:

(i) $p(x,\nabla\Phi(x))=\lambda$,

(ii) $|\nabla\Phi(x)|\sim|x|^l$, $0<l$, *as* $|x|\to\infty$ *and* $|D^\alpha(\nabla\Phi(x))|\le C_\alpha|x|^{l-|\alpha|}$,

(iii) $|\partial_x^\alpha\partial_\xi^\beta p(x,\nabla\Phi(x))|\le C_{\alpha\beta}|x|^{k-|\alpha|-|\beta|l}$.

If $k<l+1$, then $\lambda\in\sigma_e(p(x,D))$.

We will give its proof in next section. First, we notice the following remark.

Remark. The condition (i) means that there is a Lagrangian manifold $\Lambda^n \subset p^{-1}(\lambda)$, such that Λ^n admits diffeomorphic projection on x-space. In the one dimensional case, any smooth curve $(x, \phi(x))$ in $\mathbb{R}^2$ is a Lagrangian manifold which admits diffeomorphic projection on x-space. Therefore, if there is a function $\phi(x)$ satisfies the conditions in Theorem 2.1 with ϕ instead of $\bigtriangledown\Phi$, then we also have $\lambda \in \sigma_e(P(x,D))$.

From the remark above, we get the following theorem in one dimensional case.

Theorem 2.2. *Suppose $n = 1$. Let $p(\xi) \in S^m$, $m > 0$, and*

$$p(\xi) \geq C|\xi|^m, \quad |p'(\xi)| \sim |\xi|^{m-1}, \tag{2.5}$$

for $|\xi|$ sufficiently large. Assume that $q(x)$ is a real valued function and, for $k > 0$, satisfies

$$|D^\alpha q(x)| \leq C_\alpha |x|^{k-\alpha}, \quad q(x) \leq -C|x|^k. \tag{2.6}$$

If $\frac{1}{m} + \frac{1}{k} > 1$, then $\sigma_e(p(D) + q(x)) = \mathbb{R}$.

This result improves the result in [5] in one dimensional case. The proof is easy. For any $\lambda \in \mathbb{R}$, by (2.5), (2.6), there is a function ϕ such that $p(\phi(x)) + q(x) = \lambda$, and $\phi(x)$ satisfies the conditions in Theorem 2.1 with ϕ instead of $\bigtriangledown\Phi$. Therefore the results follows from the remark.

For the relativistic Schrödinger operator $\sqrt{-\Delta + m^2} + v(x)$, we have the following result.

Theorem 2.3. *If $v(x) \in C^\infty(\mathbb{R}^n)$ and satisfies*

$$|D^\alpha v(x)| \leq C|x|^{k-|\alpha|} \quad \text{and} \quad v(x) \leq -C|x|^k \tag{2.7}$$

for some $k > 0$ and any $\alpha \geq 0$ when $|x|$ is sufficiently large, then

$$\sigma_e(\sqrt{-\Delta + m^2} + v(x)) = \mathbb{R}.$$

Proof. To prove this theorem, it is sufficient to show that for any $\lambda \in \mathbb{R}$ there is a function Φ satisfying the conditions in Theorem 2.1 with $p(x,\xi) = \sqrt{\xi^2 + m^2} + v(x)$. We use $d(x, x_0)$ to denote the distance between x and x_0 under the Agmon metric $((\lambda - v(x))^2 - m^2)_+ dx^2$. By (2.7), when $|x|$ is sufficiently large, it is easy to see that $\Phi(x) = d(x, x_0)$ satisfies the conditions in Theorem 2.1.

§3. The Proofs of the Results

In this section, we will give the proofs of the results in section 2. We assume that the symbol $p(x,\xi)$ satisfies the conditions in Theorem 2.1. To prove Theorem 2.1, we will use the following criterion on the essential spectrum.

Proposition 3.1. *Let A be a closed linear operator on Hilbert space $\mathcal{H}$. If there exists $\{x_n\} \subset D(A)$ such that*

(i) *$\|x_n\| = 1$, $(A - \lambda)x_n \to 0$, $n \to \infty$,*

(ii) *no convergent subsequence exists in $\{x_n\}$,*

then $\lambda \in \sigma_e(A)$.

The proof of this proposition can be found in [9]. We notice that if $\{x_n\}$ is an othonormal sequence, then $f_n \to 0$ weakly. Therefore no convergent subsequence exists in $\{x_n\}$. If we can construct an orthonormal sequence $\{f_s\}$ in $L^2(\mathbb{R}^n)$ such that

$$(p(x,D)-\lambda)f_s \to 0, \quad s\to\infty,$$

then from Proposition 3.1 we have $\lambda\in\sigma_e(p(x,D))$. Usually the sequence which satisfies (i) and (ii) in Proposition 3.1 is called singular sequence.

In [5], we constructed a singular sequence $\{f_s\}$ such that the function f_s is concentrated in a box Q_s, and

$$Q_s\subset p^{-1}([\lambda,\lambda+|x_s|^{-\delta}]),\quad |Q_s|=c,$$

where the sequence $x_s\to\infty$, as $s\to\infty$ and c is a constant. Here the construction of f_s also follows this idea, but we should concentrate f_s in a curved box $\tilde{Q}_s$ in phase space, because there are no disjoint boxes with volume larger than a constant c contained in $p^{-1}([\lambda,\lambda+|x_s|^{-\delta}])$ for s sufficiently large.

Since $k-l<1$, for any $M>0$, and $\delta>0$ sufficiently small such that $0<k-l+\delta<1$, exists a sequence $\{x_s\}\in\mathbb{R}^n$ such that

$$B_s=\{x:|x-x_s|\le 2M|x_s|^{k-l+\delta}\}$$

are disjoint from each other. Let

$$Q_s=\{(x,\xi):|x-x_s|\le M|x_s|^{k-l+\delta},|\xi|\le M|x_s|^{l-k-\delta}\}=Q_{1s}\times Q_{2s},$$

and

$$\tilde{Q}_s=\{(x,\xi):|x-x_s|\le M|x_s|^{k-l+\delta},|\xi-\bigtriangledown\Phi(x)|\le M|x_s|^{l-k-\delta}\}.$$

Put $\Phi_s:\tilde{Q}_s\to Q_s$ defined by $\Phi_s(y,\eta)=(y,\eta-\bigtriangledown\Phi(y))$. Then

$$\Phi_s^{-1}(x,\xi)=(x,\xi+\bigtriangledown\Phi(x)).$$

It is easy to see that Φ_s is a canonical transformation and the function $S_s(y,\xi)=y\cdot\xi+\Phi(y)$ is a generating function.

Lemma 3.1. *Assume that $p(x,\xi)$ satisfies the conditions in Theorem 2.1. Then*

$$|p(x,\xi)\chi_{\tilde{Q}_s}(x,\xi)-\lambda|\le C(M)|x_s|^{-\delta}. \tag{3.1}$$

Proof. For $(x,\xi)\in\tilde{Q}_s$,

$$\begin{aligned}|p(x,\xi)-P(x,\bigtriangledown\Phi(x)| &= |\bigtriangledown_\xi\, p(x,\bigtriangledown\Phi(x)+t\cdot\xi)(\xi-\bigtriangledown\Phi(x))|\\ &\le C(M)|x_s|^{k-l}|\xi-\bigtriangledown\Phi(x)|\\ &\le C(M)|x_s|^{k-l}|x_s|^{l-k-\delta}\\ &\le C(M)|x_s|^{-\delta}.\end{aligned}$$

This is (3.1).

In order to study the concentrate in a curved box, we should consider the Fourier integral operator associated to Φ_s. That is

$$U_sf(y)=\iint e^{2\pi i(S_s(y,\xi)-x\cdot\xi)}a_s(y,\xi)f(x)dxd\xi, \tag{3.2}$$

where $a_s(y,\xi)\in C^\infty$ supported in $Q_s^\star$, $a_s(y,\xi)=1$ for $(y,\xi)\in Q_s$, $Q_s^\star$ denotes the box which has the same center as Q_s and double side lengths. For the studies about this operator, one can see [1].

Lemma 3.2. *There exists* $\{f_s\} \subset L^2(\mathbb{R}^n)$ *with the following properties:*
(i) $\|f_s\| = 1$,
(ii) $\|U_s f_s\|_{L^2(\mathbb{R}^n)} \geq c > 0$,
(iii) $(U_{s_1} f_{s_1}, U_{s_2} f_{s_2}) = 0$, *for* $s_1 \neq s_2$.

Proof. For $s_1 \neq s_2$, the support $U_{s_1} f_{s_1} \cap$ support $U_{s_2} f_{s_2} = \emptyset$. We have $(U_{s_1} f_{s_1}, U_{s_2} f_{s_2}) = 0$, this is (iii). Next, we will prove (i) and (ii).

For the integral

$$U_s f(y) = \iint e^{2\pi i (S_s(y,\xi) - x\cdot\xi)} a_s(y,\xi) f(x) dx d\xi,$$

the critical point is $x = y, \xi = 0$. By the stationary method, we have

$$U_s f_s(y) = e^{2\pi i \Phi(y)} a_s(y, 0) f_s(y) + O(\partial_\xi a_s(y,\xi) \cdot \partial_x f_s(x)).$$

Denote by $l_{Q_{1s}}$ and $l_{Q_{2s}}$ the radias of the balls Q_{1s} and Q_{2s} respectively. We can choose a function $f_s(y)$ such that

$$\|f_s\| = 1, \quad \|e^{2\pi i \Phi(y)} a_s(y,0) f_s(y)\| \geq c,$$

and

$$\|\partial f_s\| \leq \frac{c}{l_{Q_{1s}}}.$$

We notice that in (3.2) one can choose $a_s(x,\xi)$ such that

$$|\partial_\xi a_s(y,\xi)| \leq \frac{c}{2 l_{Q_{2s}}}.$$

Therefore

$$\begin{aligned} \|U_s f_s(y)\| &\geq \|e^{2\pi i \Phi(y)} a_s(y,0) f_s(y)\| - \frac{c}{l_{Q_{1s}} l_{Q_{2s}}} \\ &\geq c - \frac{c_0}{M^2}. \end{aligned}$$

When M is sufficiently large, we get $\|U_s f_s\| \geq c$. This finishes the proof.

In order to study the action of $p(x,D)$ on $U_s f_s(x)$, we will study the composition of $p(x,D)$ and U_s. It is easy to see that

$$\begin{aligned} & p(x,D) U_s f(x) \\ = & \int e^{2\pi i (x-y)\cdot\zeta + 2\pi i (S_s(y,\xi) - x'\xi)} a_s(y,\xi) p(x,\zeta) f(x') dx' d\zeta d\xi dy \\ = & \int e^{2\pi i x\xi} \tilde{p}(x,\xi) \hat{f}(\xi) d\xi, \end{aligned}$$

where

$$\tilde{p}(x,\xi) = \int e^{2\pi i ((x-y)\zeta + S_s(y,\xi) - x\xi)} a_s(y,\xi) p(x,\zeta) d\zeta dy. \tag{3.3}$$

Let $T = (x-y)\zeta - x\xi + S_s(y,\xi)$. The critical point of T with respect to (y,ζ) is $(x, \bigtriangledown\Phi(x) + \xi)$. We can change the coordinates (y,ζ) to $(\tilde{y}, \tilde{\zeta})$, so that

$$T = -\tilde{y}\tilde{\zeta} + \Phi(x),$$

the Jacobi $J_{(x,\xi)}(\tilde{y},\tilde{\zeta})=1$, and

$$|\frac{\partial y}{\partial \tilde{y}}|\leq c, \quad |\frac{\partial \zeta}{\partial \tilde{\zeta}}|\leq c,$$
$$|\frac{\partial y}{\partial \tilde{\zeta}}|=0, \quad |\frac{\partial^\alpha \zeta}{\partial \tilde{y}^\alpha}|\leq |D^\alpha \bigtriangledown \Phi|\leq C|x_s|^{l-|\alpha|}.$$

Lemma 3.3. *With the above notations, one has*

(i) $\widetilde{P}(x,\xi)\chi_{2Q_{1s}}(x,\xi)=p(x,\bigtriangledown\Phi(x)+\xi)\chi_{2Q_{1s}}(x,\xi)+R_s(x,\xi)$,

where $\mathrm{suppt}R_s(x,\xi)\subset 2Q_s$, *and for some* $\delta_0>0$,

$$|R_s(x,\xi)|\leq C|x_s|^{-\delta_0}.$$

(ii) $|\tilde{P}(x,\xi)(1-\chi_{2Q_{1s}}(x,\xi))|\leq C_r|x-x_s|^{-2r}(M|x_s|^{k-l+\delta})^n$, *for* r *sufficiently large.*

Proof. (i) From (3.3), and $J_{(x,\xi)}(\tilde{y},\tilde{\zeta})=1$, one has

$$\tilde{p}(x,\xi)=\iint e^{-2\pi i(\tilde{y}\tilde{\zeta}-\Phi(x))}\sigma(\tilde{y},\tilde{\zeta})d\tilde{\zeta}d\tilde{y}, \tag{3.4}$$

where $\sigma(\tilde{y},\tilde{\zeta})=a_s(y,\xi)p(x,\zeta)$. By the stationary method, we get

$$\widetilde{P}(x,\xi)\chi_{2Q_{1s}}(x,\xi)=e^{2\pi i\Phi(x)}p(x,\bigtriangledown\Phi(x)+\xi)\chi_{2Q_{1s}}(x,\xi)+R_s(x,\xi)$$

with

$$|R_s(x,\xi)|\sim|\frac{\partial^2\sigma}{\partial\tilde{y}\partial\tilde{\zeta}}|.$$

Consequently, from the hypothesis (iii) in Theorem 2.1, one has

$$\begin{aligned}|R_s(x,\xi)|&\leq|\frac{\partial a_s}{\partial y}||\frac{\partial p}{\partial \zeta}|+|\frac{\partial^2 p}{\partial \zeta^2}||\frac{\partial \zeta}{\partial \tilde{y}}|\\&\leq C((M|x_s|^{k-l+\delta})^{-1}|x_s|^{k-l}+|x_s|^{k-2l}|x_s|^{l-1})\\&\leq C(M^{-1}|x_s|^{-\delta}+|x_s|^{k-l-1})\\&\leq C|x_s|^{-\delta_0}\end{aligned}$$

for some $\delta_0>0$.

(ii) To estimate $\tilde{P}(x,\xi)(1-\chi_{2Q_{1s}}(x,\xi))$, we substitute

$$e^{2\pi i\tilde{y}\tilde{\zeta}}=(2\pi)^{-2r}|\tilde{y}|^{-2r}\partial_{\tilde{\zeta}}^{2r}e^{-2\pi i\tilde{y}\tilde{\zeta}}$$

in the integral representation (3.4) of $\tilde{p}(x,\xi)$. We have

$$\begin{aligned}&\widetilde{P}(x,\xi)(1-\chi_{2Q_{1s}}(x,\xi))\\&=\int|\tilde{y}|^{-2r}\partial_{\tilde{\zeta}}^{2r}e^{-2\pi i\tilde{y}\tilde{\zeta}}(1-\chi_{2Q_{1s}}(x,\xi))\sigma(\tilde{y},\tilde{\zeta})d\tilde{\zeta}d\tilde{y}\\&=(2\pi)^{-2r}\int(1-\chi_{2Q_{1s}}(x,\xi))|\tilde{y}|^{-2r}e^{-2\pi i\tilde{y}\tilde{\zeta}}\partial_{\tilde{\zeta}}^{2r}\sigma(\tilde{y},\tilde{\zeta})d\tilde{\zeta}d\tilde{y}\\&=(2\pi)^{-2r}\int(1-\chi_{2Q_{1s}}(x,\xi))|\tilde{y}|^{-2r}e^{-2\pi i\tilde{y}\tilde{\zeta}}a_s(y,\xi)\partial_{\tilde{\zeta}}^{2r}p(x,\zeta)d\tilde{\zeta}d\tilde{y}.\end{aligned}$$

Therefore

$$
\begin{aligned}
&|\widetilde{P}(x,\xi)(1-\chi_{2Q_{1s}}(x,\xi))| \\
&\le C|x-x_s|^{-2r}(M|x_s|^{(k-l+\delta)})^n \int |\partial_{\tilde{\zeta}}^{2r} p(x,\zeta)|d\tilde{\zeta} \\
&\le C_r|x-x_s|^{-2r}(M|x_s|^{k-l+\delta})^n.
\end{aligned}
$$

This finishes the proof of the lemma.

Proof of Theorem 2.1. To prove Theorem 2.1, by Lemma 3.2 and Proposition 3.1, it suffices to prove that

$$
\|(p(x,D)-\lambda)U_s f_s\| \to 0, \quad s\to\infty. \tag{3.5}
$$

Let $p_s(x,\xi)$ denote the symbol of operator $(p(x,D)-\lambda)U_s$. Then from (3.2), (3.3), one has

$$
\begin{aligned}
&p_s(x,\xi) \\
&= \tilde{p}(x,\xi) - \lambda e^{2\pi i\Phi(x)} a_s(x,\xi) \\
&= (\tilde{p}(x,\xi)\chi_{2Q_{1s}}(x,\xi) - \lambda e^{2\pi i\Phi(x)} a_s(x,\xi)) + \tilde{p}(x,\xi)(1-\chi_{2Q_{1s}}(x,\xi)) \\
&= p_s^I(x,\xi) + p_s^{II}.
\end{aligned}
$$

By the results in (i) of Lemma 3.3 and (3.1) in Lemma 3.1, one has

$$
\|p_s^I(x,D)\|_{L^2} \le C|x_s|^{-\delta_2} \tag{3.6}
$$

for some $\delta_2>0$ sufficiently small. From (ii) of Lemma 3.3, we have

$$
\begin{aligned}
&\iint |p_s^{II}(x,\xi)|^2 dx d\xi \\
&\le |x_s|^{2n(k-l+\delta)} \int_{x\notin 2Q_{1s}} |x-x_s|^{-4r}dx \int_{Q_{2s}} d\xi \\
&\le C|x_s|^{2n(k-l+\delta)}|x_s|^{(k-l+\delta)(n-4r)}|x_s|^{n(l-k-\delta)} \\
&\le C|x_s|^{(2n-4r)(k-l+\delta)}.
\end{aligned}
$$

When r is sufficiently large, we obtain

$$
\iint |p_s^{II}(x,\xi)|^2 dx d\xi \le C|x_s|^{-2\delta_2}.
$$

Therefore the operator $p_s^{II}(x,D)$ is a Hilbert-Schmidt operator with H-S norm

$$
\|p_s^{II}(x,D)\|_{HS} \le C|x_s|^{-\delta_2}.
$$

One obtains

$$
\|p_s^{II}(x,D)\|_{L^2} \le C|x_s|^{-\delta_2}. \tag{3.7}
$$

From (3.7), (3.6), we have

$$
\|p_s(x,D)\| \le C|x_s|^{-\delta_2}.
$$

Notice that from the (i) of Lemma 3.2 we have

$$
\begin{aligned}
\|(p(x,D)-\lambda)U_s f_s\| &\le C|x_s|^{-\delta_2}\|f_s\| \\
&\le C|x_s|^{-\delta_2}.
\end{aligned}
$$

From this we get (3.5). This finishes the proof of Theorem 2.1.

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