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# ON MONOTONE CONVERGENCE OF NONLINEAR MULTISPLITTING RELAXATION METHODS***

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**Abstract**

A class of parallel nonlinear multisplitting AOR methods is set up by directly multisplitting the nonlinear mapping $F : D \subset R^n \to R^n$ for solving the nonlinear system of equations $F(x) = 0$. The different choices of the relaxation parameters can yield all the known and a lot of new relaxation methods as well as a lot of new relaxation parallel nonlinear multisplitting methods. The two-sided approximation properties and the influences on convergence from the relaxation parameters about the new methods are shown, and the sufficient conditions guaranteeing the methods to converge globally are discussed. Finally, a lot of numerical results show that the methods are feasible and efficient.

**Keywords** Nonlinear system of equations, Nonlinear multisplitting, Monotonicity, Global convergence.

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## §1. Introduction

The parallel nonlinear multisplitting methods for solving nonlinear system of equations

$$F(x) = 0, \qquad F : D \subset R^n \to R^n \tag{1.1}$$

are constructed mainly in two ways. One is to multisplit the coefficient matrix of the linearized system of equations resulted from approximately solving the nonlinear system of Equations (1.1) and then we can form a class of nonlinear multisplitting iteration methods as pointed by White in [1]. This approach covers many particular nonlinear multisplitting methods such as the nonlinear multisplitting Newton-Jacobi, Newton-Gauss-Seidel as well as Newton-SOR methods, etc. The other, as shown by Frommer in [2], is to directly multisplit the nonlinear mapping $F$ and then a class of nonlinear multisplitting iteration methods is also obtained, which as well as whose some variants still include a lot of special nonlinear multisplitting relaxed methods such as the nonlinear multisplitting Gauss-Seidel-Newton as well as SOR-Newton methods, etc. The authors have proved the local convergence of these methods.

Though the global convergence of a nonlinear iteration method is much noticeable, it is rather hard to study global convergence problem of iteration method for a general smooth mapping. Moreover, for some more particular mapping classes such as the $M$-function, etc., presently there have been several literatures[3-5] discussing the monotonous as well as the

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global convergence with respect to the Newton method, nonlinear Gauss-Seidel method, nonlinear SOR method and so on, respectively. These methods are shown to be rather efficient when they are used to numerically solve the solutions of discretized nonlinear boundary value problems or nonlinear network flow problems which have the characterizations of $M$-functions. Paper [3] has studied this in a more detailed manner.

In this paper, we are absorbed to study the monotonous and the global convergence of nonlinear multisplitting methods. By directly multisplitting the nonlinear mapping we first set up a class of nonlinear multisplitting AOR methods, which have parallel computation functions and two-sided approximation properties, for solving the nonlinear system of Equations (1.1). The different choices of the relaxation parameters not only can yield Gauss-Seidel method, SOR method, etc. in the sense of nonlinear multisplitting, but also can improve the convergence properties of these methods. That the methods can converge to the maximum as well as the minimum solutions of nonlinear system of Equations (1.1) from either side respectively is proved. In addition, we further discuss several sufficient conditions which can guarantee the two-sided convergence and different convergence properties of the methods which are resulted by different choices of the relaxation parameters. At last, we enumerate a class of nonlinear boundary value problems and do numerical tests for them. The computation results thoroughly coincide with our theory.

## §2. The Nonlinear Multisplitting and Relaxation Methods

Throughout this paper the $i$-th component of a column vector $x \in R^n$ is denoted by $x_i$ and $e^i$ represents the $i$-th unit basis vector of $R^n$ for $i = 1(1)n$. The natural partial ordering "$\leq$" and "$<$" on $R^n$ are to be understood componentwise. We use the symbol $N$ to denote the set $\{1, 2, \cdots, n\}$.

For $k = 1, 2, \cdots, \alpha$ ($\alpha \leq n$ an integer), take $S^k$ to be a nonempty subset of $N$ satisfying $\bigcup_{k=1}^{\alpha} S^k = N$ and $E_k = \text{diag}(e_1^k, e_2^k, \cdots, e_n^k) \in L(R^n)$ a nonnegatively diagonal matrix defined by

$$e_i^k = \begin{cases} e_i^k \geq 0, & \text{for } i \in S^k, \\ 0, & \text{for } i \notin S^k, \end{cases} \quad i = 1(1)n \tag{2.1}$$

with $\sum_{k=1}^{\alpha} E_k = I$ ($I \in L(R^n)$ identity). It deserves to be mentioned that these $S^k (k = 1, 2, \cdots, \alpha)$ may overlap each other.

Consider nonlinear mapping $F : D \subset R^n \to R^n$. For $k = 1, 2, \cdots, \alpha$, if there exists a mapping $f^{(k)} : D \times D \subset R^n \times R^n$ such that $f^{(k)}(x; x) = F(x)$ for all $x \in D$, then the collection of pairs $(f^{(k)}, E_k), k = 1, 2, \cdots, \alpha$, is called a nonlinear multisplitting of $F$.

For $k = 1, 2, \cdots, \alpha$ if we particularly take

$$f_i^{(k)}(x, y) = \begin{cases} F_i((I - P^{(k)})x + P^{(k)}y), & \text{for } i \in S^k, \\ F_i(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n), & \text{for } i \notin S^k, \end{cases} \quad i = 1(1)n, \tag{2.2}$$

where $P^{(k)} : D \subset R^n \to R^n$ is a projection operator given by

$$P^{(k)}(x) = \begin{cases} x_i, & \text{for } i \in S^k, \\ 0, & \text{for } i \notin S^k, \end{cases} \quad i = 1(1)n, \quad \text{for all } x \in D, \tag{2.3}$$

then (2.2)–(2.3) with the preceding defined weighting matrices $E_k(k = 1, 2, \cdots, \alpha)$ form a special but practical case of the nonlinear multisplitting. Of course, there are a lot of meaningful examples of nonlinear multisplittings and it is unnecessary to write them one by one here.

Making use of the above concepts, we can now set up the following parallel nonlinear multisplitting relaxation method.

**METHOD:**

Given a starting vector $x^0 \in D \subset R^n$, for $m = 0, 1, 2, \cdots$, compute

$$x^{m+1} = \frac{\omega}{r} \sum_k E_k x^{m,k} + \Big(1 - \frac{\omega}{r}\Big) x^m \tag{2.4}$$

with $r, \omega \in (0, +\infty)$, where for $k = 1, 2, \cdots, \alpha$ the $i$-th element $x_i^{m,k}$ of $x^{m,k}$ satisfies

$$x_i^{m,k} = \begin{cases} r\hat{x}_i^{m,k} + (1-r)x_i^m, & \text{for } i \in S^k, \\ x_i^m, & \text{for } i \notin S^k, \end{cases} \quad i = 1(1)n, \tag{2.5}$$

while $\hat{x}_i^{m,k}$ satisfies

$$\begin{cases} f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{x}_i^{m,k}, x_{i+1}^m, \cdots, x_n^m) = 0, \\ i \in S^k, \quad i = 1(1)n. \end{cases} \tag{2.6}$$

Here, we call $r$ a relaxation factor and $\omega$ an acceleration factor.

In this method, (2.4) can be equivalently varied in another simplified form

$$x_i^{m+1} = \omega \sum_k e_i^k \hat{x}_i^{m,k} + (1-\omega)x_i^m, \qquad i = 1(1)n,$$

which will be used in stead of (2.4) in the remainder statements of this section.

More concretely, we consider linear mapping

$$F(x) = Ax - b, \quad A \in L(R^n) \text{ nonsingular}, \quad b \in R^n.$$

When $\alpha = 1$ the above method obviously reduces to the famous AOR method while for $1 < \alpha \le n$ it clearly becomes the parallel matrix multisplitting AOR method studied in [8] for solving the linear system of equations $Ax = b$. Therefore, it is reasonable for us to call the above method nonlinear multisplitting AOR-type method. For the convenience of our subsequent discussion, from now on, we simply denote the above method as $\mathcal{F}_{NMAOR}(r, \omega)$-method.

Evidently, when $\alpha = 1$, the $\mathcal{F}_{NMAOR}(r, \omega)$-method reduces to nonlinear AOR method (simply denoted as $\mathcal{F}_{NAOR}(r, \omega)$-method ) in the sense of nonlinear splitting. If we specially choose the relaxation parameters $r$ and $\omega$ in this type of method, we can obtain all the known nonlinear relaxation methods as well as many new ones. For example,

if $r = \omega$, we get nonlinear SOR method ( $\mathcal{F}_{NSOR}(\omega, \omega)$-method);

if $r = 1$, we get extrapolated nonlinear Gauss-Seidel method ( $\mathcal{F}_{ENGS}(1, \omega)$-method);

if $r = \omega = 1$, we get nonlinear Gauss-Seidel method ( $\mathcal{F}_{NGS}(1, 1)$-method);

if $r = 0$, we get extrapolated nonlinear Jacobi method ( $\mathcal{F}_{ENJ}(0, \omega)$-method);

if $r = \omega = 0$, we get nonlinear Jacobi method ( $\mathcal{F}_{NJ}(0, 0)$-method).

There have been many literatures[3–5] studying the local, the monotonous as well as the global convergence of the somewhat particular forms about the $\mathcal{F}_{NSOR}(\omega, \omega)$-method, the $\mathcal{F}_{NGS}(1, 1)$-method, and the $\mathcal{F}_{NJ}(0, 0)$-method.

When $\alpha > 1$, by particularly selecting the relaxation parameters $r$ and $\omega$ in the new type of method we can also get all the known nonlinear multisplitting relaxation methods as well as many new ones. For instance,

if $r = \omega$, we obtain nonlinear mutisplitting SOR method ($\mathcal{F}_{NMSOR}(\omega, \omega)$-method);

if $r = 1$, we obtain extrapolated nonlinear mutisplitting Gauss-Seidel method ($\mathcal{F}_{ENMGS}(1, \omega)$-method);

if $r = \omega = 1$, we obtain nonlinear mutisplitting Gauss-Seidel method ($\mathcal{F}_{NMGS}(1, 1)$-method);

if $r = 0$, we obtain extrapolated nonlinear mutisplitting Jacobi method ($\mathcal{F}_{ENMJ}(0, \omega)$-method);

if $r = \omega = 0$, we obtain nonlinear mutisplitting Jacobi method ($\mathcal{F}_{NMJ}(0, 0)$-method).

The build of the $\mathcal{F}_{NMAOR}(r, \omega)$-method affords a generally theoretical model for us to systematically discuss nonlinear multisplitting relaxation methods. Notice that the $\mathcal{F}_{NMAOR}(r, \omega)$-method is merely an implicit iteration method due to the implicitly nonlinear system of Equations (2.6). In practical computation we will rather implement it approximately by using some efficiently numerical methods such as Newton method, chord method, Steffensen method and so on than solve it exactly.

## §3. Basic Concepts and Facts

We first recall the following concepts from [3] and [5].

**Definition 3.1.** *$F : D \subset R^n \to R^n$ is called*

(i) *isotone (antitone) on $D$ if $x, y \in D, x \leq y$ implies that $F(x) \leq F(y)(F(x) \geq F(y))$;*

(ii) *strictly isotone (strictly antitone) on $D$ if $x, y \in D, x < y$ implies that $F(x) < F(y)(F(x) > F(y))$;*

(iii) *inverse isotone (strictly inverse isotone) on $D$ if $x, y \in D, F(x) \leq F(y)(F(x) < F(y))$ implies that $x \geq y(x > y)$.*

It is known that $F : D \subset R^n \to R^n$ is inverse isotone if and only if $F$ is injective on $D$ and $F^{-1} : F(D) \subset R^n \to R^n$ , the inverse mapping of $F$, is isotone.

**Definition 3.2.** *$F : D \subset R^n \to R^n$ is called*

(i) *off-diagonally antitone on $D$ if for all $x \in D$ the functions*

$$\phi_{ij} : \{t \in R^1 \mid x + te^j \in D\} \to R^1, \quad \phi_{ij}(t) = F_i(x + te^j), \quad i, j = 1(1)n, \quad i \neq j$$

*are antitone;*

(ii) *diagonally isotone (strictly diagonally isotone) on $D$ if for all $x \in D$ the functions*

$$\phi_{ii} : \{t \in R^1 \mid x + te^i \in D\} \to R^1, \quad \phi_{ii}(t) = F_i(x + te^i), \quad i = 1(1)n$$

*are isotone (strictly isotone);*

(iii) *surjectively diagonally isotone if $F$ is strictly diagonally isotone on $R^n$ and for any $x \in R^n$ each mapping $\phi_{ii}$ defined in* (ii) *is surjective.*

A direct fact about the above definitions is that a continuous, surjective and inverse isotone mapping $F : R^n \to R^n$ is a homeomorphism from $R^n$ onto itself.

**Definition 3.3.** *$F : D \subset R^n \to R^n$ is called an $M$-function if $F$ is inverse isotone and off-diagonally antitone.*

Obviously, the definition of $M$-function represents an extention of the $M$-matrix concept. In other words, if $A \in L(R^n)$ is an $M$-matrix, then the induced linear mapping $A : R^n \to R^n$ is an $M$-function and vice versa. In addition, if $\phi : R^n \to R^n$ is an isotone, diagonal mapping and $A \in L(R^n)$ is an $M$-matrix, then $A + \phi : R^n \to R^n$ is an $M$-function, too.

Evidently, a continuous, surjerctive $M$-function $F : R^n \to R^n$ is a homeomorphism from $R^n$ onto itself.

## §4. Convergence Analysis of the $\mathcal{F}_{NMAOR}(r,\omega)$-Method

We first set up one theorem which describes the two-sided approximation characterization of the $\mathcal{F}_{NMAOR}(r,\omega)$-method.

**Theorem 4.1.** *Let $(f^{(k)}, E^k)(k = 1, 2, \cdots, \alpha)$ be a nonlinear multisplitting of $F : D \subset R^n \to R^n$. Suppose that for each $k \in \{1, 2, \cdots, \alpha\}$, $f^{(k)}(x; y) : D \times D \subset R^n \times R^n \to R^n$ is continuous, antitone for $x$ on $D$, off-diagonally antitone and strictly diagonally isotone for $y$ on $D$. If there exist $x^0, y^0 \in D$ such that*

$$x^0 \le y^0, \qquad J = \{x \in R^n \mid x^0 \le x \le y^0\} \subset D,$$
$$F(x^0) \le 0 \le F(y^0),$$

*then the sequences $\{x^m\}$ and $\{y^m\}$ generated by the $\mathcal{F}_{NMAOR}(r,\omega)$-method starting from $x^0$ and $y^0$ respectively are well-defined on $J$ and satisfy*

(a) $x^0 \le x^m \le x^{m+1} \le y^{m+1} \le y^m \le y^0$;

(b) $\lim\limits_{m\to\infty} x^m = x^* \le y^* = \lim\limits_{m\to\infty} y^m$;

(c) $x^*, y^* \in J$ *are solutions of* (1.1);

(d) *for any solution* $\tilde{x} \in J$ *of* (1.1) *there holds* $x^* \le \tilde{x} \le y^*$

*provided $r \in [0, 1]$ and $\omega \in (0, 1]$.*

**Proof.** The proof of (a) is proceeded by induction.

Suppose that for some $m \ge 0, i \ge 1$ and $k = 1, 2, \cdots, \alpha$:

$$x^0 \le x^{m-1} \le x^m \le y^m \le y^{m-1} \le y^0; \tag{4.1}$$

$$x_j^0 \le \hat{x}_j^{m-1,k} \le \hat{x}_j^{m,k} \le \hat{y}_j^{m,k} \le \hat{y}_j^{m-1,k} \le y_j^0, \quad j = 1(1)(i-1) \text{ and } j \in S^k; \tag{4.2}$$

$$x_j^0 \le x_j^{m-1,k} \le x_j^{m,k} \le y_j^{m,k} \le y_j^{m-1,k} \le y_j^0, \qquad j = 1(1)(i-1); \tag{4.3}$$

$$x_j^0 \le x_j^m \le x_j^{m+1} \le y_j^{m+1} \le y_j^m \le y_j^0, \qquad j = 1(1)(i-1), \tag{4.4}$$

where we stipulate that $z^{-1} = z^0$ for $z \in R^n$ and (4.2)–(4.4) are vacuous for $i = 1$.

Obviously, (4.1)–(4.4) hold for $m = 0$ and $i = 1$.

For $m$ fixed we now begin to prove that (4.1)–(4.4) still hold for $j = i$.

When $i \in S^k$, by the antitonicity of $f^{(k)}(x; y)$ for $x$ and the off-diagonal antitonicity of it for $y$ we have

$$\begin{aligned}
&f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{x}_i^{m-1,k}, x_{i+1}^m, \cdots, x_n^m)\\
&\le f_i^{(k)}(x^{m-1}; x_1^{m-1,k}, \cdots, x_{i-1}^{m-1,k}, \hat{x}_i^{m-1,k}, x_{i+1}^{m-1}, \cdots, x_n^{m-1}) = 0,\\
&f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{y}_i^{m-1,k}, x_{i+1}^m, \cdots, x_n^m)\\
&\ge f_i^{(k)}(y^{m-1}; y_1^{m-1,k}, \cdots, y_{i-1}^{m-1,k}, \hat{y}_i^{m-1,k}, y_{i+1}^{m-1}, \cdots, y_n^{m-1}) = 0.
\end{aligned}$$

Making use of the continuity of $f^{(k)}(x;y)$ and the strictly diagonal isotonicity of it for $y$, we can therefore conclude that there exists unique $\hat{x}_i^{m,k} \in [\hat{x}_i^{m-1,k}, \hat{y}_i^{m-1,k}]$ such that

$$f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{x}_i^{m,k}, x_{i+1}^m, \cdots, x_n^m) = 0. \tag{4.5}$$

Similarly, we can conclude that there exists unique $\hat{y}_i^{m,k} \in [\hat{x}_i^{m-1,k}, \hat{y}_i^{m-1,k}]$ such that

$$f_i^{(k)}(y^m; y_1^{m,k}, \cdots, y_{i-1}^{m,k}, \hat{y}_i^{m,k}, y_{i+1}^m, \cdots, y_n^m) = 0. \tag{4.6}$$

Notice that

$$\begin{aligned} &f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{y}_i^{m,k}, x_{i+1}^m, \cdots, x_n^m) \\ &\geq f_i^{(k)}(y^m; y_1^{m,k}, \cdots, y_{i-1}^{m,k}, \hat{y}_i^{m,k}, y_{i+1}^m \cdots, y_n^m) = 0. \end{aligned} \tag{4.7}$$

From (4.5) and (4.7) by using the strictly diagonal isotonicity of $f^{(k)}(x;y)$ for $y$ again we obtain

$$\hat{x}_i^{m,k} \leq \hat{y}_i^{m,k}, \tag{4.8}$$

which means that (4.2) is true for $j = i$.

We now turn to (4.3).

Noticing (2.5) and $r \in [0,1]$, when $i \in S^k$ we have

$$x_i^{m,k} = (1-r)x_i^m + r\hat{x}_i^{m,k} \geq (1-r)x_i^{m-1} + r\hat{x}_i^{m-1,k} = x_i^{m-1,k},$$

$$y_i^{m,k} = (1-r)y_i^m + r\hat{y}_i^{m,k} \leq (1-r)y_i^{m-1} + r\hat{y}_i^{m-1,k} = y_i^{m-1,k}$$

and $\quad x_i^{m,k} = (1-r)x_i^m + r\hat{x}_i^{m,k} \leq (1-r)y_i^m + r\hat{y}_i^{m,k} = y_i^{m,k};$

when $i \notin S^k$ we easily get

$$x_i^{m-1,k} = x_i^{m-1} \leq x_i^m = x_i^{m,k} \leq y_i^{m,k} = y_i^m \leq y_i^{m-1} = y_i^{m-1,k}.$$

Therefore, we know that (4.3) holds for $j = i$, too.

By (2.4) and (2.5) we have

$$\begin{aligned} x_i^{m+1} &= \omega \sum_k e_i^k \hat{x}_i^{m,k} + (1-\omega)x_i^m \\ &\geq \omega \sum_k e_i^k \hat{x}_i^{m-1,k} + (1-\omega)x_i^m \\ &= (x_i^m - (1-\omega)x_i^{m-1}) + (1-\omega)x_i^m \\ &= x_i^m + (1-\omega)(x_i^m - x_i^{m-1}) \geq x_i^m. \end{aligned}$$

Analogously, we have $y_i^{m+1} \leq y_i^m$. Making use of (4.8) additionally, we obtain

$$\begin{aligned} x_i^{m+1} &= \omega \sum_k e_i^k \hat{x}_i^{m,k} + (1-\omega)x_i^m \\ &\leq \omega \sum_k e_i^k \hat{y}_i^{m,k} + (1-\omega)y_i^m = y_i^{m+1}. \end{aligned}$$

Thus, (4.4) is proved for $j = i$.

The above discussion shows us that (4.2)–(4.4) hold for all $i \in N$, therefore

$$x^m \leq x^{m+1} \leq y^{m+1} \leq y^m.$$

By induction we know that (a) is true.

(b) is clearly direct from (a).

We only take $x^*$ as a sample to prove (c).

Through the proving process of (a) we know that the sequences $\{\hat{x}_j^{m,k}\}(j=1(1)n, j\in S^k)$ and $\{x_j^{m,k}\}(j=1(1)n)$ are upper bounded and monotone nondecreasing with respect to $m$ for $k=1,2,\cdots,\alpha$; thus they converge as $m$ tends to infinity. We denote their corresponding limit points by $\hat{x}_j^k(j=1(1)n, j\in S^k)$ and $\tilde{x}_j^k(j=1(1)n)$ respectively, for $k=1,2,\cdots,\alpha$.

Write

$$K(i)=\{k \mid i\in S^k, k=1,2,\cdots,\alpha\}, \qquad i=1(1)n.$$

Take limits for (2.4)–(2.6), we accordingly have

$$\begin{cases} x_j^* &= \sum_{k=1}^{\alpha} e_j^k \hat{x}_j^k = \sum_{k\in K(j)} e_j^k \hat{x}_j^k, \\ x_j^* &= \sum_{k=1}^{\alpha} e_j^k \tilde{x}_j^k = \sum_{k\in K(j)} e_j^k \tilde{x}_j^k, \end{cases} \qquad j=1(1)n; \tag{4.9}$$

$$\begin{cases} f_i^{(k)}(x^*;\tilde{x}_1^k,\cdots,\tilde{x}_{i-1}^k,\hat{x}_i^k,x_{i+1}^*,\cdots,x_n^*)=0, \\ i\in S^k, k=1,2,\cdots,\alpha, \end{cases} \qquad i=1(1)n \tag{4.10}$$

and

$$\begin{cases} \tilde{x}_i^k = \begin{cases} r\hat{x}_i^k+(1-r)x_i^*, & \text{for } i\in S^k, \\ x_i^*, & \text{for } i\notin S^k, \end{cases} \\ k=1,2,\cdots,\alpha, \end{cases} \qquad i=1(1)n. \tag{4.11}$$

For each $j\in N$, define

$$\begin{cases} \hat{x}_j^{k_0(j)} = \min_{k\in K(j)} \hat{x}_j^k, \\ \hat{x}_j^{k_1(j)} = \max_{k\in K(j)} \hat{x}_j^k. \end{cases}$$

As $\sum_{k\in K(i)} e_i^k = 1$, it is obvious that

$$\begin{cases} \hat{x}_j^{k_0(j)} \le x_j^* \le \hat{x}_j^{k_1(j)}, \\ j\in S^{k_0(j)}\cap S^{k_1(j)}, \quad j=1(1)n. \end{cases} \tag{4.12}$$

We now show that the following facts are correct by induction:

$$F_j(x^*)=0, \quad j=1(1)n; \tag{4.13}$$

$$\hat{x}_j^k = x_j^*, \quad j=1(1)n \text{ and } j\in S^k, k=1,2,\cdots,\alpha; \tag{4.14}$$

$$\tilde{x}_j^k = x_j^*, \quad j=1(1)n, k=1,2,\cdots,\alpha. \tag{4.15}$$

When $j=1$, by (4.12) and the strictly diagonal isotonicity of $f^{(k)}(x;y)(k=1,2,\cdots,\alpha)$ for $y$ we have

$$\begin{aligned} F_1(x^*) &= f_1^{(k_1(1))}(x^*;x_1^*,\cdots,x_n^*) \\ &\le f_1^{(k_1(1))}(x^*;\hat{x}_1^{k_1(1)},\cdots,x_n^*)=0 \end{aligned}$$

and

$$\begin{aligned} F_1(x^*) &= f_1^{(k_0(1))}(x^*;x_1^*,\cdots,x_n^*) \\ &\ge f_1^{(k_0(1))}(x^*;\hat{x}_1^{k_0(1)},\cdots,x_n^*)=0. \end{aligned}$$

Thus, $F_1(x^*)=0$, which demonstrates that (4.13) holds for $j=1$.

If for $k=1,2,\cdots,\alpha$, $1\in S^k$, noticing (4.10) we can get

$$f_1^{(k)}(x^*;\hat{x}_1^k,x_2^*,\cdots,x_n^*)=0=F_1(x^*)=f_1^{(k)}(x^*;x^*),$$

by the strictly diagonal isotonicity of $f^{(k)}(x;y)(k=1,2,\cdots,\alpha)$ with respect to $y$ again we can conclude that $\hat{x}_1^k=x_1^*$, which means that (4.14) is true for $j=1\in S^k$, $k=1,2,\cdots,\alpha$.

Now, by (4.11) the correctness of (4.15) for $j=1$ and $k=1,2,\cdots,\alpha$ is obvious.

Suppose that for all $j\le i-1$, (4.13)–(4.15) are valid. When $j=i$, by (4.12) and the strictly diagonal isotonicity of $f^{(k)}(x;y)(k=1,2,\cdots,\alpha)$ with respect to $y$ we obtain

$$\begin{aligned}F_i(x^*)&=f_i^{(k_1(i))}(x^*;x_1^*,\cdots,x_n^*)\\&\le f_i^{(k_1(i))}(x^*;x_1^*,\cdots,x_{i-1}^*,\hat{x}_i^{k_1(i)},x_{i+1}^*,\cdots,x_n^*)\\&=f_i^{(k_1(i))}(x^*;\tilde{x}_1^{k_1(i)},\cdots,\tilde{x}_{i-1}^{k_1(i)},\hat{x}_i^{k_1(i)},x_{i+1}^*,\cdots,x_n^*)=0\end{aligned}$$

and

$$\begin{aligned}F_i(x^*)&=f_i^{(k_0(i))}(x^*;x_1^*,\cdots,x_n^*)\\&\ge f_i^{(k_0(i))}(x^*;x_1^*,\cdots,x_{i-1}^*,\hat{x}_i^{k_0(i)},x_{i+1}^*,\cdots,x_n^*)\\&=f_i^{(k_0(i))}(x^*;\tilde{x}_1^{k_0(i)},\cdots,\tilde{x}_{i-1}^{k_0(i)},\hat{x}_i^{k_0(i)},x_{i+1}^*,\cdots,x_n^*)=0.\end{aligned}$$

Therefore, $F_i(x^*)=0$, which indicates that (4.11) is valid for $j=i$.

If for $k=1,2,\cdots,\alpha,i\in S^k$, making use of (4.10) we can get

$$\begin{aligned}&f_i^{(k)}(x^*;x_1^*,\cdots,x_{i-1}^*,\hat{x}_i^k,x_{i+1}^*,\cdots,x_n^*)\\&=f_i^{(k)}(x^*;\tilde{x}_1^k,\cdots,\tilde{x}_{i-1}^k,\hat{x}_i^k,x_{i+1}^*,\cdots,x_n^*)=0\\&=F_i(x^*)=f_i^{(k)}(x^*;x^*),\end{aligned}$$

by the strictly diagonal isotonicity of $f^{(k)}(x;y)(k=1,2,\cdots,\alpha)$ for $y$ we get $\hat{x}_i^k=x_i^*$, which tells us that (4.14) is true for $j=i$ and $k=1,2,\cdots,\alpha$.

The correctness of (4.15) for $j=i$ and $k=1,2,\cdots,\alpha$ up to now is trivial by noticing (4.11).

By induction we know $F(x^*)=0$, i.e., $x^*$ is a solution of $F(x)=0$. $x^*\in J$ is a direct conclusion from the proof of (a).

The proof of (d) is also completed by induction.

Because $x^0\le\tilde{x}\le y^0$ and $F(\tilde{x})=0$, by using (a) we know that the sequences $\{x^m\}$ and $\{y^m\}$ generated by the $\mathcal{F}_{NMAOR}(r,\omega)$-method starting from $x^0$ and $y^0$ , respectively, satisfy

$$x^0\le x^m\le x^{m+1}\le y^{m+1}\le y^m\le y^0. \tag{4.16}$$

Suppose that for some $m\ge 0$ and $i\ge 1$:

$$x^m\le\tilde{x}\le y^m; \tag{4.17}$$

$$\hat{x}_j^{m,k}\le\tilde{x}\le\hat{y}_j^{m,k};\quad j=1(1)(i-1),\quad j\in S^k; \tag{4.18}$$

$$x_j^{m,k}\le\tilde{x}\le y_j^{m,k};\quad j=1(1)(i-1); \tag{4.19}$$

$$x_j^{m+1}\le\tilde{x}\le y_j^{m+1};\quad j=1(1)(i-1) \tag{4.20}$$

hold for $k=1,2,\cdots,\alpha$. Here we stipulate that (4.18)–(4.20) are vacuous for $i=1$.

Obviously, (4.17)–(4.20) hold for $m = 0$ and $i = 1$.

For $m$ fixed we now start to prove that (4.18)–(4.20) also hold for $j = i$.

When $i \in S^k$ for $k = 1, 2, \cdots, \alpha$, by the off-diagonal antitonicity and the strictly diagonal isotonicity of $f^{(k)}(x; y)$ with respect to $y$ we have

$$\begin{aligned} &f_i^{(k)}(x_m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{x}_i^{m,k}, x_{i+1}^m, \cdots, x_n^m) \\ &= 0 = f_i^{(k)}(\tilde{x}; \tilde{x}) \le f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \tilde{x}_i, x_{i+1}^m, \cdots, x_n^m) \end{aligned}$$

and therefore, $\hat{x}_i^{m,k} \le \tilde{x}_i$. Similarly, we can get $\hat{y}_i^{m,k} \ge \tilde{x}_i$. Hence, (4.18) is true for $j = i$.

We now prove (4.19).

Noting (2.5) and $r \in [0,1]$, when $i \in S^k$ we have

$$\begin{aligned} x_i^{m,k} &= (1-r)x_i^m + r\hat{x}_i^{m,k} \le (1-r)\tilde{x}_i + r\tilde{x}_i = \tilde{x}_i, \\ y_i^{m,k} &= (1-r)y_i^m + r\hat{y}_i^{m,k} \ge (1-r)\tilde{x}_i + r\tilde{x}_i = \tilde{x}_i; \end{aligned}$$

when $i \notin S^k$ we easily get

$$x_i^{m,k} = x_i^m \le \tilde{x}_i \le y_i^m = y_i^{m,k}.$$

Therefore, we know that (4.19) holds for $j = i$, too.

By (2.4) and (2.5) we have

$$x_i^{m+1} = \omega \sum_k e_i^k \hat{x}_i^{m,k} + (1-\omega)x_i^m \le \omega \sum_k e_i^k \tilde{x}_i + (1-\omega)\tilde{x}_i = \tilde{x}_i.$$

Similarly, we have $y_i^{m+1} \ge \tilde{x}_i$. Thus, (4.20) is proved for $j = i$.

The above discussion shows that (4.18)–(4.20) hold for all $i \in N$; therefore

$$x^{m+1} \le \tilde{x} \le y^{m+1}.$$

By induction we know that (d) is correct.

The following theorem characterizes the dependence of $\mathcal{F}_{NMAOR}(r, \omega)$ -method upon the choice of the acceleration factor.

**Theorem 4.2.** *Suppose that the conditions of Theorem 4.1 are satisfied and let $\omega, \omega' \in (0,1]$, $r \in [0,1]$ be given for which*

$$0 \le \omega \le \omega' \le 1. \tag{4.21}$$

*In all cases with $y^0$ as starting point, let $\{y^m\}$ and $\{y'^m\}$ be sequences yielded by $\mathcal{F}_{NMAOR}$ $(r,\omega)$-method with $\{r,\omega\}$ and $\{r,\omega'\}$ being the iteration parameters respectively. By Theorem 4.1 these sequences are well-defined, momotonously non-increasingly converge to a solution $y^*$ of $F(x) = 0$ in $J$ and any other solution $\tilde{x} \in J$ of $F(x) = 0$ satisfies $\tilde{x} \le y^*$. Moreover, it holds that*

$$y^m \ge y'^m \ge y^*, \quad m = 0, 1, 2, \cdots. \tag{4.22}$$

*The corresponding results, with all inequalities reversed, hold for the sequences starting from $x^0$.*

**Proof.** Suppose that for some $m \ge 0$ and $i \ge 1$,

$$y^m \ge y'^m; \tag{4.23}$$

$$\hat{y}_j^{m,k} \ge \hat{y}'^{m,k}_j, \quad j = 1(1)(i-1) \text{ and } j \in S^k; \tag{4.24}$$

$$y_j^{m,k} \geq y'^{m,k}_j, \quad j = 1(1)(i-1); \tag{4.25}$$

$$y_j^{m+1} \geq y'^{m+1}_j, \quad j = 1(1)(i-1) \tag{4.26}$$

hold for $k = 1, 2, \cdots, \alpha$. Here we stipulate that (4.24)–(4.26) are vacuous for $i = 1$. They are clearly valid for $m = 0$ and $i = 1$. Then for $i \in S^k$, by the antitonicity of $f^{(k)}(x; y)$ for $x$ and off-diagonal antitonicity of it for $y$ we have

$$\begin{aligned} &f_i^{(k)}(y^m; y_1^{m,k}, \cdots, y_{i-1}^{m,k}, \hat{y}_i^{m,k}, y_{i+1}^m, \cdots, y_n^m) \\ &= 0 = f_i^{(k)}(y'^m; y'^{m,k}_1, \cdots, y'^{m,k}_{i-1}, \hat{y}'^{m,k}_i, y'^m_{i+1}, \cdots, y'^m_n) \\ &\geq f_i^{(k)}(y^m; y_1^{m,k}, \cdots, y_{i-1}^{m,k}, \hat{y}'^{m,k}_i, y_{i+1}^m, \cdots, y_n^m). \end{aligned}$$

The strict diagonal isotonicity of $f^{(k)}(x; y)$ for $y$ implies that $\hat{y}_i^{m,k} \geq \hat{y}'^{m,k}_i$. So we have tested (4.24) for $j = i \in S^k$.

For (4.25) when $i \in S^k$ we easily have

$$y_i^{m,k} = r\hat{y}_i^{m,k} + (1-r)y_i^m \geq r\hat{y}'^{m,k}_i + (1-r)y'^m_i = y'^{m,k}_i;$$

when $i \notin S^k$ there obviously hold $y_i^{m,k} = y_i^m \geq y'^m_i = y'^{m,k}_i$. So (4.25) has also been tested for $j = i$.

Now, we turn to (4.26).

$$\begin{aligned} y_i^{m+1} &= \omega \sum_k e_i^k \hat{y}_i^{m,k} + (1-\omega) y_i^m \\ &= y'^{m+1}_i + (\omega' - \omega)\Big(y_i^m - \sum_k e_i^k \hat{y}'^{m,k}_i\Big). \end{aligned} \tag{4.27}$$

If $\omega = \omega'$, we easily have $y_i^{m+1} = y'^{m+1}_i$; when $\omega' > \omega \geq 0$, by noticing

$$\hat{y}'^{m,k}_i \leq \hat{y}'^{m-1,k}_i, \quad \text{for } i \in S^k$$

and

$$\sum_k e_i^k \hat{y}'^{m-1,k}_i = \frac{1}{\omega'} y'^m_i + (1 - \frac{1}{\omega'}) y'^{m-1}_i,$$

we have

$$y'^m_i - \sum_k e_i^k \hat{y}'^{m,k}_i \geq (1 - \frac{1}{\omega'})(y'^m_i - y'^{m-1}_i) \geq 0.$$

From (4.27) we get

$$y_i^{m+1} \geq y'^{m+1}_i + (\omega' - \omega)(y'^m_i - \sum_k e_i^k \hat{y}'^{m,k}_i) \geq y'^{m+1}_i.$$

The above discussion indicates that (4.24)–(4.26) hold for all $i \in N$. Therefore, $y^{m+1} \geq y'^{m+1}$.

By induction we have completed the proof of this theorem.

Besides the conditions of Theorem 4.1, if we assume furthermore that $F$ is surjective, inverse isotone, we can deal with the global convergence of $\mathcal{F}_{NMAOR}(r, \omega)$-method. This fact is preciesely stated by the following theorem.

**Theorem 4.3.** *Let $(f^{(k)}, E^k)(k = 1, 2, \cdots, \alpha)$ be a nonlinear multisplitting of a surjective, inverse isotone function $F : R^n \to R^n$. Suppose that for each $k \in \{1, 2, \cdots, \alpha\}$,*

*$f^{(k)}(x;y): R^n \times R^n \to R^n$ is continuous, antitone for $x$ on $R^n$, off-diagonally antitone and surjectively diagonally isotone for $y$ on $R^n$. Then any sequence generated by $\mathcal{F}_{NMAOR}(r,\omega)$-method starting from any $x^0 \in R^n$ converges to the unique solution $x^* \in R^n$ of $F(x) = 0$ provided $\{r,\omega\} \subseteq [0,1](\omega \neq 0)$.*

**Proof.** The existence and the uniqueness of the solution of $F(x) = 0$ in $R^n$ is obvious from the conditions of the theorem.

For given $x^0 \in R^n$, we define

$$a_i = \min\{F_i(x^0), 0\}, \quad b_i = \max\{F_i(x^0), 0\}, \quad i = 1(1)n,$$
$$u^0 = F^{-1}(a), \qquad v^0 = F^{-1}(b).$$

By the inverse isotonicity of $F$ we then have

$$F(u^0) \leq 0 \leq F(v^0), \quad u^0 \leq x^0 \leq v^0, \text{ and } u^0 \leq x^* \leq v^0.$$

Let $\{u^m\}, \{v^m\}$ and $\{x^m\}$ be sequences generated by $\mathcal{F}_{NMAOR}(r,\omega)$-method starting from $u^0, v^0$ and $x^0$ respectively with the same parameters $r$ and $\omega$. By the surjectively diagonal isotonicity of $f^{(k)}(x;y)$ with respect to $y$ on $R^n$ for each $k = 1, 2, \cdots, \alpha$, we know that all the solutions $\{\hat{u}_i^{m,k}\}$, $\{\hat{v}_i^{m,k}\}$ and $\{\hat{x}_i^{m,k}\}(i = 1(1)n, i \in S^k)$ correspondingly determined by the following systems of nonlinear equations

$$f_i^{(k)}(u^m; u_1^{m,k}, \cdots, u_{i-1}^{m,k}, \hat{u}_i^{m,k}, u_{i+1}^m, \cdots, u_n^m) = 0, \quad i = 1(1)n, i \in S^k; \tag{4.28}$$

$$f_i^{(k)}(v^m; v_1^{m,k}, \cdots, v_{i-1}^{m,k}, \hat{v}_i^{m,k}, v_{i+1}^m, \cdots, v_n^m) = 0, \quad i = 1(1)n, i \in S^k; \tag{4.29}$$

$$f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{x}_i^{m,k}, x_{i+1}^m, \cdots, x_n^m) = 0, \quad i = 1(1)n, i \in S^k \tag{4.30}$$

uniquely exist for $k = 1, 2, \cdots, \alpha$ and $m = 0, 1, 2, \cdots$, which indicates that the corresponding $\mathcal{F}_{NMAOR}(r,\omega)$-method are well-defined.

By Theorem 4.1 we get

$$u^0 \leq u^m \leq u^{m+1} \leq v^{m+1} \leq v^m \leq v^0, \quad m = 0, 1, 2, \cdots \tag{4.31}$$

and

$$\lim_{m\to\infty} u^m = \lim_{m\to\infty} v^m = x^*. \tag{4.32}$$

Now, suppose that for some $m \geq 0$ and $i \geq 1$,

$$u^m \leq x^m \leq v^m; \tag{4.33}$$

$$\hat{u}_j^{m,k} \leq \hat{x}_j^{m,k} \leq \hat{v}_j^{m,k}, \quad j = 1(1)(i-1) \text{ and } j \in S^k; \tag{4.34}$$

$$u_j^{m,k} \leq x_j^{m,k} \leq v_j^{m,k}, \quad j = 1(1)(i-1); \tag{4.35}$$

$$u_j^{m+1} \leq x_j^{m+1} \leq v_j^{m+1}, \quad j = 1(1)(i-1) \tag{4.36}$$

hold for $k = 1, 2, \cdots, \alpha$. Here we stipulate that (4.34)–(4.36) are vacuous for $i = 1$, so (4.33)-(4.36) are clearly valid for $m = 0$ and $i = 1$.

For $m$ fixed we want now to prove that (4.34)–(4.36) hold for $j = i$, too.

When $i \in S^k$, by the antitonicity of $f^{(k)}(x;y)$ for $x$ and the off-diagonal antitonicity of it for $y$ we have

$$\begin{aligned} & f_i^{(k)}(v^m; v_1^{m,k}, \cdots, v_{i-1}^{m,k}, \hat{x}_i^{m,k}, v_{i+1}^m, \cdots, v_n^m) \\ & \leq f_i^{(k)}(x^m; x_1^{m,k}, \cdots, x_{i-1}^{m,k}, \hat{x}_i^{m,k}, x_{i+1}^m, \cdots, x_n^m) \\ & = 0 = f_i^{(k)}(v^m; v_1^{m,k}, \cdots, v_{i-1}^{m,k}, \hat{v}_i^{m,k}, v_{i+1}^m, \cdots, v_n^m). \end{aligned}$$

Using the strictly diagonal isotonicity we get $\hat{x}_i^{m,k} \leq \hat{v}_i^{m,k}$. Similarly, we can obtain $\hat{u}_i^{m,k} \leq \hat{x}_i^{m,k}$. Then (4.34) holds for $j = i \in S^k$.

Easily, we can also show that (4.35)–(4.36) hold for $j = i$.

Therefore, (4.34)–(4.36) hold for all $i \in N$ and we can then conclude that

$$u^{m+1} \leq x^{m+1} \leq v^{m+1}.$$

By induction we have show that (4.33)–(4.36) are valid. Now, taking limit in (4.33) for $m$ to infinity we obtain

$$\lim_{m\to\infty} x^m = x^*,$$

which fulfills our proof.

At the end of this section, we make the following three remarks.

**Remark 4.1.** There are examples which show that each assumption in Theorem 4.3 is indispensable for ensuring its conclusion to hold. For the length of this paper, we will not list them here.

**Remark 4.2.** For the special nonlinear multisplitting defined by (2.2)–(2.3), replace the condition '$f^{(k)}(x;y)(k = 1, 2, \cdots, \alpha)$ are antitone for $x$ on $D$ and off-diagonal antitone for $y$ on D' by '$F : D \subset R^n \to R^n$ is off-diagonally antitone on $D$', all the conclusions of Theorem 4.1 and Theorem 4.2 still hold.

**Remark 4.3.** For the above mentioned special nonlinear multisplitting, the assumption '$F : R^n \to R^n$ is a surjective $M$-function' is sufficient for guaranteeing the conclusion of Theorem 4.3.

## §5. Numerical Results

In order to test the correctnesses of the preceding results, we consider the following two point boundary value problem of weak nonlinear ordinary differential equation

$$\begin{cases} \frac{d^2u}{dx^2} = \frac{1}{2}u^2, \quad 0 < t < 1, \\ u(0) = 1, \quad u(1) = 2. \end{cases} \tag{5.1}$$

By discretizing it with equidistant step size $h = 1/(n+1)$, we get a class of special system of $n$-unkown $(u_1, \cdots, u_n)$ nonlinear equations

$$F(u) = Au + \phi(u) = 0, \tag{5.2}$$

where

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}, \quad \phi(u) = \frac{1}{2}h^2 \begin{bmatrix} u_1^3 - \frac{2}{h^2} \\ u_2^3 \\ \vdots \\ u_{n-1}^3 \\ u_n^3 - \frac{4}{h^2} \end{bmatrix}.$$

Obviously, we see that here $A$ is an $M$-matrix and $\phi(u)$ is strictly diagonally isotone. Therefore, the system of equations (5.2) has unique solution on $R^n$. For the system of equations (5.2), by taking $n = 6$ and three kinds of different multisplitting cases

a) subsets

$$S^1 = \{1,2\}, \quad S^2 = \{3,4\}, \quad S^3 = \{5,6\}$$

and corresponding weighting matrices

$$\begin{aligned} E_1 &= \text{diag}\,(1,1,0,0,0,0), \\ E_2 &= \text{diag}\,(0,0,1,1,0,0), \\ E_3 &= \text{diag}\,(0,0,0,0,1,1). \end{aligned}$$

b) subsets

$$S^1 = \{1,2,3\}, \quad S^2 = \{2,3,4,5\}, \quad S^3 = \{4,5,6\}$$

and corresponding weighting matrices

$$\begin{aligned} E_1 &= \text{diag}\,(1,1/2,1/2,0,0,0), \\ E_2 &= \text{diag}\,(0,1/2,1/2,1/2,1/2,0), \\ E_3 &= \text{diag}\,(0,0,0,1/2,1/2,1), \end{aligned}$$

c) subsets

$$S^1 = S^2 = S^3 = \{1,2,3,4,5,6\}$$

and corresponding weighting matrices $E_1 = E_2 = E_3 = \frac{1}{3}I$, we do numerously numerical experiments about the $\mathcal{F}_{NMAOR}$-method combined with (2.6) being solved approximately by Newton method for choosing different pairs of the parameters $r$ and $\omega$. The stopping criteria and the starting vectors are

$$\|u^{m+1} - u^m\|_\infty \le 2 \times 10^{-6} \quad \text{or} \quad \|F(u^{m+1})\|_\infty \le 2 \times 10^{-4}$$

and

$$u^0 = [4,4,4,4,4,4]^T \in R^n$$

(obviously, $F(u^0) \ge 0$), respectively. All the results obtained are satisfactory and closely coincide with our theory. For the length of our paper, we just select the best one for each case and list them in the following tables. Here, we denote "IT" as iteration numbers.

Table(1): $\mathcal{F}_{NMAOR}$(0.9,0.95)-method (case a)

| IT | 10 | 40 | 60 | 90 | 110 | 130 | 150 | 160 | 170 |
|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | 0.3 | −1.0 | −1.2 | −1.280 | −1.292 | −1.296 | −1.2984 | −1.2989 | −1.299 |
| $u_2$ | 1.5 | −1.1 | −1.43 | −1.54 | −1.562 | −1.571 | −1.574 | −1.575 | −1.576 |
| $u_3$ | 2.0 | −1.3 | −1.6 | −1.76 | −1.79 | −1.806 | −1.811 | −1.812 | −1.813 |
| $u_4$ | 1.9 | −1.4 | −1.80 | −1.94 | −1.971 | −1.983 | −1.987 | −1.988 | −1.989 |
| $u_5$ | 1.0 | −1.6 | −1.93 | −2.04 | −2.070 | −2.080 | −2.084 | −2.0851 | −2.0857 |
| $u_6$ | −0.3 | −1.8 | −2.00 | −2.06 | −2.081 | −2.086 | −2.088 | −2.0890 | −2.0893 |

Table(2): $\mathcal{F}_{NMAOR}$(0.99,1.0)-method (case b)

| IT | 20 | 40 | 70 | 100 | 120 | 140 | 150 | 160 |
|---|---|---|---|---|---|---|---|---|
| $u_1$ | −0.6 | −1.1 | −1.26 | −1.292 | −1.297 | −1.2988 | −1.2992 | −1.2994 |
| $u_2$ | −0.3 | −1.2 | −1.51 | −1.563 | −1.572 | −1.575 | −1.5760 | −1.5765 |
| $u_3$ | −0.4 | −1.4 | −1.73 | −1.79 | −1.808 | −1.812 | −1.8131 | −1.8137 |
| $u_4$ | −0.4 | −1.5 | −1.90 | −1.972 | −1.984 | −1.988 | −1.989 | −1.99 |
| $u_5$ | −0.9 | −1.7 | −2.02 | −2.072 | −2.081 | −2.084 | −2.085 | −2.086 |
| $u_6$ | −1.3 | −1.90 | −2.05 | −2.081 | −2.086 | −2.0888 | −2.0893 | −2.0895 |

Table(3): $\mathcal{F}_{NMAOR}$(0.8,0.99)-method (case c)

| IT | 10 | 30 | 50 | 70 | 100 | 120 | 140 | 150 |
|---|---|---|---|---|---|---|---|---|
| $u_1$ | 0.2 | −0.9 | −1.20 | −1.26 | −1.292 | −1.296 | −1.2987 | −1.299 |
| $u_2$ | 1.2 | −0.9 | −1.3 | −1.51 | −1.563 | −1.571 | −1.575 | −1.576 |
| $u_3$ | 1.5 | −1.0 | −1.5 | −1.73 | −1.79 | −1.807 | −1.812 | −1.813 |
| $u_4$ | 1.6 | −1.2 | −1.7 | −1.90 | −1.972 | −1.983 | −1.988 | −1.989 |
| $u_5$ | 0.6 | −1.4 | −1.8 | −2.01 | −2.071 | −2.081 | −2.084 | −2.085 |
| $u_6$ | −0.4 | −1.7 | −1.9 | −2.05 | −2.081 | −2.086 | −2.0887 | −2.089 |

For the convenience we just list several decimal parts of our numerical results which are sufficient for us to illustrate the monotonicity of our new method. The numerical results in these tables further verify that when $(r,\omega) \subseteq [0,1] \times (0,1]$, the $\mathcal{F}_{NMAOR}(0.9, 0.95)$-method, $\mathcal{F}_{NMAOR}(0.99, 1.0)$-method and $\mathcal{F}_{NMAOR}(0.8, 0.99)$-method corresponding to the above weighting matrices converge monotonously to the unique solution of the system of equations (5.2) with iterations 174, 160 and 158, respectively.

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