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CONSTRUCTION OF INDECOMPOSABLE DEFINITE HERMITIAN FORMS**

Zhu Fuzu*

Abstract

This paper gives a method to construct indecomposable positive definite integral Hermitian forms over an imaginary quadratic field $Q(\sqrt{-m})$ with given discriminant and given rank. It is shown that for any natural numbers n and a, there are n-ary indecomposable positive definite integral Hermitian lattices over $Q(\sqrt{-1})$ (resp. $Q(\sqrt{-2})$) with discriminant a, except for four (resp. one) exceptions. In these exceptional cases there are no lattices with the desired properties.

Keywords Indecomposable lattice (form), Minimum of a lattice (form), Minimal vector, Irreducible vector.

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§1. Introduction and Main Results

Let $F = Q(\sqrt{-m})$ be an imaginary quadratic field and R_m ($m > 0$ and square free) the ring of algebraic integers of F. The aim of this paper is to construct indecomposable positive definite Hermitian forms (hereafter simply written as H-forms) over R_m with given rank and given discriminant. We shall establish the following theorems.

Theorem 1.1. *For any natural numbers n and a, there are indecomposable positive definite Hermitian R_1-lattices of rank n with discriminant a, except for the four exceptions: $a = 1$, $n = 2, 3, 5$; $a = 3$, $n = 3$. In these exceptional cases, there are no lattices with the desired properties.*

Theorem 1.2. *For any natural numbers n and a, there are indecomposable positive definite Hermitian R_2-lattices of rank n with discriminant a, except for one exception: $a = 1$, $n = 3$. In this exceptional case there are no lattices with the desired properties.*

By an integral H-form we mean one whose associated Hermitian matrix has integral entries or, in the language of lattices, one whose associated Hermitian lattice has integral scale. We shall assume that all lattices are integral.

Our use of the word decomposition or splitting is the geometric one, i.e., L has a non-trivial expression of the form $L = M \perp N$ with $M \neq 0$ and $N \neq 0$. If no such expression exists, we call L indecomposable. Indecomposable lattices describe completely all definite

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*Department of Mathematics, East China Normal University, Shanghai 200062, China.
**An outline of this paper has appeared in Chinese Sci. Bull. (see [19]). In Table 2 (p. 372) of [19], on the right hand of the line $n = 8$, the value of $dL = a$ should read: $2, 6, 10$

lattices in view of the absolute uniqueness of the indecomposable splittings. There is another kind of decomposition, a more algebraic one. Consider the H-form

$$h(X_1,\cdots,X_n)=\sum_{j,k=1}^{n} a_{jk}X_jX_k$$

with $a_{jk}\in R_m$ and $\bar{a}_{jk}=a_{kj}$. Call a positive definite H-form h non-decomposable if there is no expression, except the trivial one, of the type

$$h(X_1,\cdots,X_n)=f(X_1,\cdots,X_n)+g(X_1,\cdots,X_n)$$

with f and g positive semidefinite. It follows directly from the definitions that

$$h(X_1,\cdots,X_n) \text{ (or the associated lattice } L) \text{ non-decomposable}$$
$$\Rightarrow h(X_1,\cdots,X_n) \text{ (or } L) \text{ indecomposable.}$$

But the converse does not always hold. Nevertheless we can prove the following theorem.

Theorem 1.3[1]. *If the positive definite Hermitian R_m-lattice L is unimodular, then L is nondecomposable if and only if L is indecomposable.*

Positive definite integral H-forms have an interesting history with applications and connections in many branches of mathematics including number theory, geometry of numbers, Lie group, Lie algebra, algebraic geometry and algebraic coding theory. First consider the problem of indecomposability for n-ary positive definite H-forms of discriminant 1 over R_m, i.e., the unimodular case over R_m. Recently the author shows the following theorem.

Theorem 1.4.* 1.[1−3] *For any natural number n, we can construct explicitly indecomposable positive definite unimodular odd Hermitian R_m-lattices of rank n ($m=1,2,3,5,7,11,13,15,19,43,67,163$), except for a finite number of exceptions (see Table 1). In the exceptional cases there are no lattices with the desired properties.*

2.[4,5] *For any given natural numbers n and square-free m satisfying the following conditions*

$$m\equiv 1 \pmod 4 \text{ and } 4|n,$$

or (1.1)

$$m\equiv 2 \pmod 4 \text{ and } 2|n,$$

we can construct explicitly indecomposable positive definite unimodular even Hermitian R_m-lattices of rank n. Moreover, these conditions on m and n are necessary for the existence of such lattices.

Theorem 1.4 is an analogy of the well-known theorem of Erdös-Ko-Zhu[6−9], and Theorems 1.1 and 1.2 are analogies of the Theorem of O'Meara-Zhu-Shao[10−12] on quadratic forms.

Remark 1.1. The method of the proof given in Theorem 1.4 can solve completely the problem of constructing indecomposable positive definite unimodular odd and even Hermitian R_m-lattices of any rank n and any square-free m.

*The author proved recently a more genreral result, which generalizes the first part of Theorem 1.4 for any R_m, and showed that there are no exceptions in case $m\not\equiv 3 \pmod 4$ if $m\geq 5$ and in case $m\equiv 3 \pmod 4$ if $m\geq 15$. See [20]. (Note added May 17, 1994).

Table 1

$m \setminus n$	exceptions
1	$2, 3, 4, 5$
2	$2, 3$
3	$2, 3, 4, 5, 7$
7	2
11	3
$5, 13, 15$ $19, 43, 67, 163$	no exceptions

§2. Some Lemmas

Lemma 2.1[10,12]. *For any natural numbers a and $n \geq 2$, there are indecomposable positive definite quadratic $\mathbb{Z}$-lattices of rank n with discriminant a, except for a finite number of exceptions. For $n \geq 3$, there are exactly 44 exceptions, which are listed in the following table.*

Table 2

n	$dL = a$
3	$1, 2, 3, 5, 6, 9, 11, 14, 15$
4	$1, 2, 3, 6, 7, 10, 14, 26$
5	$1, 2, 3, 5, 7, 10, 13$
6	$1, 2, 5, 6, 14$
7	$1, 3, 7$
8	$2, 6, 10$
9	$1, 2, 3, 5$
10	$1, 2$
11	1
12	2
13	1

In these exceptional cases, there are no lattices with the desired properties.

Lemma 2.2[3]. *If L is an indecomposable positive definite quadratic $\mathbb{Z}$-lattice, then $L \otimes_{\mathbb{Z}} R_m$ is an indecomposable positive definite Hermitian R_m-lattice.*

By Lemmas 2.1 and 2.2, we have

Lemma 2.3. *For any natural numbers n and a, there exists an indecomposable positive definite Hermitian R_m-lattice of rank n with discriminant a, except possibly for a finite number of lattices with $n = 2$ and those (n, a) as listed in Table 2.*

In order to determine whether the Hermitian lattices with values (n, a) in Table 2 are indecomposable or not, we require following lemmas. First we introduce some definitions.

Let h be a Hermitian form with associated sesquilinear form ϕ. Then $h(x) = \phi(x, x)$ for any vector x. The minimum of a Hermitian R_m-lattice with respect to its associated

Hermitian form h is

$$\min L = \min_h L = \{|h(x)| \mid 0 \neq x \in L\}.$$

A vector $v \in L$ is called minimal if v satisfies $|h(v)| = \min L$. A nonzero vector $x \in L$ is called reducible in L if there exist nonzero vectors $y, z \in L$ such that $x = y + z$ and $\phi(y, z) = 0$, otherwise x is irreducible.

We shall use h's, with or without subscripts, to denote Hermitian forms, and similarly ϕ's for the associated sesquilinear forms. If more than one form is present, we may prefix certain adjectives with the relevant form: h-irreducible vectors, h-indecomposible lattices. We will write (L, h) if otherwise confusion might arise over which form on L is being considered.

Let V be an n-dimensional vector space over F. A Hermitian form on V is positive definite if $h(x) > 0$ for any nonzero $x \in V$ and positive semidefinite or non-negative if $h(x) \geq 0$ for any nonzero $x \in V$. If h_1 and h_2 are positive H-forms on V, then so is $h = h_1 + h_2$. If h_1 is positive definite and h_2 is positive, then $h = h_1 + h_2$ is positive definite and $h(x) \geq h_1(x) > 0$ for any nonzero $x \in V$. If h_1 and h_2 are positive forms on the lattice L, then

$$\min_h L = \min_{h_1} L + \min_{h_2} L$$

holds if and only if h_1 and h_2 have a common minimal vector in L.

Lemma 2.4. *Let L be an R_m-lattice on a positive definite Hermitian space over F, and let $\mathcal{X} = \{x_1, \cdots, x_n\}$ be a basis for L. If*

$1°$ L is indecomposable, then

$2°$ $\forall\ y, z \in \mathcal{X}$, there is an ordering $y = v_1, \cdots, v_t = z$ of $\mathcal{X}$ such that $\phi(v_j, v_{j+1}) \neq 0$ for $1 \leq j \leq t-1$.

Moreover, if $\mathcal{X}$ consists of irreducible vectors, then $1°$ and $2°$ are equivalent.

This Lemma is easily proved by an analogy of a well-known theorem of Kneser[13].

Lemma 2.5. *Let L be a Hermitian R_m-lattice, on which a positive definite form h_0 and a positive semidefinite form h_1 have been defined, and let $h = h_0 + h_1$.*

$1°$ Let $\mu_0 = \min_{h_0} L$ and $\mu_1 = \min_{h_1}(h_1(L) - 0)$. Suppose that x is an h_0-irreducible vector in L with $h(x) < 2(\mu_0 + \mu_1)$. Then x is h-irreducible.

$2°$ Suppose that L is h_0-indecomposable and has a basis $\mathcal{X}$ of vectors satisfying the condition on x in $1°$. Suppose $(L, h_1) \simeq$ diag $(c_1, \cdots, c_n)$ with respect to $\mathcal{X}$. Then L is h-indecomposable.

Proof. $1°$ Suppose that x is h-reducible, say $x = y + z$ with $y \neq 0$, $z \neq 0$ and $\phi(y, z) = 0$. Then

$$h(x) = h(y) + h(z) = h_0(y) + h_1(y) + h_0(z) + h_1(z) < 2(\mu_0 + \mu_1)$$

with $h_0(y) + h_0(z) \geq 2\mu_0$. Hence $h_1(y) + h_1(z) < 2\mu_1$. Without loss of generality we can assume $h_1(y) < \mu_1$ and so $h_1(y) = 0$. Since h_1 is positive semidefinite, by Schwarz inequality we have

$$|\phi_1(y, z)|^2 \leq h_1(y) h_1(z) = 0,$$

and hence $\phi_1(y, z) = 0$. But

$$0 = \phi(y, z) = \phi_0(y, z) + \phi_1(y, z) = \phi_0(y, z).$$

Hence x is h_0-reducible, contrary to our hypothesis.

2° Let $\mathcal{X} = \{z_1, \cdots, z_n\}$. By 1° the vectors $z_j (1 \le j \le n)$ are h-irreducible. Since L is h_0-indecompos-
able, condition 2° of Lemma 2.4 holds for ϕ_0. But $\phi_0(z_j, z_k) = \phi_0(z_j, z_k) + \phi_1(z_j, z_k)$ for $j \ne k$. Hence L is $(h_0 + h_1)$-indecomposable.

Lemma 2.6[1,2]. *Let* $h(Y) = \sum\limits_{j=1}^{n} Y_j \bar{Y}_j + (\sum\limits_{j=1}^{n} Y_j)(\sum\limits_{j=1}^{n} \bar{Y}_j)$, $L(Y) = \sum\limits_{j=1}^{n} \alpha_j Y_j$ *with* $\alpha_j \in R_m$. *Then the Hermitian form*

$$h(Y) - L(Y)\overline{L(Y)}$$

is positive semidefinite if and only if $L(Y) = Y_j$ $(j = 1, \cdots, n)$ *or* $\sum\limits_{j=1}^{n} Y_j$.

We now generalize [1, Lemma 7] and [2, Theorem 2] as follows.

Lemma 2.7. *The n-ary Hermitian form*

$$h(X) = aX_1\bar{X}_1 + \beta X_1\bar{X}_2 + \bar{\beta}\bar{X}_1X_2 + 2\sum_{j=2}^{n} X_j\bar{X}_j + \sum_{j=2}^{n-1} \bar{X}_jX_{j+1} + \sum_{j=2}^{n-1} X_j\bar{X}_{j+1}$$

over R_m *with discriminant* Δ_n $(0 < \Delta_n < n)$ *is positive definite and non-decomposable* (*and therefore indecomposable*), *where* $n \ge 2$, $a \ge 2$, $\beta = b_1 + b_2\theta$ *with* $b_j \in \mathbb{Z}$, $0 \le b_j < n$, *and* $\theta = \sqrt{-m}$ *if* $m \not\equiv 3 \pmod 4$ *or* $\theta = \frac{1}{2}(1 + \sqrt{-m})$ *if* $m \equiv 3 \pmod 4$, *and the following conditions are satisfied:*

1. $a < \beta\bar{\beta}$,
2. $a - \beta\bar{\beta} + 2b_1 - n < 0$,
3. $a - \beta\bar{\beta} + m(2b_2 - n) < 0$,
4. $a - \beta\bar{\beta} + 2b_1 - n + m(2b_2 - n) < 0$

if $m \not\equiv 3 \pmod 4$; *and*

1° $a < \beta\bar{\beta}$,
2° $a - \beta\bar{\beta} + 2b_1 + b_2 - n < 0$,
3° $a - \beta\bar{\beta} + b_1 + \frac{1}{2}(m+1)b_2 - \frac{1}{4}(m+1)n < 0$,
4° $a - \beta\bar{\beta} + 3b_1 + \frac{1}{2}(m+3)b_2 - \frac{1}{4}(m+9)n < 0$

if $m \equiv 3 \pmod 4$.

Proof. We need to prove only the case with $m \not\equiv 3 \pmod 4$ since the proof of the case with $m \equiv 3 \pmod 4$ is similar. Clearly $h(X)$ is positive definite, since its discriminant $\Delta_n = na - (n-1)\beta\bar{\beta} > 0$ and clearly all its principal minors are positive.

First, we shall show that nondecomposition of $h(X)$ involving a linear norm exists. By the unimodular transformation

$$X_1 = -Y_1, \quad X_2 = -Y_2, \quad X_j + X_{j+1} = (-1)^{j-1}Y_{j+1} \quad (j = 2, \cdots, n-1),$$

$h(X)$ is replaced by

$$h'(Y) = aY_1\bar{Y}_1 + \beta Y_1\bar{Y}_2 + \bar{\beta}\bar{Y}_1Y_2 + \sum_{j=2}^{n} Y_j\bar{Y}_j + \Big(\sum_{j=2}^{n} Y_j\Big)\Big(\sum_{j=2}^{n} \bar{Y}_j\Big).$$

By Lemma 2.6, it follows that the only norms which need to be considered are

1° $(\alpha_1 Y_1)(\bar{\alpha}_1\bar{Y}_1)$,

2° $(\alpha_1 Y_1 + Y_2)(\bar{\alpha}_1 \bar{Y}_1 + \bar{Y}_2)$,

3° $(\alpha_1 Y_1 + Y_j)(\bar{\alpha}_1 \bar{Y}_1 + \bar{Y}_j)$, $(j = 3, \cdots, n)$,

4° $\Big(\alpha_1 Y_1 + \sum\limits_{j=2}^{n} Y_j\Big)\Big(\bar{\alpha}_1 \bar{Y}_1 + \sum\limits_{j=2}^{n} \bar{Y}_j\Big)$,

where $\alpha_1 \neq 0$, $\alpha_1 \in R_m$.

The case 1° is ruled out, since the determinant of the matrix of the form $h'(Y) - (\alpha_1 Y_1)(\bar{\alpha}_1 \bar{Y}_1)$ is $\Delta_n - n\alpha_1\bar{\alpha}_1 < 0$. For the cases 3° and 4°, we need to consider only the norm $(\alpha_1 Y_1 + Y_3)(\bar{\alpha}_1 \bar{Y}_1 + \bar{Y}_3)$, since $h'(Y)$ is symmetrical in $Y_3, \cdots, Y_n$, and the transformation

$$Y_3 \to -\sum_{k=2}^{n} Y_k, \qquad Y_j \to Y_j \qquad (j = 1, 2, 4, 5, \cdots, n) \tag{1}$$

permutes $Y_3\bar{Y}_3$ and $\Big(\sum\limits_{j=2}^{n} Y_j\Big)\Big(\sum\limits_{j=2}^{n} \bar{Y}_j\Big)$.

Consider first the H-form

$$\begin{aligned} K(Y) &= h'(Y) - (\alpha_1 Y_1 + Y_2)(\bar{\alpha}_1 \bar{Y}_1 + \bar{Y}_2) \\ &= (a - \alpha_1\bar{\alpha}_1) Y_1 \bar{Y}_1 + (\beta - \alpha_1) Y_1 \bar{Y}_2 + (\bar{\beta} - \bar{\alpha}_1) \bar{Y}_1 Y_2 + \sum_{j=3}^{n} Y_j \bar{Y}_j + \Big(\sum_{j=2}^{n} Y_j\Big)\Big(\sum_{j=2}^{n} \bar{Y}_j\Big). \end{aligned}$$

By the transformation

$$Y_2 = -\sum_{j=2}^{n} Z_j, \qquad Y_j = Z_j \qquad (j = 1, 3, 4, \cdots, n),$$

$K(Y)$ is replaced by

$$\begin{aligned} K_1(Z) &= (a - \alpha_1\bar{\alpha}_1) Z_1 \bar{Z}_1 - (\beta - \alpha_1) Z_1 \Big(\sum_{j=2}^{n} \bar{Z}_j\Big) - (\bar{\beta} - \bar{\alpha}_1) \bar{Z}_1 \Big(\sum_{j=2}^{n} Z_j\Big) + \sum_{j=2}^{n} Z_j \bar{Z}_j \\ &= \sum_{j=2}^{n} (Z_j - (\beta - \alpha_1) Z_1)(\bar{Z}_j - (\bar{\beta} - \bar{\alpha}_1)\bar{Z}_1) \\ &\quad + (a - \alpha_1\bar{\alpha}_1 - (n-1)(\beta - \alpha_1)(\bar{\beta} - \bar{\alpha}_1)) Z_1 \bar{Z}_1. \end{aligned}$$

Write $\alpha_1 = a_1 + a_2\sqrt{-m}$ and $\beta = b_1 + b_2\sqrt{-m}$, where a_1, a_2, b_1, b_2 are rational integers. Then the coefficient of $Z_1\bar{Z}_1$ can be expressed as

$$A = a - (a_1^2 + a_2^2 m) - (n-1)((b_1 - a_1)^2 + (b_2 - a_2)^2 m).$$

Consider $A = A(a_1, a_2)$ as a real function of two variables. A has a maximum for $a_1 = \frac{n-1}{n} b_1$, $a_2 = \frac{n-1}{n} b_2$. Since $0 \leq b_j/n < 1\,(j = 1, 2)$, we have for $(a_1, a_2) = (b_1, b_2)$, $(b_1, b_2 - 1)$, $(b_1 - 1, b_2)$ and $(b_1 - 1, b_2 - 1)$, respectively,

$$\begin{aligned} &A = a - (b_1^2 + m b_2^2) = a - \beta\bar{\beta} < 0, \\ &A = a - (b_1^2 + m(b_2 - 1)^2) - (n-1)m = a - \beta\bar{\beta} + m(2b_2 - n) < 0, \\ &A = a - ((b_1 - 1)^2 + m b_2^2) - (n-1) = a - \beta\bar{\beta} + 2b_1 - n < 0, \end{aligned}$$

and

$$\begin{aligned} A &= a - ((b_1 - 1)^2 + m(b_2 - 1)^2) - (n-1)(1 + m) \\ &= a - \beta\bar{\beta} + (2b_1 - n) + m(2b_2 - n) < 0, \end{aligned}$$

so that $K_1(Z)$ is indefinite. This settles the case 2°.

Consider next the H-form

$$\begin{aligned} G(Y) &= h'(Y) - (\alpha_1 Y_1 + Y_3)(\bar{\alpha}_1 \bar{Y}_1 + \bar{Y}_3) \\ &= (a - \alpha_1\bar{\alpha}_1)Y_1\bar{Y}_1 + \beta Y_1\bar{Y}_2 + \bar{\beta}\bar{Y}_1 Y_2 + Y_2\bar{Y}_2 + \sum_{j=4}^{n} Y_j\bar{Y}_j \\ &\quad + \Big(\sum_{j=2}^{n} Y_j\Big)\Big(\sum_{j=2}^{n} \bar{Y}_j\Big) - \alpha_1 Y_1\bar{Y}_3 - \bar{\alpha}_1\bar{Y}_1 Y_3. \end{aligned}$$

By the transformation (1), $G(Y)$ is replaced by

$$\begin{aligned} G_1(Z) &= (a - \alpha_1\bar{\alpha}_1)Z_1\bar{Z}_1 + \beta Z_1\bar{Z}_2 + \bar{\beta}\bar{Z}_1 Z_2 + \alpha_1 Z_1\Big(\sum_{j=2}^{n}\bar{Z}_j\Big) + \bar{\alpha}_1\bar{Z}_1\Big(\sum_{j=2}^{n} Z_j\Big) + \sum_{j=2}^{n} Z_j\bar{Z}_j \\ &= \sum_{j=3}^{n}(Z_j + \alpha_1 Z_1)(\bar{Z}_j + \bar{\alpha}_1\bar{Z}_1) + (Z_2 + (\beta + \alpha_1)Z_1)(\bar{Z}_2 + (\bar{\beta} + \bar{\alpha}_1)\bar{Z}_1) \\ &\quad + (a - \alpha_1\bar{\alpha}_1 - (\beta + \alpha_1)(\bar{\beta} + \bar{\alpha}_1) - (n-2)\alpha_1\bar{\alpha}_1)Z_1\bar{Z}_1. \end{aligned}$$

Write $\alpha_1 = a_1 + a_2\sqrt{-m}$, $\beta = b_1 + b_2\sqrt{-m}$, where a_1, a_2, b_1, b_2 are rational integers. Then the coefficient of $Z_1\bar{Z}_1$ can be expressed as

$$B = a - (a_1^2 + ma_2^2) - ((b_1 + a_1)^2 + m(b_2 + a_2)^2) - (n-2)(a_1^2 + ma_2^2).$$

Consider $B = B(a_1, a_2)$ as a real function of two variables. B has a maximum for $a_1 = -b_1/n$, $a_2 = -b_2/n$. Since $-1 < -b_j/n \le 0$ $(j = 1, 2)$, we have for $(a_1, a_2) = (0,0)$, $(-1,0)$, $(0,-1)$ and $(-1,-1)$, respectively,

$$\begin{aligned} B &= a - (b_1^2 + b_2^2 m) = a - \beta\bar{\beta} < 0, \\ B &= a - 1 - ((b_1 - 1)^2 + mb_2^2 - (n-2) = a - \beta\bar{\beta} + 2b_1 - n < 0, \\ B &= a - m - (b_1^2 + m(b_2 - 1)^2) - m(n-2) = a - \beta\bar{\beta} + m(2b_2 - n) < 0, \end{aligned}$$

and

$$\begin{aligned} B &= a - (1+m) - ((b_1 - 1)^2 + m(b_2 - 1)^2) - (n-2)(1+m) \\ &= a - \beta\bar{\beta} + (2b_1 - n) + m(2b_2 - n) < 0, \end{aligned}$$

so that $G_1(Z)$ is indefinite. Cases 3° and 4° are also settled.

Suppose now that there is a decomposition

$$h(X) = f(X) + g(X),$$

where $f(X)$ and $g(X)$ are non-negative over R_m. No term $X_j\bar{X}_j$ $(j \ge 2)$ can occur in either $f(X)$ or $g(X)$, for then a norm can be taken out of $h(X)$. Hence we can assume that $g(X)$, say, has a term $2X_n\bar{X}_n$. Then $g(X)$ must also contain $X_{n-1}\bar{X}_n + \bar{X}_{n-1}X_n$, for otherwise $f(X)$ assumes negative values by choice of X_n. Then $g(X)$ contains also $2X_{n-1}\bar{X}_{n-1}$, for otherwise $g(X)$ will assume negative values by choice of X_{n-1}. Proceeding in this way, $g(X)$ will contain all the terms of $h(X)$ involving $X_n, X_{n-1}, \cdots, X_2$. Hence $f(X) = aX_1\bar{X}_1$, and so a norm $X_1\bar{X}_1$ can be taken out of $h(X)$, which contradicts what we have proved.

This completes the proof of Lemma 2.7.

§3. Proofs of Main Theorems

Proof of Theorem 1.1. First, consider the case with $m = 1$, i.e., positive definite H-forms over the domain of Gaussian integers $R_1 = Z[i]$, $(i^2 = -1)$. The method given here can be applied to other cases with any R_m.

Proposition 3.1. *The n-ary positive definite R_1-lattice $E_n \simeq \begin{pmatrix} 2 & 1+i \\ 1-i & \Gamma_{n-1} \end{pmatrix}$ of discriminant 2 is indecomposable, where $\Gamma_1 = 2$, $\Gamma_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \cdots, \Gamma_s = \begin{pmatrix} 2 & 1 \\ 1 & \Gamma_{s-1} \end{pmatrix}$.*

Proof. Let $\{z_1, \cdots, z_n\}$ be the basis with respect to which E_n has the above matrix. Suppose, if possible, $E_n = P \perp M$ is an orthogonal splitting of E_n. Since $h(z_j) = 2$ $(1 \le j \le n)$, $\phi(z_j, z_{j+1}) \ne 0$ $(1 \le j \le n-1)$, and $1 \notin h(E_n)$, all $z_1, \cdots, z_n$ belong to the same component P, say, of the splitting. Hence $\operatorname{rank}(P) = n$ and so $E_n = P$ is indecomposable.

Proposition 3.2. *For any natural number $a > 1$, there exist binary Hermitian indecomposable R_1-lattices of discriminant a.*

Proof. By [10, Theorem 4.11], for any not square-free a, there are indecomposable positive definite binary quadratic $\mathbb{Z}$-lattices of discriminant a, except for two exceptions: $a = 4$ and 18. Hence by Lemma 2.2, we need only to show that if $a = 2^e p_1 p_2 \cdots p_s$ with $e = 0, 1$ and odd primes $p_1 < p_2 < \cdots < p_s$, and $a = 4, 18$ there are binary indecomposable positive definite Hermitian R_1-lattices of discriminant a.

For $a = 4$ and 18, we have the indecomposable lattices: $L_4 \simeq \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$ and $L_{18} \simeq \begin{pmatrix} 4 & 1+i \\ 1-i & 5 \end{pmatrix}$ respectively. Next, we consider the general case.

1° If $e = 0$, i.e., $a = p_1 \cdots p_s$ is odd, then the lattice $L_a \simeq \begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2}(a+1) \end{pmatrix}$ is indecomposable. In fact, if $L_a = \langle b \rangle \perp \langle c \rangle$, then $bc = a$, so that b and c must be odd and ≥ 1. This cannot represent 2 unless $b = c = 1$. But $a \ge 3$. This contradiction shows that L_a is indecomposable.

2° If $e = 1$, i.e., $a = 2p_1 \cdots p_s = 2k$, say, with odd k, then $L_a \simeq \begin{pmatrix} 2 & 1+i \\ 1-i & k+1 \end{pmatrix}$ is an indecomposable Hermitian R_1-lattice. In fact, if $L_a = \langle b \rangle \perp \langle c \rangle$, then $bc = a = 2k$, so that at least one of b, c must be odd. But L_a is even. This contradiction shows that L_a is indecomposable.

In the following we need to consider only those cases with (n, a) and $a \ge 3$, $3 \le n \le 9$ listed in Table 2.

1° $n = 3$, $a = 3, 5, 6, 9, 11, 14, 15$.

From the proof of [14, Theorem 1] there are only two classes of ternary positive definite H-forms with discriminant 3 over R_1, which both represent 1, and hence there are no indecomposable R_1-lattices in the case $n = a = 3$.

Clearly each of the following ternary R_1-lattices are indecomposable:

$$L_{3,5} \simeq \begin{pmatrix} 3 & 1+i & \\ 1-i & 2 & 1 \\ & 1 & 2 \end{pmatrix}, \quad L_{3,6} \simeq \begin{pmatrix} 2 & 1+i & \\ 1-i & 3 & 1 \\ & 1 & 2 \end{pmatrix}, \quad L_{3,9} \simeq \begin{pmatrix} 3 & 1+i & \\ 1-i & 2 & 1 \\ & 1 & 3 \end{pmatrix},$$

$$L_{3,11} \simeq \begin{pmatrix} 3 & 1+i & \\ 1-i & 3 & 1 \\ & 1 & 2 \end{pmatrix}, \quad L_{3,14} \simeq \begin{pmatrix} 2 & 1+i & \\ 1-i & 5 & 1 \\ & 1 & 2 \end{pmatrix}.$$

We can prove this by Lemma 2.5 or by direct method. For instance, in Lemma 2.5 we take

$$h_0 \simeq \begin{pmatrix} 2 & 1+i & \\ 1-i & 2 & 1 \\ & 1 & 2 \end{pmatrix},$$

which is indecomposable by Proposition 3.1, and take $h_1 \simeq \mathrm{diag}(1,0,0)$, $\mathrm{diag}(0,1,0)$, $\mathrm{diag}(1,0,1)$, $\mathrm{diag}(1,1,0)$, $\mathrm{diag}(0,3,0)$ respectively. Then each of the lattices $L_{3,a}$ $(a = 5,6,9,11,14)$ are indecomposable. For $n=3$ and $a=15$, we put

$$L_{3,15} \simeq \begin{pmatrix} 3 & 1+i & \\ 1-i & 3 & 1+i \\ & 1-i & 3 \end{pmatrix}.$$

Suppose $L_{3,15} = P \perp M$. Since $h(e_j) = 3$ for $j = 1,2,3$, $\phi(e_j, e_{j+1}) \neq 0$ for $j = 1,2$, and $L_{3,15}$ does not represent 1, (where $\{e_1, e_2, e_3\}$ is the basis with respect to which $L_{3,15}$ has the above matrix), all e_j $(j = 1,2,3)$ fall in the same component P, say, of the splitting. Hence $L_{3,15} = P$ is indecomposable.

2° $n = 4$, $a = 3,6,7,10,14,26$.

For $L_{4,3}$ and $L_{4,7}$ we can take $h_0 \simeq \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1+i & \\ & 1-i & 2 & 1 \\ & & 1 & 2 \end{pmatrix}$, which is indecomposable by [1, Theorem 5], and $h_1 \simeq \mathrm{diag}(0,0,0,1)$ and $\mathrm{diag}(0,0,1,0)$ respectively. By Lemma 2.5, $L_{4,3} \simeq \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1+i & \\ & 1-i & 2 & 1 \\ & & 1 & 3 \end{pmatrix}$ and $L_{4,7} \simeq \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1+i & \\ & 1-i & 3 & 1 \\ & & 1 & 2 \end{pmatrix}$ are both (h_0+h_1)-indecomposable.

For $L_{4,6}$, $L_{4,10}$, $L_{4,14}$ and $L_{4,26}$ we can take $h_0 \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 2 \end{pmatrix}$, which is indecomposable by Proposition 3.1, and $h_1 \simeq \mathrm{diag}(0,0,1,0)$, $\mathrm{diag}(0,0,2,0)$, $\mathrm{diag}(0,2,0,0)$ and $\mathrm{diag}(0,1,1,1)$ respectively. Then by Lemma 2.5,

$$L_{4,6} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & 2 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 2 \end{pmatrix}, \quad L_{4,10} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & 2 & 1 & \\ & 1 & 4 & 1 \\ & & 1 & 2 \end{pmatrix},$$

$$L_{4,14} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & 4 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & 2 \end{pmatrix}, \quad L_{4,26} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & 3 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 3 \end{pmatrix}$$

are all (h_0+h_1)-indecomposable.

3° $n = 5$, $a = 3,5,7,10,13$.

Clearly, $L_{5,3} \simeq \begin{pmatrix} 7 & 2(1+i) \\ 2(1-i) & \Gamma_4 \end{pmatrix}$ is indecomposable by Lemma 2.7, and $L_{5,5} \simeq$

$\begin{pmatrix} \Gamma_2 & 1+i & & \\ 1-i & 2 & 1 & \\ & 1 & 3 & 1+i \\ & & 1-i & 3 \end{pmatrix}$ is indecomposable by the method for $L_{3,15}$. For $L_{5,7}$, $L_{5,10}$ and $L_{5,13}$ we take $h_0 \simeq \begin{pmatrix} 2 & 1+i \\ 1-i & \Gamma_4 \end{pmatrix}$, which is indecomposable by Proposition 3.1, and $h_1 \simeq \mathrm{diag}(1,0,0,0,0)$, $\mathrm{diag}(0,0,0,1,1)$ and $\mathrm{diag}(1,0,0,0,1)$ respectively. By Lemma 2.5, the lattices

$$L_{5,7} \simeq \begin{pmatrix} 3 & 1+i \\ 1-i & \Gamma_4 \end{pmatrix}, \qquad L_{5,10} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & \Gamma_2 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 3 \end{pmatrix},$$

$$L_{5,13} \simeq \begin{pmatrix} 3 & 1+i & \\ 1-i & \Gamma_3 & 1 \\ & 1 & 3 \end{pmatrix}$$

are (h_0+h_1)-indecomposable.

4° $n=6$, $a=5,6,14$.

It is clear that $L_{6,5} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & \Gamma_3 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 3 \end{pmatrix}$ is indecomposable. For $L_{6,6}$ and $L_{6,14}$ we take $h_0 \simeq \begin{pmatrix} 2 & 1+i \\ 1-i & \Gamma_5 \end{pmatrix}$, which is indecomposable by Proposition 3.1, and $h_1 \simeq \mathrm{diag}(0,0,0,0,1,0)$ and $\mathrm{diag}(0,0,0,1,0,1)$ respectively. Then by Lemma 2.5, the lattices

$$L_{6,6} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & \Gamma_3 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 2 \end{pmatrix}, \qquad L_{6,14} \simeq \begin{pmatrix} 2 & 1+i & & & \\ 1-i & \Gamma_2 & 1 & & \\ & 1 & 3 & 1 & \\ & & 1 & 2 & 1 \\ & & & 1 & 3 \end{pmatrix}$$

are (h_0+h_1)-indecomposable.

5° $n=7$, $a=3,7$.

Clearly $L_{7,3} \simeq \begin{pmatrix} 9 & 3+i \\ 3-i & \Gamma_6 \end{pmatrix}$ is indecomposable by Lemma 2.7 and

$$L_{7,7} \simeq \begin{pmatrix} 3 & 1+i & & & \\ 1-i & \Gamma_2 & 1 & & \\ & 1 & 3 & 1 & \\ & & 1 & 2 & 1+i \\ & & & 1-i & \Gamma_2 \end{pmatrix}$$

is indecomposable by the method for $L_{3,15}$.

6° $n=8$, $a=6,10$.

$L_{8,6} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & \Gamma_5 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 2 \end{pmatrix}$ and $L_{8,10} \simeq \begin{pmatrix} 2 & 1+i & & \\ 1-i & \Gamma_5 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 3 \end{pmatrix}$ are indecomposable by the method for $L_{3,15}$, or by Lemma 2.5 with $h_0 \simeq \begin{pmatrix} 2 & 1+i \\ 1-i & \Gamma_7 \end{pmatrix}$.

7° $n = 9$, $a = 3, 5$.

Clearly $L_{9,5} \simeq \begin{pmatrix} 29 & 4(1+i) \\ 4(1-i) & \Gamma_8 \end{pmatrix}$ is indecomposable by Lemma 2.7. Taking $h_0 \simeq \begin{pmatrix} 9 & 3+i \\ 3-i & \Gamma_8 \end{pmatrix}$, which is indecomposable by Lemma 2.7, and $h_1 \simeq \mathrm{diag}(0,0,0,0,0,0,0,0,1)$, we see that, by Lemma 2.5, $L_{9,3} \simeq \begin{pmatrix} 9 & 3+i & \\ 3-i & \Gamma_7 & 1 \\ & 1 & 3 \end{pmatrix}$ is $(h_0 + h_1)$-indecomposable.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Next, consider the case with $m = 2$, i.e., positive definite H-forms over $R_2 = \mathbb{Z}[\sqrt{2}i]$. In view of the proof of Theorem 1.1, we can obtain analogous results for the case with R_2, if we replace the pair $(1+i, 1-i)$ in the proof of Theorem 1.1 by $(\sqrt{2}i, -\sqrt{2}i)$ and make some supplement. For instance, by changing $1+i$ (resp. $1-i$) into $\sqrt{2}i$ (resp. $-\sqrt{2}i$), $E_n \simeq \begin{pmatrix} 2 & 1+i \\ 1-i & \Gamma_{n-1} \end{pmatrix}$ reduces to $E'_n \simeq \begin{pmatrix} 2 & \sqrt{2}i \\ -\sqrt{2}i & \Gamma_{n-1} \end{pmatrix}$ and Proposition 3.1 becomes a proposition for R_2-lattices. Moreover, the R_2-lattices $M_{9,3} \simeq \begin{pmatrix} 11 & 2+2\sqrt{2}i \\ 2-2\sqrt{2}i & \Gamma_8 \end{pmatrix}$ and $M_{7,3} \simeq \begin{pmatrix} 15 & 3+2\sqrt{2}i \\ 3-2\sqrt{2}i & \Gamma_6 \end{pmatrix}$ are indecomposable by Lemma 2.7, and $M_{3,3} \simeq \begin{pmatrix} 2 & 1 & \\ 1 & 2 & 1+\sqrt{2}i \\ & 1-\sqrt{2}i & 3 \end{pmatrix}$ is indecomposable by the method of proof for $L_{3,15}$.

This completes the proof of Theorem 1.2.

Remark 3.1. 1. By the same argument as in the proof of Theorem 1.2, we can show that, for any natural numbers n and a, there are n-ary indecomposable positive definite Hermitian lattices over R_7 $(= \mathbb{Z}[\theta]$ with $\theta = \frac{1}{2}(1+\sqrt{7}i))$ with discriminant a, except for one exception: $n = 2$ and $a = 1$. In this exceptional case, there are no lattices with the desired properties. In fact, in the proof of Theorem 1.1, we need only to replace $1+i$ and $1-i$ by θ and $\bar{\theta}$ respectively, and then analogous results on $R_7 = \mathbb{Z}[\theta]$ are obtained. Moreover, we need only to supply the following facts:

1° Clearly the R_7-lattice $N_{3,3} \simeq \begin{pmatrix} 3 & 1+\theta & \\ 1+\bar{\theta} & 2 & 1 \\ & 1 & 3 \end{pmatrix}$ is indecomposable.

2° By Lemma 2.7, $N_{7,1} \simeq \begin{pmatrix} 7 & 2+\theta \\ 2+\bar{\theta} & \Gamma_6 \end{pmatrix}$ and $N_{9,3} \simeq \begin{pmatrix} 11 & 3+2\theta & \\ 3+2\bar{\theta} & 3 & 1 \\ & & \Gamma_7 \end{pmatrix}$ are indecomposable. Then take $h_0 \simeq N_{7,1}$ and $h_1 \simeq \mathrm{diag}(0, \cdots, 0, 1)$. By Lemma 2.5, the R_7-lattice $N_{7,3} \simeq \begin{pmatrix} 7 & 2+\theta & \\ 2+\bar{\theta} & \Gamma_5 & 1 \\ & 1 & 3 \end{pmatrix}$ is $(h_0 + h_1)$-indecomposable.

This completes the proof of our claim.

2. By establishing an algebraic criterion for the indecomposability of positive definite Hermitian R_m-lattices, the author proves recently analogy as Theorem 1.1 for the case $R_3 = \mathbb{Z}[\theta_3]$ with $\theta_3 = \frac{1}{2}(1+\sqrt{-3})$, except for ten exceptions: $a = 1$, $n = 2, 3, 4, 5, 7$; $a = 2$,

$n = 2, 3$; $a = 4$, $n = 2$; $a = 5$, $n = 3$; $a = 10$, $n = 2$. This result will appear elsewhere[15].

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