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THE SPECTRAL THEORY OF CONTRACTIONS ON PONTRYAGIN SPACES $\prod_K$ (II)**

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Abstract

This is the continuation of [8]. The main purpose of this paper is to give both general form of any unitary extension and unitary dilation of a contraction of Π_K associated with its triangle model.

Keywords Pontryagin space, Contraction, Unitary dilation, Unitary extension, Triangle model.

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§1. Preliminary

Sz. Nagy and C. Foias established the harmonic analysis theory of contractions of Hilbert spaces[7]. We hope to find the relevent theory on spaces with indefinite metrics, specially on Pontryagin space Π_K. So it is very important to research the general form of unitary dilation of contraction on a space with an indefinite metric. Davis, ch.[6] gave the proof of the existence of J-unitary dilation of any bounded operator in a Hilbert spaces. Yang Shaozong[2] gave the general forms of J-unitary dilations. We also see [3, 9] in this way. All these research works have been done in the framework of regular decomposition in a Krein space and can not show the property of degenerate part of contraction. Yan Shaozong, Chen Xiaoman[8] founded the triangle model of contraction on Pontryagin space Π_K. It is an important role in the harmonic analysis theory of contraction on Pontryagin space Π_K. We characterize the general form of unitary dilation of contraction on Pontryagin space Π_K and any unitary dilation of contraction on Pontryagin space Π_K relative to its triangle model. Moreover in the third section we also get a theorem which perfectly depicts a characterization of minimal dilation, that is, the unitary part in a minimal unitary dilation associated with the nondegenerate part of the triangle model of a contraction on Π_K becomes a minimal unitary dilation of the nongenerate part.

The results and symbols used in our paper come from Yan Shaozong[1] without explanation. As this is a continued paper of [9], we recall some theorems for convenience sake.

At first we give some definitions in the following.

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Definition 1.1. *If T is a contraction on Π_K and U is a unitary operator from $\Pi_K \oplus H_1$ onto $\Pi_K \oplus H_2$ such that*

$$P_{\Pi_K} U|_{\Pi_K} = T, \tag{1.1}$$

where H_1 and H_2 are two Hilbert spaces and P_{Π_K} is an orthogonal projection from $\Pi_K \oplus H_2$ onto Π_K, then we call the unitary operator U a unitary extension of T.

Obviously the unitary extension of contraction on Π_K is not unique.

Definition 1.2. *If T is a contraction on Π_K and U is a unitary operator on $\Pi_K \oplus H$ such that*

$$P_{\Pi_K} U^n|_{\Pi_K} = T^n \text{ for } n = 1, 2, \cdots, \tag{1.2}$$

then U is called a unitary dilation of T. Moreover, if $\Pi_K \oplus H =$ C.L.S.$\{(U-z)^{-1}\Pi_K | z \in \rho(U)\}$, where C.L.S. means closure of linear span, then U is called a minimal unitary of T.

Definition 1.3. *If T is a contraction on Pontryagin space Π_K and there exists a standard decomposition $\Pi_K = N \oplus \{Z \dot{+} Z^*\} P$ such that*

$$T = \begin{bmatrix} S & F & G & B \\ & T_N & T_1 & C \\ & & T_P & D \\ & & & S^{*-1} \end{bmatrix} \begin{matrix} Z \\ N \\ P \\ Z^* \end{matrix}, \tag{1.3}$$

where S is injective on Z and $\begin{bmatrix} T_N & T_1 \\ & T_P \end{bmatrix}$ is a contraction of $N \oplus P$, then (1.3) is called a triangle model of T, denoted by $T = \{S, T_N, T_P, T_1, C, D, F, G, B\}$. $\begin{bmatrix} T_N & T_1 \\ & T_P \end{bmatrix}$ is called the nondegenerate part of T, denoted by $T_0 = \begin{bmatrix} T_N & T_1 \\ & T_P \end{bmatrix}$.

The following Theorem 1.1 and Theorem 1.2 can be found in [1].

Theorem 1.1. *Let $\Pi_{K_0} = N \oplus P$ be a regular decomposition such that*

$$T_0 = \begin{bmatrix} T_N & T_1 \\ & T_P \end{bmatrix} \tag{1.4}$$

and $T_0 N = N$. Let $T_N = VR$ be a polar decomposition. Then T_0 must have a unitary extension and the general form is the following formula

$$U = \begin{matrix} N \\ P \\ H_2 \end{matrix} \begin{bmatrix} T_N & T_1 & E' \\ & T_P & H' \\ X' & Q' & L' \end{bmatrix} \begin{matrix} N \\ P \\ H_1 \end{matrix} \tag{1.5}$$

and the general solutions of E', H', L', Q', X' are the following formulae

$$E' = -T_1 T_P^* (I - T_P T_P^*)^{-1/2} I_2^* + V(R^2 - I)^{1/2} (I - K_0 K_0^*)^{1/2} V_1^*, \tag{1.6}$$

$$H' = (I - T_P T_P^*)^{1/2} U_2^*, \tag{1.7}$$

$$L' = -[U_4 R K_0 + V_0 (I - K_0^* K_0)^{1/2}] T_P^* U_2^* + [U_4 R (I - K_0 K_0^*)^{1/2} - V_0 K_0^*] V_1^* + W, \tag{1.8}$$

$$Q' = U_4 (R^2 - I)^{-1/2} R V^* T_1 + V_5 (I - K_0^* K_0)^{1/2} (I - T_P^* T_P)^{1/2}, \tag{1.9}$$

$$X' = U_4 (R^2 - I)^{1/2} \tag{1.10}$$

and two Hilbert spaces H_1 and H_2 have orthogonal decompositions

$$H_1 = (\mathcal{R}(U_2) \oplus \mathcal{R}(V_1)) \oplus (\mathcal{R}(U_2) \oplus \mathcal{R}(V_1))^{\perp}, \tag{1.11}$$

$$H_2 = (\mathcal{R}(U_4) \oplus \mathcal{R}(V_5)) \oplus (\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))^{\perp}, \tag{1.12}$$

where $K_0 = (R^2-I)^{-1/2}V^*T_1(I-T_P^*T_P)^{-1/2}$ is a contraction from $(\overline{(I-T_P^*T_P)^{1/2}P}, (\cdot,\cdot))$ to $((R^2-I)^{1/2}N, -(\cdot,\cdot))$ and U_4, U_2, V_5, V_1 and W are metric-preserving bijections

$U_4 : (\overline{\mathcal{R}(R^2-I), -(\cdot,\cdot)}) \to (\mathcal{R}(U_4), [\cdot,\cdot]_2)$,

$U_2 : (\overline{\mathcal{R}((I-T_PT_P^*)^{1/2}, (\cdot,\cdot)}) \to (\mathcal{R}(U_2), [\cdot,\cdot]_1)$,

$V_5 : (\overline{\mathcal{R}((I-K_0^*K_0)^{1/2}(I-T_P^*T_P)^{1/2})}, (\cdot,\cdot)) \to (\mathcal{R}(U_5), [\cdot,\cdot]_2)$,

$V_1 : (\mathcal{R}((I-K_0K_0^*)^{1/2}(I-K_0K_0^*)^{1/2}), (\cdot,\cdot)) \to (\mathcal{R}(V_1), [\cdot, cdot]_1)$,

$W : (\mathcal{R}(U_2) \oplus \mathcal{R}(V_1))^{\perp}, [\cdot,\cdot]_1) \to ((\mathcal{R}(U_4) \oplus \mathcal{R}(5))^{\perp}, [\cdot,\cdot]_2)$.

Theorem 1.2. *Let T_0 be as above. Then the general form of unitary dilation of T_0 is the following*

$$U_0 = \begin{matrix} N \\ P \\ H \end{matrix} \begin{bmatrix} T_N & T_1 & E' \\ & T_P & H' \\ X' & Q' & L' \end{bmatrix} \begin{matrix} N \\ P \\ H \end{matrix} \tag{1.13}$$

where general solutions of E', H', L', Q' and X' are as in Theorem 1.1 and $H_1 = H_2 = H$ must be the following general form

$$\begin{aligned} H = & \left[\overset{\infty}{\underset{K=1}{\oplus}} W^{*K}(\mathcal{R}(U_2)) \oplus \mathcal{R}(V_1)) \right] \oplus (\mathcal{R}(U_2) \oplus \mathcal{R}(U_1)) \oplus (\mathcal{R}(U_4) \oplus \mathcal{R}(V_5)) \\ & \left[\overset{\infty}{\underset{K=1}{\oplus}} W^{K}(\mathcal{R}(U_4)) \oplus \mathcal{R}(V_5)) \right] \oplus H_0 \end{aligned} \tag{1.14}$$

and restriction W_0 of W on H_0 is a unitary operator on H_0.

The following theorem is the result of [9]. We refine this result.

Theorem 1.3. *If T is a contraction on Pontryagin space Π_K, then there exists a standard decomposition $\Pi_K = N \oplus \{Z \dot{+} Z^*\} \oplus P$ such that $T = \{S, T_N, T_P, T_1, C, D, F, G\}$ is a triangle model of T_1, and we have*

(1) *S is injective on Z, $T_N = VR$ is a polar decomposion on T_N $R \geq I$, V is injective on N, T_P is a contraction of $(P, (\cdot,\cdot))$ and $B = (1/2)S(C^*C - D^*D - Q)$, where $\mathrm{Re}Q = (1/2)(Q+Q^*) > 0$.*

(2) *$K_0 = (R^2-I)^{-1/2}V^*T_1(I-T_P^*T_P)^{-1/2}$ is a contractive operator from Hilbert space $(\overline{\mathcal{R}((I-T_P^*T_P)^{1/2})}, (\cdot,\cdot))$ to Hilbert space $(N, -(\cdot,\cdot))$.*

(3) *$K_1 = (\mathrm{Re}Q)^{-1/2}(S^{-1}F - C^*T_N)(R^2-I)^{-1/2}$ is a contractive operator from Hilbert space $(\mathcal{R}((R^2-I)^{1/2}), -(\cdot,\cdot))$ to Hilbert space $(Z, \langle\cdot,\cdot\rangle)$.*

(4) *$K_2 = (\mathrm{Re}Q)^{-1/2}(S^{-1}F - C^*T_N)R(R^2-I)^{-1/2}K_0(I-K_0^*K_0)^{-1/2} + (\mathrm{Re}Q)^{-1/2}(S^{-1}G - C^*T_1 + D^*T_P)(I-T_P^*T_p)^{-1/2}(I-K_0^*K_0)^{-1/2}$ is a contractive operator from Hilbert space $(\overline{\mathcal{R}((I-K_0^*K_0)^{1/2}(I-T_P^*T_P)^{1/2}}, (\cdot,\cdot))$ to Hilbert space $(Z, \langle\cdot,\cdot\rangle)$.*

(5) *Let $K(n+p) = K_1n + K_2p$, where*

$$n \in \mathcal{R}((R^2-I)^{1/2}), -(\cdot,\cdot)),\ p \in \overline{(\mathcal{R}((I-K_0^*K_0)^{1/2}(I-T_P^*T_P)^{1/2})}, (\cdot,\cdot)).$$

Then K is a contractive operator from

$$(\mathcal{R}((R^2-I)^{1/2}) \oplus \overline{\mathcal{R}((I-K_0^*K_0)^{1/2}(I-T_P^*T_P)^{1/2})}, -(\cdot,\cdot)_N \oplus (\cdot,\cdot)_P) \text{ to } (Z, \langle\cdot,\cdot\rangle).$$

Moreover if (1), (2), (3), (4) *and* (5) *are satisfied, then* T *is a contraction on* Π_K.

Proof. The proof of Theorem 1.3 is finished if we prove (3), (4) and (5). Obviously, (iv) in Theorem 2.2 of [8] is written as

$$\mathcal{R}(S^{-1}F - C^*T_N) \cup \mathcal{R}(S^{-1}G - C^*T_1 + D^*T_P) \subset (\operatorname{Ker}\operatorname{Re}Q)^{\perp},$$

and for any $n \in N$, $p \in P$,

$$\begin{aligned}&\|(\operatorname{Re}Q)^{-1/2}[S^{-1}F - C^*T_N)n + (S^{-1}G - C^*T_1 + D^*T_P)p]\|^2_{\langle\cdot,\cdot\rangle}\\ &- \|(I - T_P^*T)^{1/2}\|^2 - \|(R^2 - I)^{1/2}n + R(R^2 - I)^{-1/2}V^*Y_1P\|^2\\ &+ \|(R^2 - I)^{-1/2}V^*T_1p\|^2 \le 0,\end{aligned} \tag{1.15}$$

where $\|\cdot\|_{\langle\cdot,\cdot\rangle}$ is the norm induced by $\langle\cdot,\cdot\rangle$. Let $p = 0$ in (1.15). Then

$$\|(\operatorname{Re}Q)^{-1/2}(S^{-1}F - C^*T_N)n\|^2 \le \| * R^2 - I)^{1/2}n\|^2.$$

As $\operatorname{Ker}((R^2 - I)^{1/2} \subset \operatorname{Ker}(\operatorname{Re}Q)^{-1/2}(S^{-1}F - C^*T_N)$ we can define $(\operatorname{Re}Q)^{-1/2}(S^{-1}F - C^*T_N)(R^2 - I)^{-1/2}$ as a contractive operator from $(\mathcal{R}((R^2 - I)^{-1/2}), -(\cdot,\cdot))$ to $(Z, \langle\cdot,\cdot\rangle)$. Then (3) is proved. Let $n = -R(R^2 - I)^{-1}V^*T_1$. Then (1.15) becomes

$$\begin{aligned}&\|(\operatorname{Re}Q)^{-1/2}[-S^{-1}F - C^*T_N)R * R^2 - I)^{-1}V^*T_1P + (S^{-1}G - C^*T_1 + D^*T_P)p]\|^2_{<\cdot,\cdot>}\\ &\le \|(I - T_P^*T_P)^{1/2}P\|^2 - \|K_0(I - T_P^*T_P)^{1/2}P\|^2\\ &= \|(I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2}P\|^2\end{aligned}$$

for any $p \in P$. We have

$$\begin{aligned}&\operatorname{Ker}((I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2}\\ &\subset \operatorname{Ker}(\operatorname{Re}Q)^{-1/2}[-(S^{-1}F - C^*T_N)R(R^2 - I)^{-1}V^*T_1 + (S^{-1}G - C^*T_1 + D^*T_P)]\end{aligned}$$

and define

$$\begin{aligned}&-(\operatorname{Re}Q)^{-1/2}(S^{-1}F - C^*T_N)R(R^2 - I)^{-1}V^*T_1(I - T_P^*T_P)^{-1/2}(I - K_0^*K_0)^{-1/2}\\ &+ (\operatorname{Re}Q)^{-1/2}(S^{-1}G - C^*T_1 + D^*T_P)(I - T_P^*T_P)^{-1/2}(I - K_0^*K_0)^{-1/2}\end{aligned}$$

as a contractive operator from the Hilbert space

$$\overline{(\mathcal{R}((I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2})}, (\cdot,\cdot)) \text{ to } (Z, \langle\cdot,\cdot\rangle).$$

By reorganizing the contractive operator, (4) is proved.

The left side of the inequality (1.15)

$$\begin{aligned}&= \|(\operatorname{Re}Q)^{-1/2}(S^{-1}F - C^*T_N)(R^2 - I)^{-1/2}((R^2 - I)^{1/2}n + R(R^2 - I)^{-1/2}V^*T_1p)\\ &(\operatorname{Re}Q)^{-1/2}[-(S^{-1}F - C^*T_N)R(R^2 - I)^{-1}V^*T_1p + (S^{-1}G - C^*T_1 + D^*T_P)p]\|^2_{\langle\cdot,\cdot\rangle}\\ &- \|(R^2 - I)n + R(R^2 - I)^{-1/2}V^*T_P\|^2 + \|(R^2 - I)^{-1/2}V^*T_1p\|^2 - \|(I - T_P^*T_P)^{1/2}\|^2\\ &= \|K_1((R^2 - I)^{1/2}n + R(R^2 - I)^{-1/2}V^*T_1p) + K_2(I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2}p\|^2_{\langle\cdot,\cdot\rangle}\\ &- \|(R^2 - I)^{1/2}n + R(R^2 - I)^{1/2}V^*T_1p\|^2 - \|(I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2}p\|^2\\ &\le 0.\end{aligned}$$

Let

$$K_3 = \begin{bmatrix} (R^2 - I)^{1/2} & R(R^2 - I)^{-1/2}V^*T_1 \\ & (I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2} \end{bmatrix} \begin{matrix} N \\ P \end{matrix},$$

$$\overline{K_3(N \oplus P)} = \mathcal{R}((R^2 - I)^{1/2} \oplus \overline{\mathcal{R}((I - K_0^* K_0)^{1/2}(I - T_P^* T_P)^{1/2})}.$$

Then K is a contractive operator from

$$(\mathcal{R}((R^2 - I)^{1/2} \oplus \overline{\mathcal{R}((I - K_0^* K_0)^{1/2}(I - T_P^* T_P)^{1/2})}, (\cdot, \cdot)_N \oplus (\cdot, \cdot)_P) \text{ to } (Z, \langle \cdot, \cdot \rangle),$$

which is equivalent to (1.15). The proof is finished.

In the following we give a lemma, which is used to compute general solutions in the second section.

Lemma 1.1. *Let T_1 be a linear operator from Hilbert space H_1 to Hilbert space H_2 and T_2 be a linear operator from H_1 to Hilbert space H_2. And let $\|T_1 x\| \le \|T_2 x\|$ for any $x \in H_1$. Then the general solution of operator-valued equation $T_1 = V^* T_2$ where V is an isometric operator from $\overline{\mathcal{R}(T_1)}$ to $H (\supset \overline{\mathcal{R}(T_2)})$ is as follows: $H = \overline{\mathcal{R}(T_2)} \oplus \overline{\mathcal{R}(T_2)}^{\perp}$ and $Vx = T^* x + V_0 (I - TT^*)^{1/2} x$, where $Ty = T_1 T_2^{-1} y$ for any $x \in H$ and $y \in \mathcal{R}(T_2)$, T is a contractive operator which is uniquely extented on $\overline{\mathcal{R}(T_2)}$ and V_0 is a metric-preserving operator from $\overline{\mathcal{R}((I - TT^*)^{1/2})}$ to $\overline{\mathcal{R}(T_2)}$.*

Proof. As $\|T_1 x\| \le \|T_2 x\|$ for any $x \in H_1$, the operator equation $T_1 = V^* T_2$ is equivalent to $T_1 T_2^{-1} = V | \mathcal{R}(T_2)$. And as T is a unique extension on $\overline{\mathcal{R}(T_2)}$ of $T_1 T_2^{-1}$, the operator equation is equivalent to $T = V^* | \overline{\mathcal{R}(T_2)}$. It is equivalent to $T^* = P_{\overline{\mathcal{R}(T_2)}} V$. Let V be a metric-preserving operator from $\overline{\mathcal{R}(T_1)}$ to $\overline{\mathcal{R}(T_2)} \oplus \overline{\mathcal{R}(T_2)}^{\perp}$ and $V = A + B$ where A and B are bounded operators from $\overline{\mathcal{R}(T_1)}$ to $\overline{\mathcal{R}(T_2)}$ and $\overline{\mathcal{R}(T_2)}^{\perp}$ respectively. $T^* = P_{\overline{\mathcal{R}(T_2)}} V$ is equivalent to $A = T^*$. So $\|(I - TT^*)^{1/2} x\| = \|Bx\|$ for $x \in \overline{\mathcal{R}(T_1)}$. Then $B = V_0 (I - TT^*)^{1/2}$ and V_0 is a metric-preserving operator from $\overline{\mathcal{R}((I - TT^*)^{1/2}}$ into $\overline{\mathcal{R}(T_2)}^{\perp}$. The above process is invertible, and then the lemma is proved.

§2. The General Form of Unitary Extension of Contraction on Π_K Associated With Its Triangle Model

In this section we only consider unitary extension of contraction on Π_K associated with its triangle model.

Theorem 2.1. *Let T be a contraction on Π_K and its triangle model as in* (1.3). *Then the general form of its unitary extension (U, H_1, H_2) is*

$$U = \begin{matrix} Z \\ N \\ P \\ Z^* \\ H_2 \end{matrix} \begin{bmatrix} S & F & G & B & A' \\ & T_N & T_1 & C & E' \\ & & T_P & D & H' \\ & & & S^{*-1} & 0 \\ 0 & X' & Q' & M' & L' \end{bmatrix} \begin{matrix} Z \\ N \\ P \\ Z^* \\ H_1 \end{matrix} \tag{2.1}$$

where

$$U_0 = \begin{matrix} N \\ P \\ H_2 \end{matrix} \begin{bmatrix} T_N & T_1 & E' \\ & T_P & H' \\ X' & Q' & L' \end{bmatrix} \begin{matrix} N \\ P \\ H_1 \end{matrix} \tag{2.2}$$

and U_0 is a unitary extension of $T_0 = \begin{bmatrix} T_N & T_1 \\ & T_P \end{bmatrix}$. And so the general solutions of E', H', L', N' and X' are (1.6)-(1.10) *of Theorem* 1.1. *Moreover,*

$$M' = V_0 (\mathrm{Re} Q)^{1/2}, \tag{2.3}$$

$$
\begin{aligned}
A' = {} & SC^*[-T_1T_P^*(I-T_PT_P^*)^{-1/2}U_2^* + V(R^2-I)(I-K_0K_0^*)^{1/2}V_1^* \\
& - SD^*(I-T_PT_P^*)^{1/2}U_2^* \\
& + S(\mathrm{Re}Q)^{1/2}V_6^*[U_4RK_0 + V_5(I-K_0^*K_0)^{1/2}]T_P^*U_2^* \\
& - S(\mathrm{Re}Q)^{1/2}V_6^*[U_4R(I-K_0K_0^*)^{1/2} - V_5K_0^*]V_1^* \\
& - S(\mathrm{Re}Q)^{1/2}V_6^*W, \qquad (2.4)
\end{aligned}
$$

where V_1, U_2, U_4, V_5, W are as in Theorem 1.1, *and V_6 satisfies the following form*

$$V_6 = V_6^{(0)} + V_6^{(1)}, \qquad (2.5)$$

$$V_6^{(0)}x = K'x + V_0(I-K'K'^*)^{1/2}x \quad \text{for any } x \in \mathcal{R}(K) \qquad (2.6)$$

where $V_6^{(1)}$ is a metric-preserving operator from $\mathcal{R}(\mathrm{Re}Q) \ominus \mathcal{R}(K)$ to $(\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))^\perp \cap \overline{(V_6^{(0)})^\perp}$, $K' = KU_4^* + KV_5^*$, $K(n+p) = K_1n + K_2p$ for any $n \in \mathcal{R}((R^2-I)^{1/2})$, $p \in \overline{\mathcal{R}((I-T_P^*T_P)^{1/2})}$ and V_0 is a metric-preserving operator from $\mathcal{R}(I-K_0K_0^*)^{1/2}$ to $(\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))^\perp$.

Proof. Let unitary extension of T be

$$
U = \begin{matrix} Z \\ N \\ P \\ Z^* \\ H_2 \end{matrix}
\begin{bmatrix}
S & F & G & B & A' \\
 & T_N & T_1 & C & E' \\
 & & T_P & D & H' \\
 & & & S^{*-1} & J' \\
Y' & X' & Q' & M' & L'
\end{bmatrix}
\begin{matrix} Z \\ N \\ P \\ Z^* \\ H_1 \end{matrix}, \qquad (2.7)
$$

$$
U^\dagger = \begin{matrix} Z \\ N \\ P \\ Z^* \\ H_1 \end{matrix}
\begin{bmatrix}
S^{-1} & -C^* & D^* & B^* & M'^* \\
 & T_N^* & 0 & -F^* & -X^* \\
 & -T_1^* & T_P^* & G^* & Q'^* \\
 & & & S^* & Y'^* \\
J'^* & -E'^* & H'^* & A'^* & L'^*
\end{bmatrix}
\begin{matrix} Z \\ N \\ P \\ Z^* \\ H_2 \end{matrix}. \qquad (2.8)
$$

The equation $U^\dagger U = I$ is equivalent to the following five series of operator equations:

$$
(1) \quad \begin{cases}
I + M'^*Y' = I, \\
-X'^*Y' = 0, \\
Q'^*Y' = 0, \\
Y'^*Y' = 0, \\
L'^*Y' = 0;
\end{cases}
$$

$$
(2) \quad \begin{cases}
S^{-1}F - C^*T_N + M'^*X' = 0, \\
T_N^*T_N - X'^*X' = I, \\
-T_1^*T_N + Q'^*X' = 0, \\
Y'^*X' = 0, \\
J'^*F - E'^*T_N + L'^*X' = 0;
\end{cases}
$$

$$
(3) \quad \begin{cases}
S^{-1}G - C^*T_1 + D^*T_P + M'^*Q' = 0, \\
T_N^*T_1 - X'^*Q' = 0, \\
-T_1^*T_1 + T_P^*T_P + Q'^*Q' = I, \\
Y'^*Q' = 0, \\
J'^*G - E'^*T_1 + H'^*T_P + L'^*Q' = 0;
\end{cases}
$$

$$(4)\quad \begin{cases} S^{-1}B - C^*C + D^*D + B^*S^{-1} + M'^*M' = 0, \\ T_N^*C - F^*S^{*-1} - X'^*M' = 0, \\ -T_1^*C + T_P^*D + G^*S^{*-1} + Q'^*M' = 0, \\ I + Y'^*M' = I, \\ J'^*B - E'^*C + H'^*D + A'^*S^{*-1} + L'^*M' = 0; \end{cases}$$

$$(5)\quad \begin{cases} S^{-1}A' - C^*E' + D^*H' + B^*J' + M'^*L' = 0, \\ T_N^*E' - F^*J' - X'^*L' = 0, \\ -T_1^*E' + T_P^*H' + G^*J' + Q'^*L' = 0, \\ S^*J' + Y'^*L' = 0, \\ J'^*A' - E'^*E' + H'^*H' + A'^*J' + L'^*L' = I. \end{cases}$$

And $UU^\dagger = I$ is equivalent to the following five series of operator equations:

$$(1')\quad \begin{cases} I + A'J'^* = I, \\ E'J'^* = 0, \\ H'J'^* = 0, \\ J'J'^* = 0, \\ L'J'^* = 0; \end{cases}$$

$$(2')\quad \begin{cases} -SC^* + FT_N^* - GT_1^* - A'E'^* = 0, \\ T_N^*T_N^* - T_1T_1^* - E'E'^* = I, \\ -T_PT_1^* - H'E'^* = 0, \\ -J'E'^* = 0, \\ -Y'C^* + X'T_N^* - Q'T_1^* - L'E'^* = 0; \end{cases}$$

$$(3')\quad \begin{cases} SD^* + GT_P^* + A'H'^* = 0, \\ T_1T_P^* + E'H'^* = 0, \\ T_PT_P^* + H'H'^* = I, \\ J'H'^* = 0, \\ Y'D^* + Q'T_P^* + L'H'^* = 0; \end{cases}$$

$$(4')\quad \begin{cases} SB^* - FF^* + GG^* + BS^* + A'A'^* = 0, \\ -T_NF^* + T_1G^* + CS^* + E'A'^* = 0, \\ T_PG^* + DS^* + H'A'^* = 0, \\ I + J'A'^* = I, \\ Y'B^* - X'F^* + Q'G^* + M'S^* + L'A'^* = 0; \end{cases}$$

$$(5')\quad \begin{cases} SM'^* - FX'^* + GQ'^* + BY'^* + A'L'^* = 0, \\ -T_NX'^* + T_1Q'^* + CY'^* + E'L'^* = 0, \\ T_PQ'^* + DY'^* + H'L'^* = 0, \\ S^{*-1}Y'^* + J'L'^* = 0, \\ Y'M'^* - X'X'^* + Q'Q'^* + M'Y'^* + L'L'^* = I. \end{cases}$$

From above ten series of operator equations we have

$$Y' = 0, \tag{2.9}$$

$$J' = 0. \tag{2.10}$$

And by observing and arranging these equations we get two series of twenty equations at last:

$$
\text{(L)}\quad \begin{cases} T_N^* T_N - X^* X = I, \\ T_P^* T_P - T_1^* T_1 + Q'^* Q' = I, \\ -E'^* E' + H'^* H' + L'^* L' = I, \\ T_N^* T_1 - X'^* Q' = 0, \\ T_N^* E' - X'^* L' = 0, \\ T_P^* H' - T_1^* E' + Q'^* L' = 0, \\ S^{-1} F - C^* T_N + M'^* X' = 0, \\ S^{-1} G - C^* T_1 + D^* T_P + M'^* Q' = 0, \\ S^{-1} B - C^* C + D^* D + B^* S^{*-1} + M'^* M' = 0, \\ S^{-1} A' - C^* E' + D^* H' + M'^* L' = 0; \end{cases} \qquad (2.11\text{-}2.20)
$$

$$
\text{(R)}\quad \begin{cases} T_N T_N^* - T_1 T_1^* - E' E'^* = I, \\ T_P T_P^* + H' H'^* = I, \\ -X' X'^* + Q' Q'^* + L' L'^* = I, \\ T_1 T_P^* + E' H'^* = 0, \\ T_N X'^* - T_1 Q'^* - E' L' = 0, \\ T_P Q'^* + H' L'^* = 0, \\ SC^* - F T_N^* + G T_1^* + A' E'^* = 0, \\ SD^* + G T_P^* + A' H'^* = 0, \\ SB^* - FF^* + GG^* + BS^* + A' A'^* = 0, \\ SM'^* - F X'^* + G Q'^* + A' L'^* = 0. \end{cases} \qquad (2.11'\text{-}1.20')
$$

The equations (L) are equivalent to the equations (1)-(5) and (R) are equivalent to (1')-(5'). (2.11)-(2.16) and (2.11')-(2.16') are equivalent to $U_0^\dagger U_0 = I$ and $U_0 U_0^\dagger = I$ (see 2.2). Then the general solutions of E', H', L', Q' and X' are the same as in Theorem 1.1. We only find the general solutions of M', A' satisfying (2.17)-(2.20) and (2.17')-(2.20').

In the following we reason (2.17')-(2.20') from (2.11)-(2.20) and (2.11')-(2.16'). From (2.17)-(2.19) we have

$$F = SC^* T_N - SM'^* X', \tag{2.21}$$

$$G = SC^* T_1 - SD^* T_P - SM'^* Q', \tag{2.22}$$

$$A' = SC^* E' - SD^* H' - SM'^* L'. \tag{2.23}$$

We substitute (2.21)-(2.23) in the left sides of (2.17')-(2.20').

$$
\begin{aligned}
&\text{The left side of } (2.17') \\
&= SC^* - (SC^* T_N - SM'^* X') T_N^* + (SC^* T_1 - SD^* T_P - SM'^* Q') T_1^* \\
&\quad + (SC^* E' - SD^* H' - SM'^* L') E'^* \\
&= SC^*(I - T_N T_N^* + T_1 T_1^* + E' E'^*) - SM^*(-X' T_N^* + Q' T_1^* + L' E'^*) \\
&\quad - SD^*(T_P T_1^* + H' E'^*) \\
&= 0
\end{aligned}
$$

(reasoned from (2.11'), (2.14') and (2.15')).

The left side of $(2.18')$

$$= SD^* + (SC^*T_1 - SD^*T_P - SM'^*Q')T_P^* + (SC^*E' - SD^*H' - SM'^*L')H'^*$$
$$= SD^*(I - T_PT_P^* - H'H'^*) + SC^*(T_1T_P^* + E'H'^*) - SM'^*(Q'T_P^* + L'H'^*)$$
$$= 0$$

(reasoned from (2.12'), (2.14') and (2.16')).

The left side of $(2.19')$

$$= SB^* + BS^* - (SC^*T_N - SM'^*X')(T_N^*CS^* - X'^*M'S^*)$$
$$+ (SC^*T_1 - SD^*T_P - SM'^*Q\prime)(T_1^*CS - T_P^*DS^* - Q'^*M'S^*)$$
$$+ (SC^*E' - SD^*H' - SM'^*L')(E'^*CS^* - H'^*DS^* - L'^*M'S^*)$$
$$= S[S^{-1}B + B^*S^{*-1} + D^*(T_PT_P^* + H'H'^*)D - C^*(T_NT_N^* - T_1T_1^* - E'E'^*)C$$
$$+ M'^*(-X'X'^* + Q'Q\prime* + L'L'^*)M']S^*$$
$$+ SC^*(T_NX^* - T_1Q'^* - E'L'^*)M'S - SC^*(T_1T_P^* + E'H'^*)DS^*$$
$$+ SM'^*(X'T_N^* - Q'T_1^* - L'E'^*)CS^* + SM'^*(Q'T_P^* + L'H'^*)DS^*$$
$$- SD^*(T_PT_1^* + H'E'^*)CS^* + SD^*(T_PQ'^* + H'L'^*)M'S^*$$

(reasoned from (2.11')-(2.16'))

$$= S(S^{-1}B + B^*B^{*-1} + D^*D - C^*C + M'^*M')S^* = 0 \text{ (reasoned from (2.19))}.$$

The left side of $(2.20')$

$$= SM'^* - (SC^*T_N - SM'^*S')X'^*$$
$$+ (SC^*Y_1 - SD^*T_P - SM'^*Q')Q'^* + (SC^*E' + SD^*H' - SM'^*L')L'^*$$
$$= SM'^*(I + X'X'^* - Q'Q'^* - L'L'^*) - SC^*(T_NX'^* - T_1Q'^* - E'L'^*)$$
$$- SD^* - SD^*(T_PQ'^* + H'L'^*)$$
$$= 0$$

(reasoned from (2.13'), (2.15') and (2.16')). Then the ten series of operator equations (1)-(5) and (1')-(5') are equivalent to the following two series of equations (a) and (b):

(a) U_0 is a unitary extension of T_0.

(b) equations (2.17)-(2.10).

The general solutions of (a) are given in Theroem 1.1 and we only find the general solutions of A' and M' satisfying (2.17)-(2.20). Substituting $B = (1/2)S(C^*C - D^*D - Q)$ in (2.19), we have $M'^*M' = \text{Re}Q$, where $\text{Re}Q = (Q+Q^*)/2$. As $\text{Re}Q \geq 0$ by Theroem 1.3 there exists a metric-preserving operator V_6 from $(\mathcal{R}(\text{Re}Q), \langle\cdot,\cdot\rangle)$ to H_2 such that $M' = V_6(\text{Re}Q)^{1/2}$. Then we must determine $\mathcal{R}(V_6)$ such that V_6 satisfies (2.17) and (2.18).

We substitute $M' = V_6(\text{Re}Q)^{1/2}$ in (2.17) and (2.18) and get the following equations:

$$(S^{-1}F - C^*T_N) + (\text{Re}Q)^{1/2}V_6^*U_4(R^2 - I)^{1/2} = 0, \tag{2.24}$$

$$(S^{-1}G - C^*T_1 + D^*T_P) + (\text{Re}Q)^{1/2}V_6^*U_4(R^2 - I)^{1/2}RV^*T_1$$
$$+ (\text{Re}Q)^{1/2}V_6^*V_5(I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2} = 0. \tag{2.25}$$

In the following we solve equations (2.24) and (2.25). From (2.25) we see that (2.24) is equivalent to $V_6^*U_4 = K_1$. Substitute the expression of K_1 in (2.25). Then (2.25) is equivalent to $V_6^*V_5 = K_2$. Let $K(n+p) = K_1n + K_2p$ for any $n \in \mathcal{R}(R^2 - I)$ and $p \in \overline{\mathcal{R}((I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2})}$. K is a contractive operator from

$$(\mathcal{R}(R^2 - I) \oplus \overline{\mathcal{R}((I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2}}, -(\cdot,\cdot)_N \oplus (\cdot,\cdot)_P)$$

onto $(\mathcal{R}(K), \langle\cdot,\cdot\rangle)$ where $\mathcal{R}(K) \subset Z^*, K' = KU_4^* + KV_5^*$ is a contractive operator from $\overline{\mathcal{R}(U_4) \oplus \mathcal{R}(U_5)}$ onto $(\mathcal{R}(K), \langle\cdot,\cdot\rangle)$. Let $V'(n+p) = U_4n + V_5p$ for any $n \in \mathcal{R}(R^2 - I)$ and $p \in \overline{\mathcal{R}((I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2}}$. Then V' is a metric-preserving operator from

$$\mathcal{R}(R^2 - I) \oplus \overline{\mathcal{R}((I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2}}$$

onto $\mathcal{R}(U_4) \oplus \mathcal{R}(V_5)$. From Lemma 1.1 we know that there exists $V_5^{(0)}$ from $\mathcal{R}(K)$ to $(\mathcal{R}(U_4) \oplus \mathcal{R}(V_5)) \oplus (\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))^\perp$ and $V_6^{(0)}x = K'^*x + V_0(I - K'J'^*)^{1/2}x$, $x \in \mathcal{R}(K)$, $V_6 = V_6^{(0)} + V_6^{(1)}$, where $V_6^{(1)}$ is a metric-preserving operator from $\mathcal{R}(\mathrm{Re}Q) \ominus \mathcal{R}(K_0)$ to

$$(\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))^\perp \cap \mathcal{R}(V_6^{(0)})^\perp$$

and V_6 is the general solution of $K = V_6^*V'$, which is equivalent to (2.14) and (2.15). Theorem 2.1 is proved.

§3. The General Form of Unitary Dilation of Contraction on Π_K Associated With Its Triangle Model and a Theorem About the Minimality of Unitary Dilation

Theorem 3.1. *Let T be a contraction of Pontryagin space Π_K and U be a unitary dilation of T. Then all the general solutions are as in Theorem 2.1 in the special case that $H = H_1 = H_2$ and H must be the following form*

$$H = \left[\overset{\infty}{\underset{k=1}{\oplus}} W^{*k}(\mathcal{R}(U_2) \oplus \mathcal{R}(V_1))\right] \oplus [\mathcal{R}(U_2) \oplus \mathcal{R}(V_1))]$$
$$\oplus [\mathcal{R}(U_4) \oplus \mathcal{R}(V_5)] \oplus \left[\overset{\infty}{\underset{k=1}{\oplus}} W^k(\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))\right] \oplus H_0, \tag{3.1}$$

where W_0, which is the restriction of W on H_0, is a unitary operator. And $\mathcal{R}(V_6)$ must satisfy the following conditions:

(1) $\mathcal{R}(V_6) \perp \overset{\infty}{\underset{k=0}{\oplus}} W^{*K}(\mathcal{R}(U_2) \oplus \mathcal{R}(V_1))$,

$\mathcal{R}(V_6) \perp \overset{\infty}{\underset{k=1}{\oplus}} W^K(\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))$,

(2) $\mathcal{R}(V_6) \perp W^k\mathcal{R}(V_6)$, $k = 1, 2, \cdots$.

Proof. Obviously $T_0^n = PU_0^n|_{N\oplus}P$, for $n = 1, 2, \cdots$, where P is an orthogonal projection from $\Pi_K \oplus H$ onto $N \oplus P$ and U_0 is a unitary operator on $N \oplus P \oplus H$. We know that H must be the form (3.1) by Theorem 1.2 and all the general solutions are the same as in Theorem 2.1. From equation $T^2 = P_{\Pi_K}U^2|\Pi_K$ we reason the following equalities:

$$A'X' = 0, \tag{3.2}$$

$$A'Q' = 0, \tag{3.3}$$

$$A'M' = 0, \tag{3.4}$$

$$E'M' = 0, \tag{3.5}$$

$$H'M' = 0. \tag{3.6}$$

As $A' = SC^*E' - SD^*H' - SM'^*L'$, the equality (3.2)-(3.4) are equivalent to the following (3.7)-(3.9):

$$M'^*L'X' = 0, \tag{3.7}$$

$$M'^*L'Q' = 0, \tag{3.8}$$

$$M'^*L'M' = 0. \tag{3.9}$$

We substitute (1.6)-(1.10) and $M' = V_6(\mathrm{Re}Q)^{1/2}$ in (3.5), (3.6) and (3.7)-(3.9) and get the following equations:

$$[-T_1T_P^*(I - T_PT_P^*)^{-1/2}U_2^* + V(R^2 - I)^{1/2}(I - K_0K_0^*)^{1/2}V_1^*]V_6(\mathrm{Re}Q)^{1/2} = 0, \tag{3.10}$$

$$(I - T_PT_P^*)^{1/2}U_2^*V_6(\mathrm{Re}Q)^{1/2} = 0, \tag{3.11}$$

$$\begin{aligned}&(\mathrm{Re}Q^{1/2}V_6^*[-(U_4RK_0 + V_5(I - K_0^*K_0)^{1/2})T_P^*U_2^*\\&+ (U_4R(I - K_0K_0^*)^{1/2} - V_5K_0^*)V_1^* + W]U_4(R^2 - I)^{1/2} = 0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&(\mathrm{Re}Q^{1/2}V_6^*[-(U_4RK_0 + V_5(I - K_0^*K_0)^{1/2})T_P^*U_2^*\\&+ (U_4R(I - K_0K_0^*)^{1/2} - V_5K_0^*)V_1^* + W]U_4(R^2 - I)^{-1/2}RV^*T_1\\&+ V_5(I - K_0^*K_0)^{1/2}(I - T_P^*T_P)^{1/2} = 0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&(\mathrm{Re}Q^{1/2}V_6^*[-(U_4RK_0 + V_5(I - K_0^*K_0)^{1/2})T_P^*U_2^*\\&+ (U_4R(I - K_0K_0^*)^{1/2} - V_5K_0^*)V_1^* + W]V_6(\mathrm{Re}Q)^{1/2} = 0.\end{aligned} \tag{3.14}$$

Then the equations (3.10)-(3.14) are equivalent to the following five equations:

$$U_2^*V_6 = 0, \tag{3.15}$$

$$V_1^*V_6 = 0, \tag{3.16}$$

$$V_6^*WV_4 = 0, \tag{3.17}$$

$$V_6^*WV_5 = 0, \tag{3.18}$$

$$V_6^*WV_6 = 0. \tag{3.19}$$

So $\mathcal{R}(V_6)\perp\mathcal{R}(R_2) \oplus \mathcal{R}(V_1)$, $\mathcal{R}(V_6)\perp W(\mathcal{R}(U_4) \oplus \mathcal{R}(V_5))$, and $\mathcal{R}(V_6)\perp W\mathcal{R}(V_6)$. As $T_n = P_{\Pi_K}U^n|\Pi_K$ for $n = 3, 4, \cdots$ by induction, similar to the discussion above, we have

$$E'L'^nM' = 0,\ H'L'^nM' = 0,\ M'^*L'^{n+1}X' = 0,\ M'^*L'^{n+1}Q' = 0,\ M'^*L'^{n+1}M' = 0$$

for $n = 1, 2, \cdots$, i.e.,

$$V_2^*W^nV_6 = 0,\ V_1^*W^nV_6 = 0,\ V_6^*W^{n+1}U_4 = 0,\ V_6^*W^{n+1}V_5 = 0\, V_6^*W^{n+1}V_6 = 0$$

for $n=1,2,\cdots$. Then

$$\mathcal{R}(V_6)\perp W^{*n}(\mathcal{R}(U_2)\oplus\mathcal{R}(V_1)),\quad \mathcal{R}(V_6)\perp W^{n+1}(\mathcal{R}(U_4)\oplus\mathcal{R}(V_5)),$$

and $\mathcal{R}(V_6)\perp W^{n+1}\mathcal{R}(V_6)$ for $n=0,1,2,\cdots$. The theorem is proved.

Let $T=\{S,T_N,T_P,T_1,C,D,F,G,B\}$ be the triangle model of a contraction T on Π_K associated with a standard decomposition $\Pi_K=N\oplus\{Z\dot{+}Z^*\}\oplus P$ and U be a minimal dilation of T, i.e., $T^n=P_{\Pi_K}U^n|_{\Pi_K}$ for $n=1,2,\cdots$ and

$$\text{C.L.S.}\ \{(U-z)^{-1}\Pi_K|z\in\rho(U)\}=\Pi_K\oplus H.$$

Then such a minimal unitary dilation is unique to unitary equivalence, and suppose $U_0=P'U|_{N\oplus P\oplus H}$ where P' is an orthogonal projection from $\Pi_K\oplus H$ onto $N\oplus P\oplus H$, and $T_0=\begin{bmatrix}T_N & T_1\\ & T_P\end{bmatrix}$. Then according to Theorem 3.1, T_0 is a contraction of $N\oplus P$, and U_0 is a unitary dilation of T_0. However U_0 does not have to be a minimal unitary dilation of T_0. In the following we give a criterion to judge whether U_0 is a minimal unitary dilation of T_0.

Corollary 3.1. *Let U be the minimal unitary dilation of T. Then U_0 is the minimal unitary dilation of T_0 too if and only if K^* is a preserving-metric operator from*

$$(\mathcal{R}(\text{Re}Q),\langle\cdot,\cdot\rangle)\ \textit{to}\ (\mathcal{R}((R^2-I)^{1/2}\oplus\overline{\mathcal{R}((I-T_P^*T_P)^{1/2})},-(\cdot,\cdot)_N\oplus(\cdot,\cdot)_p).$$

Proof. $H_0=0$ is equivalent to that U_0 is the minimal unitary dilation of T_0 too. By Theorem 3.1 and Theorem 2.1 we see that K'^* is a preserving-metric operator from $(\mathcal{R}(K),\langle\cdot,\cdot\rangle)$ to $\mathcal{R}(U_4)\oplus\mathcal{R}(V_5)$ and $\mathcal{R}(K)=\mathcal{R}(\text{Re}Q)$. Then the corollary is proved.

The following corollary is easily deduced from Corrollary 3.1.

Corollary 3.2. *If U_0 is the minimal unitary dilation of T_0 too, then*

$$\dim\mathcal{R}(\text{Re}Q)\le\dim\mathcal{R}(R^2-I)+d_K, \tag{3.20}$$

where d_K is the defect number of K_0, i.e., $d_K=\dim\overline{\mathcal{R}((I-K_0^*K_0)^{1/2})}$.

Obviously we know U is the minimal unitary dilation of T it when U_0 is the minimal unitary dilation of T_0. The following corollary is in [9]. Here it is a simple corollary of our result.

Corollary 3.3. *If T is an isometric operator on Π_K, then U is the minimal unitary dilation of T if and only if U_0 is the minimal unitary dilation of T_0.*

Proof. From the triangle model of isometric operator T we have $\text{Re}Q=0$, $S^{-1}F-C^*T_N=0$ and $S^{-1}G-C^*T_1+D^*T_P=0$ ($T_1=0$ here). So $K_1=0$ and $K_2=0$ (see Theorem 1.6). Then $K=0$. K^* is satisfied with the condition of Corollary 3.1. The corollary is proved.

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