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SOME DYNAMICAL PROPERTIES OF QUADRATIC RATIONAL MAPS**

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Abstract

This paper studies the dynamics of the analytic family $z+1/z+b$ and describes the topology of the parameter space, structural stability and J-stability. The mapping class group of almost all maps of the above family is determined.

Keywords Complex dynamics, Julia set, Structural stability, J-stability, Mapping class group.

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§0. Introduction

Let $f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ be a rational function with degree greater than one. f^n denotes the n-th iterate of f. z is a stable point of f if there exists a neighborhood U, $z \in U$, such that $\mathcal{F} = \{f^n|U\}$ is a normal family. The stable set $F(f)$ is the set of stable points of f. Its complement, $J(f)$, is called the Julia set of f. $J(f)$ is perfect, completely invariant and never empty.

A periodic point of f is a point z for which there exists an integer n such that $f^n(z) = z$. The eigenvalue $\lambda(z)$ is defined by

$$\lambda(z) = (f^n)'(z).$$

The cycle is called repelling if $|\lambda(z)| > 1$, neutral if $|\lambda(z)| = 1$, attracting if $0 < |\lambda(z)| < 1$ and superattracting if $\lambda(z) = 0$.

Fatou[1] and Julia[2] proved that the Julia set $J(f)$ is the closure of its repelling periodic points.

A connected component U of the stable set $F(f)$ is called periodic if there exists an integer n such that $f^n(U) = U$. A component U of $F(f)$ is eventually periodic if there is an integer m such that $f^m(U)$ is periodic. Sullivan's finiteness theorem says that every component of the stable set is eventually periodic[3]. His proof is based on the techniques of Teichmuller theory. Different proofs of the theorem are given by L. Bers and Lu Yinian independently[4,5].

The periodic stable components are completely classified: (1) attracting (superattracting), (2) parabolic, (3) Siegel disk, (4) Herman ring[6,7].

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For quadratic polynomials z^2+c, the Mandelbrot set

$$M=\{c|c,c^2+c,(c^2+c)^2+c,\cdots,\nrightarrow\infty\}$$

is well understood primarily due to the pioneering work of A. Douady, J. H. Hubbard and D. Sullivan in the eighties[8]. For general references on complex analytic dynamics, see [9], [10], [11] and [12].

The outline of this paper is as follows.

In section one, we describe the topology of the parameter space of the family $z+1/z+b$.

In section two, we consider the structural stability and the J-stability and prove that Julia sets are null measure for some quadratic rational maps.

We determine the mapping class groups for almost all maps of the family $z+1/z+b$ in section three.

§1. The Parameter Space of $z+1/z+b$

In [13], L. Goldberg and L. Keen gave a description of the topology of the family $\lambda(z+1/z+b)$, where $|\lambda|>1$ and $b\in\mathbb{C}$.

Combining the main theorem in [14] with the ideas and methods of [13], we discuss the analytic family $z+1/z+b$ in this section.

First of all, we state some well-known facts about parabolic fixed point which revealed by Fatou and Camacho[1,15].

We consider rational functions

$$f(z)=\lambda z+a_2z^2+a_3z^3+\cdots$$

which are defined and holomorphic in some neighborhood of the origin. We suppose that the multiplier λ at the fixed point is a root of unity, $\lambda^q=1$.

The following topological result belongs to C. Camaco[15].

Theorem. *Let $f(z)$ be as above. Then either $f^q=id.$ or there is a local homeomorphism h and an integer $k>1$ such that $h(0)=0$ and*

$$h\circ f\circ h^{-1}(z)=\lambda z(1+z^{kn}).$$

Remark. h can be chosen to be quasiconformal, but cannot be conformal !

In this section, we consider the special case $\lambda=1$,

$$f(z)=z+az^{n+1}+\cdots,\quad a\neq 0.$$

Definition. *A connected open set U, with compact closure $\overline{U}$ contained in a small neighborhood of the origin, is called an attracting petal for f at the origin if*

$$f(\overline{U})\subset U\cup\{0\}\quad and\quad \bigcap_{k\geq 0}f^k(\overline{U})=\{0\}.$$

Similarly, U' is a repelling petal for f if U' is an attracting petal for f^{-1}.

Fatou Flower Theorem. *If the origin is a fixed point of*

$$f(z)=z+az^{n+1}+\cdots,a\neq 0,$$

with $n+1\geq 2$, then there exist n disjoint attracting petals U_i and n disjoint repelling petals, which together with the origin itself form a neighborhood of the origin. These petals alternate *with each other, as illustrated in Figure 1, so that each U_i intersects only U_i' and U_{i-1}'(where U_0' is to be identified with U_n').*

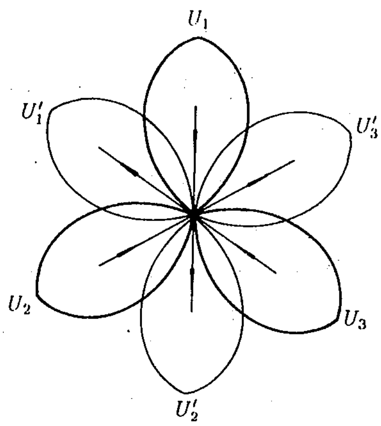

Figure 1

Each attracting petal U_i determines a parabolic basin of attraction Ω_i, consists of all z_0 for which the orbit $z_0 \longrightarrow z_1 = f(z_0) \longrightarrow \cdots \longrightarrow z_n = f^n(z_0) \longrightarrow \cdots$ eventually lands in U_i. $\Omega_1, \cdots, \Omega_n$ are disjoint open sets.

Now we restrict our interest in the parameter space of the family $f_b(z) = z + 1/z + b$ with critical points ± 1. $A_b(\infty)$ is the parabolic basin corresponding to ∞. It is PSL(2, $\mathbb{C}$) conjugate to the form $(1/z + z + b)^{-1} = z(1 - bz + (b-1)z^2 + \cdots)$ in the neighborhood of the origin. There exist two attractions for $b = 0$, i.e., $A_0(\infty)$ has two components. There exists only one attraction for $b \neq 0$, $A_b(\infty)$ is connected. The main theorem of [14] implies that $J(f_b)$ is connected iff $A_b(\infty)$ contains only one critical point for $b \neq 0$, and $J(f_0)$ is connected.

Set $U = \{b \in \mathbb{C} \,|\, J(f_b)$ is a Cantor set $\}$ and $O = \{b \in \mathbb{C} \,|$ there are integers $m, n \geq 0$ so that $f_b^m(+1) = f_b^n(-1)\}$, the set of orbit relations.

$O \subset U$ and the Mandelbrot set $M_p = \{b | J(f_b)$ is connected$\} = \mathbb{C} - U$; U contains the imaginary-$\{0\}$.

It is easy to check that $M_p \subset \{b = u + iv | -2 \leq u, v \leq 2\}$ and $[-2, 2] \subset M_p$.

There exists a conformal map φ_b which is defined in some half-plane such that $\varphi_b \circ f_b \circ \varphi_b^{-1}(z) = z + 1$, where φ_b is analytic on b[12].

The curve of Reφ_b=constant is the leaf of foliation whose singularities occur precisely along the backward orbits of $+1$ and -1. There are two possibilities for the structure of the foliation:

(1) Both critical points lie on the same leaf of the foliation. This occurs exactly when b is on the y axis.

(2) f_b has a preferred critical point. In this case, there is a distinguished leaf of the foliation which is a figure eight curve and has a preferred critical point as its cut point.

The principal loop of γ_0 passes through the parabolic fixed point ∞, and the secondary loop passes through the pole 0. The region bounded by the principal loop is the largest dynamically defined region on which φ_b is a conjugacy.

The critical points $+1$ and -1 are preferred in the right and left half b-planes respectively (R and L).

If $b \in R \cap O$, it has the form $f_b^n(+1) = f_b^m(-1), m \geq n$.

Let $Q(z) = z^2 + z$. The map Q has a single parabolic fixed point at 0 with multiplier 1, a critical point at $c_+ = -1/2$ and a critical value $w_+ = Q(c_+) = -1/4$. The filled-in Julia set K is connected and locally connected[8]. There is an analytic map

$$\varphi : \mathrm{int}(K) \mapsto \mathbb{C}$$

which conjugates $Q(z)$ to $\omega \longrightarrow \omega + 1$ for z near 0.

As for f_b, we denote by γ_0 the preferred component of the leaf containing c_+. γ_0 is a figure eight curve whose principal loop passes through 0, and secondary loop passes through the preimage -1 for 0.

Definition. *A generalized figure eight curve in $\overline{\mathbb{C}}$ is a union of simple closed curves consists of*

(i) *a primary loop which separates the sphere into two disks;*

(ii) $n\,(\geq 1)$ *secondary loops, each of which is attached to the primary loop at a unique point and all of which are contained in the same disk.*

Let $\gamma_n = Q^{-n}(\gamma_0)$. Then γ_n is a generalized figure eight curve with 2^n secondary loops and $\gamma_n \subset A(n+1) \cup Q^{-n}(0)$ for all n. Each $B(n)$ contains a single point of the set $Q^{-n}(0)$. $\Delta = \overline{Q(\gamma_0)} \subset A(\gamma_0) \cup \{0\}$ is a disk (see Figures 2(a) and 2(b)).

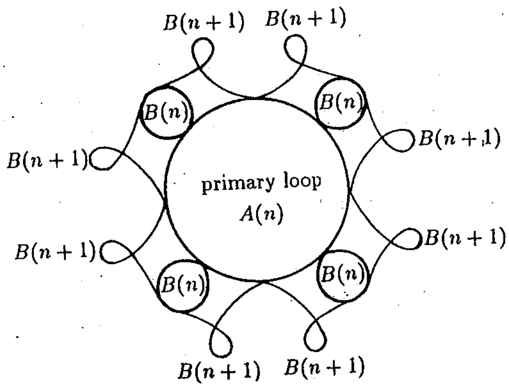

Figure 2.a

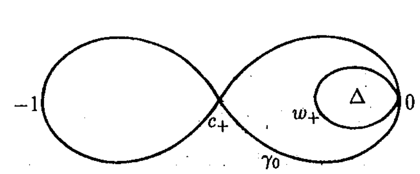

Figure 2.b

Lemma 1.1. *There is an injective holomorphic map*

$$E : R - M_p \longrightarrow \mathrm{int}(K) - \Delta$$

which has the following properties:

i) *If $f_b^n(v_+) = f_b^m(v_-), v_+ = f_b(+1)$ and $v_- = f_b(-1)$, then $E(b) = w$ satisfies*

$$Q^n(w_+) = Q^m(w_-), w_+ = \xi_b(v_+) \text{ and } w_- = \xi_b(v_-) = w.$$

ii) *As $b \longrightarrow \partial R, w = E(b) \longrightarrow \partial\Delta$.*

Proof. We define the map E in the following way.

Since $f_b'(\infty) = Q'(0) = 1$, there is a unique univalent analytic function ξ_b from a neighborhood of 0 in $\mathrm{int}(K)$ which conjugates f_b to Q and is normalized so that $\xi_b(+1) = -1/2 = c_+$.

Since $b \in R - M_p$, the map ξ_b can be analytically continued in the region bounded by the loop of the leaf foliation of v_- which passes through ∞, and we define $E(b) = \xi_b(v_-)$. Since $+1$ is preferred for $b \in R$, $\xi_b(v_-)$ lies outside of Δ. The conjugacies depend holomorphically on b, so that E is holomorphic. By the construction, E satisfies properties i) and ii).

Furthermore, E is injective. If $\xi_b(v_-(b)) = \xi_{b'}(v_-(b'))$, the map $\xi_{b'}^{-1} \circ \xi_b$ conjugates f_b to $f_{b'}$ in a neighborhood of ∞ in $A_b(\infty)$.

Lift $\xi_{b'}^{-1} \circ \xi_b | U_b$ to $f^{-1}(U_b)$ so that $\xi_{b'}^{-1} \circ \xi_b$ sends $v_-(b)$ to $v_-(b')$, etc.

$\xi_{b'}^{-1} \circ \xi_b$ can be extended to a conformal conjugacy of f_b to $f_{b'}$ on the stable set $A_b(\infty)$. From the proof of Lemma 5 in [14], we know that $\xi_{b'}^{-1} \circ \xi_b$ is continuous on $\mathbb{C}$. For $b \in U = \mathbb{C} - M_p$, the Julia set $J(f_b)$ is null measure. We should prove this fact in section 2. It follows from a theorem of [16] that the map $\xi_{b'}^{-1} \circ \xi_b$ extends further to a conformal conjugacy between f_b and $f_{b'}$ on the whole sphere $\overline{\mathbb{C}}$. The assumption $b, b' \in R$ implies that $b = b'$.

The following lemma proved in [13] will be used repeatedly.

Lemma 1.2. *Suppose that S is a Riemann surface homeomorphic to $\overline{\mathbb{C}} - n$ disks and let $w_+, w_- \in S$. There is exactly one isomorphism class of degree 2 ramified simply over w_+ and w_-. There projections are normal, and the total space is homeomorphic to $\overline{\mathbb{C}} - 2n$ disks.*

Now we state the main result in this section, which is similar to the generic case in [13].

Theorem 1.1. *The map $E : R - M_p \longrightarrow \mathrm{int}(K) - \Delta$ is a homeomorphism.*

Proof. We construct an inverse to the map using the same method as that used in [13].

For any fixed point $w_- \in \mathrm{int}(K) - \Delta$, there are two distinct elements of $Q^{-1}(w_-)$. Choose one and label it c_-. Let N be the smallest integer such that $w_- \in A(\gamma_n) \cup B(\gamma_n)$. Choose a primary loop $\partial\Omega_0$ of a leaf of foliation such that this simple analytic curve bounds a disk Ω_0 which contains γ_N, w_+, c_+ and w_- but not c_- (see Figure 3).

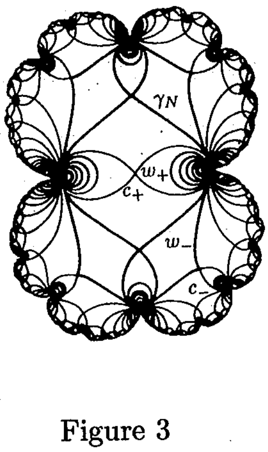

Figure 3

Lemma 1.2 implies that there exists a holomorphic degree 2 covering $\pi_1 : \Omega_1 \longrightarrow \Omega_0$ ramified over w_+ and w_- with Ω_1 doubly connected.

Since Ω_0 contains one critical point $c_+, Q(\Omega_0)$ is a disk and $Q : \Omega_0 \longrightarrow Q(\Omega_0)$ is a 2:1 covering ramified over w_+ (w_- is not in $Q(\Omega_0)$).

Set $\Omega_1' = \pi_1^{-1}(Q(\Omega_0))$. Then $\pi_1 : \Omega_1' \longrightarrow Q(\Omega_0)$ is also a 2:1 covering ramified over w_+.

There exists a homeomorphism $i_1 : \Omega_0 \longrightarrow \Omega_1'$ satisfying $\pi_1 \circ i_1 = Q$.

Define $Q_1 = i_1 \circ \pi_1 \Omega_1 :\longrightarrow \Omega_1' \subset \Omega_1$. Q_1 is a holomorphic endomorphism which is a branched covering of its image ramified over $i_1(w_+)$ and $i_1(w_-)$. The point $i_1(0)$ is a fixed point of Q_1 which lies on the boundary of Ω_1. $\Omega_1/\sim Q_1$ is a sphere with four or three punctures, where $\sim Q_1$ is the equivalent relation of grand orbit for Q_1.

We proceed inductively. The hypotheses are:

(i) For $0 \le j \le n$, $\Omega_j = \overline{\mathbb{C}} - 2^j$ disks.

(ii) For $0 \le j \le n$, $Q_j : \Omega_j \longrightarrow \Omega_j$ is a holomorphic endomorphism which is a 2:1 branched covering of its image. Each Q_j has a parabolic fixed point $z_n(0) = i_n \circ i_{n-1} \circ \cdots \circ i_1(0) \in \partial\Omega_n$.

(iii) For $0 \le j \le n-1$, there is a holomorphic embedding $i_{j+1} : \Omega_j \longrightarrow \Omega_{j+1}$ satisfying $Q_{j+1} \circ i_{j+1} = i_{j+1} \circ Q_j$. Lemma 1.2 gives a planar Riemann surface Ω_{n+1} homeomorphic to $\overline{\mathbb{C}} - 2^{n+1}$ disks and a 2:1 normal branching covering

$$\pi_{n+1} : \Omega_{n+1} \longrightarrow \Omega_n$$

ramified over $w_+(n) = i_n \circ i_{n-1} \circ \cdots \circ i_1(w_+)$ and $w_-(n) = i_n \circ i_{n-1} \circ \cdots \circ i_1(w_-)$. $Q_n(\Omega_n)$ is homeomorphic to $\overline{\mathbb{C}} - 2^{n-1}$ disks.

Set $\Omega_{n+1}' = \pi_{n+1}^{-1}(Q_n(\Omega_n))$.

Both $\pi_{n+1} : \Omega_{n+1}' \longrightarrow Q_n(\Omega_n)$ and $Q_n : \Omega_n \longrightarrow Q_n(\Omega_n)$ are 2:1 covering projections ramified over $w_+(n)$ and $w_-(n)$.

Therefore, the uniqueness of Lemma 1.2 implies that there is a holomorphic isomorphism $i_{n+1} : \Omega_n \longrightarrow \Omega_{n+1}'$ satisfying $\pi_{n+1} \circ i_{n+1} = Q_n$ on Ω_n.

Define $Q_{n+1} : \Omega_{n+1} \longrightarrow \Omega_{n+1}$ by $Q_{n+1} = i_{n+1} \circ \pi_{n+1}$. Then

$$Q_{n+1} \circ i_{n+1} = i_{n+1} \circ \pi_{n+1} \circ i_{n+1} = i_{n+1} \circ Q_n.$$

Q_{n+1} is a holomorphic endomorphism with parabolic fixed point $z_{n+1}(0) = i_{n+1} \circ z_n(0) \in \partial\Omega_{n+1}$.

We complete the inductive step.

The direct limit Ω_∞ of the system (Ω_n, i_n) is the quotient of the union $\bigcup_{n\ge 0} \Omega_n$ by all the identifications of the form $z \sim i_n(z)$.

Ω_∞ is a Riemann surface of infinite type whose fundamental group is given by

$$\pi_1(\Omega_\infty) = \lim_{n\to\infty} ((i_n)_* : \pi_1(\Omega_n) \to \pi_1(\Omega_{n+1})).$$

A two fold self-covering

$$Q_\infty : \Omega_\infty \longrightarrow \Omega_\infty$$

is defined by

$$Q_\infty([z]) = [Q_n(z)], z \in \Omega_n.$$

The map Q_∞ is holomorphic and ramified over $[w_+]$ and $[w_-]$ and has a parabolic fixed point at $[0]$.

$\Omega_\infty/\sim Q_\infty \simeq \Omega_n/\sim Q_n \simeq \Omega_0/\sim Q \simeq$ a sphere with four or three marked points.

Consider first that the grand orbits of $[w_+]$ and $[w_-]$ are disjoint. $\Omega_\infty/\sim Q_\infty$ is a sphere with four marked points.

Let Ω_∞^* be the region obtained by deleting from Ω_∞ the grand orbits of $[w_+]$ and $[w_-]$. $\Omega_\infty^*/\sim Q_\infty$ is a sphere with four punctures.

Suppose that a fixed $b \neq 0$ lies on the y axis, and the quotient space $A_b(\infty)/\sim f_b$ of the stable set $A_b(\infty)$ of f by the grand orbit equivalent relation is a sphere with four punctures, where $A_b^*(\infty)$ is the region $A_b(\infty) - \{$ grand orbits of $+1$ and $-1\}$.

Let $f^* : A_b^*(\infty)/\sim f_b \longrightarrow \Omega_\infty^*/\sim Q_\infty$ be a quasiconformal homeomorphism with Beltrami coefficient μ^*. Lifting f^* to $A_b^*(\infty)$, we get a quasiconformal homeomorphism f from $A_b^*(\infty)$ to Ω_∞^* with Beltrami coefficient μ which is a conjugacy between $f_b|_{A_b^*(\infty)}$ and $Q_\infty|_{\Omega_\infty^*}$, where μ is compatible with f_b, that is,

$$\mu(f_b(z))\overline{f_b'(z)}/f_b'(z) = \mu(z), z \in A_b^*(\infty).$$

Extend μ so that $\mu = 0$ on $\overline{\mathbb{C}} - A_b^*(\infty)$.

There is a unique quasiconformal mapping $g : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that $g \circ f_b \circ g^{-1}$ is a rational function of the form $f_{b(w_-)}$ for a unique $b(w_-) \in R - M_p$.

It is easy to check that the correspondence $w_- \longrightarrow b(w_-)$ gives an inverse of E.

If the grand orbits of $[w_+]$ and $[w_-]$ are the same, then $\Omega_\infty^*/\sim Q_\infty$ is a sphere with three punctures. The above construction defines a unique $b(w_-) \in O$.

Let $H(M_p)$ denote the set of such b-values that f_b has an attracting cycle. From the main theorem of [14], it is clear that $H(M_p)$ is contained in M_p. Applying the implicit function theorem we see that $H(M_p)$ is an open set. A connected component W of $H(M_p)$ is called a hyperbolic component of M_p. It is not known whether $H(M_p)$ equals the interior of M_p.

We prove the following theorem by means of the method of Sullivan.

Theorem 1.2. *For each hyperbolic component W of M_p, the multiplier ρ_W induces a conformal isomorphism*

$$\rho_W : W \longrightarrow D$$

which can be extended to a homeomorphism of $\overline{W}$ onto $\overline{D}$.

Proof. For $b \in W \subset R$, f_b has an attracting cycle $\{z(b), f_b(z(b)), \cdots, f_b^{k-1}(z(b))\}$.

$$\rho_W(b) = (f_b^k)'(z(b)).$$

For any $\mu \in D$, let q_μ be the mapping $z \longrightarrow z(z+\mu)/(1+\overline{\mu}z)$. It is a proper holomorphic mapping from D to D with degree 2 and $q_\mu(0) = 0$, $q_\mu'(0) = \mu$.

Suppose $b \in W$. Let $\mu_0 = \rho_W(b)$ and U be the connected component of $\text{int}(K_b)$ containing -1, where $K_b = \overline{\mathbb{C}} - A_b(\infty)$.

There exists a homeomorphism φ from U to D such that $\varphi \circ f_b^k \circ \varphi^{-1} = q_{\mu_0} : D \longrightarrow D$. Suppose $|\mu_0| < r < 1$, put $B = \overline{D_r}$ and $A_\mu = q_\mu^{-1}(B)$ for $\mu \in D$. For $|\mu| < r$, it is easy to check that $B \subset \text{int}(A_\mu)$ and $B = \overline{D_r}$ contains the critical values of q_μ. By Riemann-Hurwicz formula, A_μ is simply connected. Set $E = \varphi^{-1}(A_{\mu_0})$. We can construct a quasiconformal mapping ψ_μ from E to A_μ for each $\mu \in D_r$ such that $\psi_\mu \circ f_b^k = q_\mu \circ \psi_\mu$ on the boundary of E, and $\psi_{\mu_0} = \varphi$. For each $\mu \in D_r$, we define $g_\mu : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ by $g_\mu = f_b$ on $\overline{\mathbb{C}} - E$ and $g_\mu^k = \psi_\mu^{-1} \circ q_\mu \circ \psi_\mu$ on E. We define an almost complex structure σ_μ on E by $\sigma_\mu = \psi_\mu^* \sigma_0$, where σ_0 is the standard complex structure. Extend σ_μ to the set $\bigcup_{n \geq 1} f^{-n}(E) = \text{int}(K_b)$ satisfying $g_\mu^* \sigma_\mu = \sigma_\mu$, $\sigma_\mu = \sigma_0$ on $\overline{\mathbb{C}} - \text{int}(K_b)$. By the theorem of Ahlfors-Bers, there exists a unique quasiconformal homeomorphism Φ_μ such that $\Phi_\mu \circ g_\mu \circ \Phi_\mu^{-1}$ has the form $z + 1/z + b(\mu)$. $b(\mu)$ depends continuously on μ, and $b(\mu_0) = b$ and $b(\mu) \in W$ for $\mu \in D_r$ and $\rho_W(b(\mu)) = \mu$. $\mu \longrightarrow b(\mu)$ is a continuous section of ρ_W on D_r. Let $r \longrightarrow 1$. We conclude that $\rho_W : W \longrightarrow D$ is a covering with no branching point. Since D is simply connected, $\rho_W : W \longrightarrow D$ is a conformal isomorphism. The boundary of W is a real-analytic curve (see [8]). ρ_W can be extended to a homeomorphism of $\overline{W}$ onto $\overline{D}$.

Remark. For quadratic polynomials, the same result was proved by A. Douady and J. H. Hubbard[8].

Theorem 1.2 implies that each hyperbolic component has a unique b-value for which the attracting cycle is superattracting. We call this point the center of the hyperbolic component.

A point b is called a Misiurewicz point if the orbit of the other critical point of f_b is eventually periodic, but not periodic. One can prove that for any Misiurewicz point b the Julia set $J(f_b) = K_b = \overline{\mathbb{C}} - A_b(\infty)$.

In order to understand the complexity of the boundary of M_p, we state a proposition as follows.

Proposition 1.1. (1) *The boundary of M_p is contained in the closure of the Misiurewicz points.*

(2) *The boundary of M_p is contained in the closure of the centers of hyperbolic components.*

Proof. From the Montel's theorem, it is easy to prove this proposition.

Let W_1 be the subset of M_p for which f_b has an attracting (superattracting) fixed point. f_b is $\mathrm{PSL}(2, \mathbb{C})$ conjugate to a polynomial-like mapping in the sense of [17].

Theorem 1.3. *For any $b \in W_1$, the Julia set $J(f_b)$ of f_b is connected and locally connected.*

Proof. Suppose that $f_b(z) = z + 1/z + b$ has an attracting or superattracting fixed point $z(b)$. Let $h_b(z) = (z - z(b))^{-1}$. Then $g_b(z) = h_b \circ f_b \circ h_b^{-1}$ is of the form $\lambda(b)z - az/c(z+c)$, $|\lambda(b)| > 1$ or $\lambda(b)z^2 + z$. Choose a suitable simply connected open set U such that $g_b^{-1}(U)$ is simply connected and contained in U. $g : g^{-1}(U) \longrightarrow U$ is a polynomial-like mapping with degree 2. By the straightening Theorem, g_b is hybrid equivalent to a quadratic polynomial g which has a parabolic fixed point. $J(g)$ is connected and locally connected[8]. Therefore, $J(f_b)$ is connected and locally connected.

§2. The Structural Stability and the Measures of Julia Sets

In this section, we discuss the structural stability of the analytic family

$$f(b,z) = f_b(z) = z + 1/z + b : (\mathbb{C} - M_p) \times \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}},$$

global stability and J-stability in the sense of [16]. Using these facts, we prove that the measure of $J(f_b)$ is null for each $b \in \mathbb{C} - M_p$.

We claim that the Julia set $J(f_b)$ is contained in the y axis for b lying on the y axis.

In fact, for $\lambda = ic$, let $h(z) = iz$ be a rotation.

$$g(z) = h \circ f_b \circ h^{-1}(z) = z - 1/z + c, c \in \mathbb{R}.$$

For any $z_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, $g^{-1}(z_0) \in \overline{\mathbb{R}}$. Then $J(g) \subset$ the closure of $\{\bigcup_{n \geq 0} g^{-n}(z_0)\} \subset \overline{\mathbb{R}}$ (see [10] or [9]), that is, $J(f_b) = h(J(g))$ is contained in y axis.

For $b \in \mathbb{C} - M_p \cup O$, the critical points ± 1 of f_b lie in $A_b(\infty)$ but are not in the same grand orbit.

First of all, we prove that the set $\mathbb{C} - M_p \cup O$ is open.

Proposition 2.1. *The set* $\mathbb{C} - M_p \cup O$ *is open.*

Proof. For any $b_0 \in \mathbb{C} - M_p \cup O$ and b_1 lying on y axis, let $A_{b_0}(\infty)$ and $A_{b_1}(\infty)$ be the parabolic stable regions. Then $A^*_{b_0}(\infty)/\sim f_{b_0}$ and $A^*_{b_1}(\infty)/\sim f_{b_1}$ are homeomorphic to the sphere with four punctured points, where $\sim$ is the grand orbit equivalent.

There is a quasiconformal homeomorphism

$$h : A^*_{b_1}(\infty)/\sim f_{b_1} \longrightarrow A^*_{b_0}(\infty)/\sim f_{b_0}.$$

Lifting h to $A^*_{b_1}(\infty)$, we see that $h : A^*_{b_1}(\infty) \longrightarrow A^*_{b_0}(\infty)$ is a quasiconformal homeomorphism satisfying $h \circ f_{b_1} = f_{b_0} \circ h$. Since $\text{mes}(J(f_{b_1})) = 0$, h can be extended to $\overline{\mathbb{C}}$, conjugating f_{b_1} to f_{b_0}.

Let $\mu(z) = h_{\bar{z}}(z)/h_z(z)$ be the Beltrami coefficient of h. The Ahlfors-Bers Theorem[18] implies that there is a unique quasiconformal homeomorphism $h_t(z)$ with Beltrami coefficient $t \cdot \mu(z)$ satisfying $h_t(\infty) = \infty$, $h_t(1) = 1$, $h_t(-1) = -1$, where $|t| <\| \mu \|^{-1}$.

Since $\mu(z)$ is f_{b_0}-invariant, $h_t^{-1} \circ f_{b_0} \circ h_t$ is analytic with the form $z + 1/z + b(t)$ which is analytic on $|t| <\| \mu \|^{-1}$, $b(0) = b_0$ and $b(1) = b_1$. It is obvious that $b(t) \in \mathbb{C} - M_p \cup O$ for $|t| <\| \mu \|^{-1}$. By the Open Mapping Theorem, b_0 is an interior of $\mathbb{C} - M_p \cup O$, that is, $\mathbb{C} - M_p \cup O$ is an open set.

Corollary 2.1. *For any* $b \in \mathbb{C} - M_p \cup O$, *the Julia set* $J(f_b)$ *of* f_b *is contained in a quasidisk and hence is null measure.*

Proof. For any $b \in \mathbb{C} - M_p \cup O$ and b_1 lying on the y axis, there is a quasiconformal homeomorphism $h : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ such that $h \circ f_b = f_{b_1} \circ h$. $J(f_b) = h^{-1}(J(f_{b_1}))$ is contained in a quasidisk.

Proposition 2.2. *The analytic* $f_b(z) = z+1/z+b$ *is structural stable on sets* $\mathbb{C} - M_p \cup O$ *and* $W - \{$ *the center* $\}$, *where* W *is a hyperbolic component of* M_p.

Proof. From Theorem 1.2, Proposition 2.1 and Theorem C in [16], this proposition is a straight consequence.

Proposition 2.3. *Let* $A \in D$ *be an open annulus of infinite modulus. Then the bounded component of* $\mathbb{C} - A$ *is a point.*

Proposition 2.4. *The set* $U = \{b \in \mathbb{C} \,|\, J(f_b)$ *is a Cantor set* $\} = \mathbb{C} - M_p$ *is open.*

Proof. From the proof of Theorem 1.1, the map $E : R - M_p \cup O \longrightarrow \text{int}(K) - \Delta \cup \{z \in \text{int}(K) - \Delta | Q^m(z) = Q^n(w_+)$ for some $m \geq n > 0\}$ is conformal. For any $b \in R \cap O$, there are integers $m \geq n > 0$ such that $f_b^m(v_-) = f_b^n(v_+)$. Then $w = E(b)$ satisfies $Q^m(w) = Q^n(w_+)$, $m \geq n > 0$, $w \in \text{int}(K) - \Delta$. There is a simply connected neighborhood V of w so that $V \subset \text{int}(K) - \Delta$ and

$$(V - \{w\}) \cap \{z \in \text{int}(K) - \Delta | Q^m(z) = Q^n(w_+) \text{ for some } m \geq n > 0\} = \emptyset.$$

$E : E^{-1}(V - \{w\}) \longrightarrow V - \{w\}$ is conformal. $E^{-1}(V - \{w\})$ is an open annulus of infinite modulus. Proposition 2.4. implies that $E^{-1}(V - \{w\})$ is a punctured disk and is contained in $R - M_p \cup O$. The continuity of E implies $b = E^{-1}(w)$. b is an interior of $R - M_p$. Combine this with Proposition 2.1, we conclude that U is an open set.

Proposition 2.5. *The analytic family* $f_b(z) = z + 1/z + b$ *is J-stable on the set* $U = \mathbb{C} - M_p$. *The measure of* $J(f_b)$ *is null for* $b \in U$.

Proof. By the Theorem B in [16], the analytic family $f_b(z) = z + 1/z + b$ is J-stable on the set U. Moreover, if $b_0 \in U$, there exists a neighborhood v_0 in U of b_0 and a continuous for $m, n \geq 0$. The Riemann surface $S(f)$ has one component for each cycle region together with special marked points corresponding to orbits of critical points in the regions. For the missing detail, see [7].

Let $T(f)$ denote the Cartesian product of the polydisk of the Julia set and the Teichmuller space of $S(f)$.

In [7], D. Sullivan has proved the following theorem.

Theorem. *The Teichmuller space* Teich(f) *of quasiconformal deformations of a rational map f is in canonical one to one correspondence with the complex manifold $T(f)$.*

From $M(f) = \text{Teich}(f)/\text{MCG}(f)$ and the above theorem, it follows that $\text{MCG}(f) = \pi_1(M(f))$.

For generic, shift-like quadratic rational maps, the complete presentations of the mapping class group are given in [13].

In this section, we describe the mapping class group of the quadratic map with form $f_b(z) = z + 1/z + b$.

Let f be a rational map, Aut(f) be the group of Moebius transformations commuting with f. It is a finite group.

For any $f_b(z) = z+1/z+b$, $b \neq 0$, it is easy to check that $\text{Aut}(f_b) = id$. $\text{Aut}(f_0) = \{z, -z\}$.

Theorem 3.1. *The mapping class group* MCG(f_b) *has no non-trival subgroup for any* $b \in \mathbb{C}$.

Proof. $\text{MCG}(f_0) = \pi_1(M(f_0)) = id$. Let G be a finite subgroup of $\text{MCG}(f_b), b \neq 0$. There exists a rational map g, quasiconformally conjugate to f, such that G is realized as a subgroup of Aut(g)[21]. g is PSL$(2, \mathbb{C})$ conjugate to $f_{b'}$ for some $b' \in \mathbb{C}^*$.

$$\text{Aut}(g) = \text{Aut}(f_{b'}) = id.$$

Therefore, $G = id$.

Theorem 1.1. gives a conformal homeomorphism E between $R - M_p$ and $\text{int}(K) - \Delta$.

We conclude that the b's satisfying orbit relations are isolated points in $\mathbb{C} - M_p$ which accumulate on M_p. Therefore $\pi_1(\mathbb{C} - M_p \cup O)$ is an infinitely generated free group, which is generated by

(i) one loop enclosing each orbit relation,

(ii) one loop enclosing M_p, denoted by γ_0.

It is certainly possible to choose these loops of (i) so that each is contained either in right or in left half-plane. If $\{\gamma_j\}$ is the set of generators for $\pi_1(R - M_p \cup O)$, then $\{\gamma_j, -\gamma_j, \gamma_0\}$ are generators of $\pi_1(\mathbb{C} - M_p \cup O)$.

If $b \in \mathbb{C} - M_p \cup O$, $M(f_b) = \mathbb{C} - M_p \cup O$; if $b \in O$, $M(f_b)$ is a single point.

Let W be a hyperbolic component of M_p. Then $M(f_b) = W - \{$ the center $\}$ for $b \in W - \{$ the center $\}$; $M(f_b)$ is a single point for b being the center of W.

If f_{b_0} has an indifferent cycle of period k, we claim that $M(f_{b_0})$ is a single point.

For any quadratic rational map $g(z) = z + 1/z + b'$ quasiconformal conjugating to f_{b_0}, i.e., $[g] \in M(f_{b_0})$. Let φ be the quasiconformal mapping, $\varphi \circ g = f_{b_0} \circ \varphi$. Denote by μ(z) the Beltrami coefficient. By the Ahlfors-Bers theorem, there is a unique quasiconformal mapping φ_t with Beltrami coefficient $t\mu(z)$ so that $\varphi_t^{-1} \circ f_{b_0} \circ \varphi_t(z) = z + 1/z + b(t)$ for $|t| < \| \mu \|^{-1}$. $b(t)$ is analytic on $|t| < \| \mu \|^{-1}$, $b(0) = b_0$ and $b(1) = b'$. If $b' \neq b_0$, $b(|t| < \| \mu \|^{-1})$ is an open set. f_b has an indifferent cycle of period k for $b \in b(|t| <\| \mu \|^{-1})$. It is impossible. Therefore, $b = b_0$, i.e., $M(f_{b_0})$ is a single point.

If b is a Misurewicz point, the same result can be showed by using the above methods.

We summarize the analysis above in

Theorem 3.2. *The mapping class group* MCG(f_b) *of* f_b *is*

(i) $\mathrm{MCG}(f_b) = \pi_1(\mathbb{C} - M_p \cup O)$ *is infinitely generated with generators* $\{\gamma_j, -\gamma_j, \gamma_0\}$ *for* $b \in \mathbb{C} - M_p \cup O$; $\mathrm{MCG}(f_b) = id.$ *for* $b \in O$.

(ii) *Let* W *be a hyperbolic component of* M_p. *Then* $\mathrm{MCG}(f_b) = Z$ *for* $b \in W - \{$*the center*$\}$; $\mathrm{MCG}(f_b) = id.$ *for* b *being the center of* W.

(iii) $\mathrm{MCG}(f_b) = id.$ *if* f_b *has an indifferent cycle or* b *is a Misurewicz point.*

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