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COMPLETE EXTREMAL SURFACES OF MIXED TYPE IN 3-DIMENSIONAL MINKOWSKI SPACE**

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Abstract

Many complete extremal surfaces of mixed type are constructed with explicit expressions. It is proved that there exist complete extremal surfaces of mixed type which have a given number of time-like spans and a given number of annular ends.

Keywords Extremal surface, Mixed type, Minkowski space.
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§1. Introduction

The extremal surfaces in Minkowski space are C^2-surfaces with vanishing mean curvature. Space-like case and time-like case have been investigated by many authors (eg. [1–5] for space-like case and [6–8] for time-like case). An extremal surface of mixed type (ESMT) is a connected extremal surface which contains space-like part and time-like part simultaneously. These surfaces are determined by a quasilinear partial differential equation which is elliptic on the space-like part and hyperbolic on the time-like part. In previous papers [9,10,11], the equation for extremal surfaces is linearized via the Legendre transformation (or by using the generalized isothermal coordinates), provided that there are no flat points. Moreover, explicit expressions for these surfaces are obtained. It has been shown that (i) the bordlines between the space-like and the time-like parts should be analytic null curves, (ii) the surfaces are analytic around the bordlines, even on the time-like regions. A method to construct ESMT globally was proposed. However, only a few examples have been worked out. In the present paper a series of complete ESMT are given explicitly. Here a complete surface is an immersed C^2 surface in $R^{2,1}$ and its boundary is empty. Moreover, the possible geometrical structure is discussed in detail. Just like the Euclidean case, an ESMT may have space-like ends. Here an end means a part of the surface which extends to infinity and the normals to this part cover a neiborhood of a time-like direction and omit the direction itself. Besides, an ESMT may have time-like spans each of which is bounded by a null curve and extends to infinity. It is shown that there exist ESMT with any number of space-like annular ends[12] and any number of time-like spans. It is also found that there are infinite number of real algebraic rational surfaces which are complete ESMT.

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In § 2, we recall briefly the previous results with some additional remarks. § 3 is devoted to explicit examples. In § 4, some lemmas which are useful in the global constructions of complete ESMT are given. In § 5 we analyse firstly the case when two space-like components appear. Afterwards, we construct ESMT with given numbers of time-likes spans and space-like annular ends. In the construction, we need to eliminate the residues of some contour integrations so as to make the ends annular. In these cases there is only one space-like region in each ESMT. The appendix A is devoted to the elimination of the residues of these contour integrations and appendix B contains figures of some complete ESMT.

§ 2. General Expressions of Extremal Surfaces of Mixed Type

The partial differential equation for extremal graph $z = f(x, y)$ in $R^{2,1}$ is

$$(1 - p^2)t + 2pqs + (1 - q^2)r = 0 \tag{2.1}$$

and the first fundamental form of a surface in $R^{2,1}$ is

$$ds^2 = (1 - p^2)dx^2 - 2pqdxdy + (1 - q^2)dy^2. \tag{2.2}$$

Here

$$p = f_x, \quad q = f_y, \quad r = f_{xx}, \quad s = f_{xy}, \quad t = f_{yy}. \tag{2.3}$$

By using the Legendre transformation

$$\varphi(p, q) = px + qy - z, \qquad x = \varphi_p, \quad y = \varphi_q \tag{2.4}$$

(2.1) is linearized as

$$(1 - p^2)\varphi_{pp} - 2pq\varphi_{pq} + (1 - q^2)\varphi_{qq} = 0, \tag{2.5}$$

provided that $rt - s^2 \neq 0$. It has been seen that x, y, z, as functions of p, q satisfy the same equation

$$(1 - p^2)\psi_{pp} - 2pq\psi_{pq} + (1 - q^2)\psi_{qq} - 2p\psi_p - 2q\psi_q = 0. \tag{2.6}$$

In the region $p^2 + q^2 > 1$, where the surface is time-like, equation (2.6) can be tranformed to

$$\psi_{\xi\eta} = 0, \tag{2.7}$$

where

$$\xi = \theta + \cos^{-1}\frac{1}{\rho}, \quad \eta = \theta - \cos^{-1}\frac{1}{\rho} \tag{2.8}$$

and (ρ, θ) are the polar coordinates of the (p, q)-plane. Since x, y and z are related by (2.4), we have the general expression for the time-like part of the surface:

$$\begin{aligned} z &= \frac{1}{2}\left(\int f(\xi)d\xi + \int g(\eta)d\eta\right), \\ x &= \frac{1}{2}\left(\int f(\xi)\cos\xi d\xi + \int g(\eta)\cos\eta d\eta\right), \\ y &= \frac{1}{2}\left(\int f(\xi)\sin\xi d\xi + \int g(\eta)\sin\eta d\eta\right). \end{aligned} \tag{2.9}$$

In the space-like region $p^2+q^2<1$, equation (2.6) can be transformed to

$$\psi_{\zeta\bar{\zeta}}=0, \tag{2.10}$$

where

$$\zeta=\theta+i\sigma=\theta-i\cosh^{-1}\frac{1}{\rho}. \tag{2.11}$$

The general expression for the space-like part is

$$\begin{aligned} z&=\mathrm{Re}\left(\int h(\zeta)d\zeta\right)=\int\{\alpha(\theta,\sigma)d\theta-\beta(\theta,\sigma)d\sigma\},\\ x&=\mathrm{Re}\left(\int h(\zeta)\cos\zeta d\zeta\right)\\ &=\int\{(\alpha(\theta,\sigma)\cos\theta\cosh\sigma+\beta(\theta,\sigma)\sin\theta\sinh\sigma)d\theta\\ &\quad+(\alpha(\theta,\sigma)\sin\theta\sinh\sigma-\beta(\theta,\sigma)\cos\theta\cosh\sigma)d\sigma\}, \qquad (\sigma<0).\\ y&=\mathrm{Re}\left(\int h(\zeta)\sin\zeta d\zeta\right)\\ &=\int\{(\alpha(\theta,\sigma)\sin\theta\cosh\sigma-\beta(\theta,\sigma)\cos\theta\sinh\sigma)d\theta\\ &\quad+(-\alpha(\theta,\sigma)\cos\theta\sinh\sigma-\beta(\theta,\sigma)\sin\theta\cosh\sigma)d\sigma\}, \end{aligned} \tag{2.12}$$

Here $h(\zeta)=\alpha+i\beta$ is an analytic function of ζ. (2.12) is a map from $\mathrm{Im}\,\zeta<0$ (or a part of it) to the surface. The map may be multi-valued.

Along the bordline of the time-like part and the space-like part, we have

$$\frac{\partial z}{\partial\theta}=\frac{1}{2}\left(f(\theta)+g(\theta)\right),\quad \frac{\partial z}{\partial\theta}=\alpha(\theta) \tag{2.13}$$

from the two sides respectively, and hence $2\alpha(\theta)=f(\theta)+g(\theta)$. On the time-like part, we have

$$\frac{\partial z}{\partial\rho}=\frac{1}{2}(f(\xi)-g(\eta))\frac{1}{\rho(\rho^2-1)^{1/2}}. \tag{2.14}$$

It should be finite as $\rho\to1$, since the surface is assumed to be C^2. Hence

$$f(\theta)=g(\theta). \tag{2.15}$$

Similarly, on the space-like part we have

$$\frac{\partial z}{\partial\rho}=\beta\frac{1}{\rho(1-\rho^2)^{1/2}} \tag{2.16}$$

and hence $\beta=0$ when $\sigma=0$. Thus $h(\zeta)$ is an analytic function and real-valued when ζ is real. Consequently $f(\theta)=g(\theta)=h(\theta)$ is a real analytic function of θ and $h(\zeta)$ is the analytic continuation of $f(\theta)$ and will be denoted by $f(\zeta)$.

Theorem 2.1. *If an ESMT has no flat point on the bordline C, then the surface is analytic around C. Moreover, the bordline is an analytic null curve*

$$z=\int f(\theta)d\theta,\quad x=\int f(\theta)\cos\theta d\theta,\quad y=\int f(\theta)\sin\theta d\theta. \tag{2.17}$$

Here $f(\theta)$ is a real analytic function.

Let C_0 be an analytic curve in the Eucleden plane $R^2 = \{(x, y)\}$, θ be the angle between the tangent line and the x-axis and $f(\theta)$ be the radius of curvature. Then C_0 is the orthogonal projection of the bordline C and C is obtained from C_0 by the lift

$$z = \int f(\theta) d\theta = s,$$

where s is the arc length of the curve C_0. From the general expressions (2.9) and (2.12) with $g(\theta) = f(\theta)$, $h(\zeta) = f(\zeta)$ and Theorem 2.1, we have

Theorem 2.2. *Any ESMT free of flat points on the bordlines can be obtained from a plane analytic curve C_0 by using the following algorithm:*

(1) *Lift it to obtain a null curve C.*

(2) *Extend the curve C to obtain the time-like part of the surface by formula* (2.9) *with $g(\eta) = f(\eta)$.*

(3) *Extend the curve C to obtain the space-like part of the surface by formula* (2.12). *Here $h(\zeta)$ is the analytic continuation of $f(\theta)$.*

The time-like extension and space-like extension of an analytic null curve C

$$x = a(\tau), \quad y = b(\tau), \quad z = c(\tau) \quad \left(c(\tau) = \int \left[\left(\frac{da}{d\tau} \right)^2 + \left(\frac{db}{d\tau} \right)^2 \right]^{1/2} d\tau \right) \tag{2.18}$$

are defined as

$$\begin{aligned} x &= \frac{1}{2}(a(\tau + \upsilon) + a(\tau - \upsilon)), \\ y &= \frac{1}{2}(b(\tau + \upsilon) + b(\tau - \upsilon)), \\ z &= \frac{1}{2}(c(\tau + \upsilon) + c(\tau - \upsilon)) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} x &= \mathrm{Re}(a(\tau + i\upsilon)), \\ y &= \mathrm{Re}(b(\tau + i\upsilon)), \\ z &= \mathrm{Re}(c(\tau + i\upsilon)) \end{aligned} \tag{2.20}$$

respectively. The space-like extention is possible since $a(\tau)$, $b(\tau)$ and $c(\tau)$ are analytic functions of τ.

Remark 2.1. In the deduction of Theorem 2.2, we use the special parameters θ to construct the space-like and time-like extensions. If the parameter θ is replaced by another analytic parameter τ, i.e., $\theta = \theta(\tau)$, where $\theta(\tau)$ is an analytic function of τ and $\theta'(\tau) \neq 0$, the surface obtained by (2.19) and (2.20) is the same one.

Remark 2.2. Let

$$\mu = \begin{cases} \upsilon^2, \\ -\upsilon^2, \end{cases}$$

the time-extension and the space-extension give a unified expression of the surface

$$x = a(\theta, \mu), \quad y = b(\theta, \mu), \quad z = c(\theta, \mu),$$

here a, b, c are analytic functions of θ and μ (or τ and μ).

Remark 2.3. If we eliminate θ, μ from the time-like extension (or space-like extension) and obtain the analytic expression of the surface in the form

$$F(x, y, z) = 0,$$

then it is an expression for the whole surface, since the surface is analytic and $F(x, y, z)$ is an analytic function.

Remark 2.4. The above results are obtained from the case of graphs, but they are valid for the general case. The construction is also valid for the case when the bordline contains an arc of nonflat points.

§ 3. Examples

We have the following complete ESMT as examples. They have simple expressions. They are useful for understanding the general case.

Example 3.1. The plane curve C_0 is the unit circle[9]

$$x = \cos\theta, \quad y = \sin\theta.$$

The radius of the curvature is $f(\theta) = 1$.

The null space curve is the helix

$$x = \cos\theta, \quad y = \sin\theta, \quad z = \theta.$$

The time-like extension is

$$\begin{aligned} x &= \frac{1}{2}(\cos(\theta + \upsilon) + \cos(\theta - \upsilon)) = \cos\theta\cos\upsilon, \\ y &= \frac{1}{2}(\sin(\theta + \upsilon) + \sin(\theta - \upsilon)) = \sin\theta\cos\upsilon, \\ z &= \theta. \end{aligned}$$

Eliminating υ, we obtain the whole surface

$$z = \tan^{-1}\frac{y}{x}. \tag{3.2}$$

It is a helicoid, the locus of spiral motion of a straight line. The surface contains another null light-like curve $x = -\cos\theta$, $y = -\sin\theta$, $z = \theta$. There are two space-like regions:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \theta \qquad (r > 1)$$

and

$$x = -r\cos\theta, \quad y = -r\sin\theta, \quad z = \theta \qquad (r > 1)$$

and one time-like region. The situation is typical when the plane curve C_0 is convex and closed.

Example 3.2. The plane curve C_0 is the curve[10]

$$y = \cosh x. \tag{3.3}$$

The lift is defined by

$$z = s = \int (1 + y'^2)^{1/2} dx = \sinh x.$$

The time-like extension is

$$z = \sinh x \cosh \upsilon, \quad y = \cosh x \cosh \upsilon, \quad x = x.$$

The whole surface is

$$z = y \tanh x. \tag{3.4}$$

There is another null light-like curve

$$y = -\cosh x, \quad z = -\sinh x.$$

There are two time-like regions, called time-like spans, which are bounded by the convex cylinder $y = \pm \cosh x$ respectively and extend to infinity. In this example there is only one space-like region. In general, a time-like span is an unbounded time-like region of a surface bounded by a null curve and its projection to the x-y plane is a convex curve.

In this example the radius of curvature of the plane curve C_0 is $f(\theta) = \sec^2 \theta$. In the real axis it has poles at $\theta = \pm \frac{2n+1}{2}\pi$ $(n = 0, 1, 2, \cdots)$. Since the expressions

$$\mathrm{Re}\left\{\int f(\zeta) d\zeta\right\}, \quad \mathrm{Re}\left\{\int f(\zeta) \cos \zeta d\zeta\right\}, \quad \mathrm{Re}\left\{\int f(\zeta) \sin \zeta d\zeta\right\} \tag{3.5}$$

have period 2π, we need only $-\frac{\pi}{2} \leq \theta < \frac{3\pi}{2}$. The two time-like spans are the images of the characteristic triangles based on $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

Example 3.3. The plane curve C_0 is

$$y = -\ln(1 - x^2), \quad |x| < 1, \tag{3.6}$$

the lift is defined by

$$z = -x + \ln \frac{1+x}{1-x}, \quad |x| < 1.$$

The time-like extension is

$$\begin{aligned} &x = x, \\ &y = -\frac{1}{2} \ln((1 - x^2 - \upsilon^2)^2 - 4x^2\upsilon^2), \\ &z = -x + \ln \left[\frac{(1+x)^2 - \upsilon^2}{(1-x)^2 - \upsilon^2}\right]. \end{aligned}$$

Eliminating υ^2, we obtain the equation of the whole surface

$$z = -x + \sinh^{-1}(2xe^y). \tag{3.7}$$

There are three time-like spans bounded by the convex cylinders

$$y = -\ln |(1 - x^2)|, \quad (|x| < 1,\ x > 1 \text{ and } x < -1)$$

respectively. In this example

$$f(\theta) = \frac{1}{\cos \theta (\cos \theta + 1)}.$$

The three expressions in (2.12) are of period 2π and the time-like spans are the images of the characteristic triangles based on the intervals $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\frac{\pi}{2} < \theta < \pi$, $\pi < \theta < \frac{3\pi}{2}$ on the axis $\sigma = 0$.

Example 3.4. The plane curve C_0 is

$$y = -\ln\cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}. \tag{3.8}$$

Its lift is determined by

$$z = \int \sec x dx = \ln(\sec x + \tan x).$$

The time-like extension has expression

$$\begin{aligned} y &= -\frac{1}{2}\ln(\cos(x+v)\cos(x-v)), \\ z &= \frac{1}{2}\ln\left\{(\sec(x+v)+\tan(x+v))(\sec(x-v)+\tan(x-v))\right\}. \end{aligned}$$

Eliminating v, we obtain the equations for the whole surface[13]

$$e^z = \sin x e^y + \sqrt{e^{2y}\sin^2 x + 1},$$

or

$$z = \sinh^{-1}(e^y \sin x). \tag{3.9}$$

z is a periodic function of x and there are infinite number of time-like spans. In this case $f(\theta) = \sec\theta$. There is one space-like component.

Example 3.5. Let $f(\theta) = \dfrac{1}{(1-\cos\theta)^2}$. By integration the null curve has the expressions

$$z = -\frac{1}{6}\lambda^3 - \frac{1}{2}\lambda, \quad x = -\frac{1}{6}\lambda^3 + \frac{1}{2}\lambda, \quad y = -\frac{1}{2}\lambda^2 \quad (\lambda = \cot\frac{\theta}{2}). \tag{3.10}$$

The time-like part has the expression

$$\begin{aligned} x &= \frac{\lambda}{2}\left(-\frac{\lambda^2}{3} - v^2 + 1\right), \\ y &= -\frac{1}{2}(\lambda^2 + v^2), \\ z &= \frac{\lambda}{2}\left(-\frac{\lambda^2}{3} - v^2 - 1\right). \end{aligned}$$

Eliminating λ and v^2, we obtain the algebric surface of third degree

$$y = -\frac{2}{3}(x-z)^2 - \frac{1}{2}\left(\frac{z+x}{z-x}\right). \tag{3.11}$$

There is one time-like span bounded by the convex cylinder

$$x^2 = -\frac{2}{9}y^3 - \frac{2}{3}y^2 - \frac{1}{2}y.$$

The time-like span is the image of the characteristic triangle based on $(0, 2\pi)$.

In these examples, we see that if $f(\theta)$ is regular on $(-\infty, \infty)$, it is possible to have more space-like regions and if $f(\theta)$ has poles on the real axies, then there may be time-like spans of finite or infinite number.

§ 4. Some Lemmas

We give a few lemmas which are useful in the construction of complete ESMT.

Lemma 4.1. *If the real analytic function $f(\theta)$ has a zero point, then the surface has a point where the surface is not of C^2.*

Proof. From (2.9) and $f(\theta) = g(\theta) = h(\theta)$, on the time-like part we have

$$\begin{aligned} dz &= \frac{1}{2}(f(\xi)d\xi + f(\eta)d\eta), \\ dx &= \frac{1}{2}(f(\xi)\cos\xi d\xi + f(\eta)\cos\eta d\eta), \\ dy &= \frac{1}{2}(f(\xi)\sin\xi d\xi + f(\eta)\sin\eta d\eta). \end{aligned} \tag{4.1}$$

Eliminating $f(\xi)d\xi$ and $f(\eta)d\eta$, we obtain

$$p = \frac{\cos\dfrac{\eta+\xi}{2}}{\cos\dfrac{\eta-\xi}{2}}, \quad q = \frac{\sin\dfrac{\eta+\xi}{2}}{\cos\dfrac{\eta-\xi}{2}}. \tag{4.2}$$

Besides, we have

$$d\xi = \frac{2}{f(\xi)}\frac{\sin\eta dx - \cos\eta dy}{\sin(\eta-\xi)}, \quad d\eta = \frac{2}{f(\eta)}\frac{\sin\xi dx - \cos\xi dy}{\sin(\xi-\eta)}. \tag{4.3}$$

Express dp and dq by dx and dy, it is easily seen that t is unbounded when $\rho \to 1$ and $\theta \to 0$.

Similarly, we can prove

Lemma 4.2. *If the function $f(\theta + i\sigma)$ has a zero at $\theta_0 + i\sigma_0$ ($\sigma_0 \neq 0$), then, the surface is not of C^2 at that point corresponding to $(\theta_0 + i\sigma_0)$ in the space-like part.*

Lemma 4.3. *If the real analytic function $f(\theta)$ is not a periodic function of period 2π and is regular in an interval $[a, b]$ with $b - a > 2\pi$, then, the surface has an edge where the surface is not of C^2.*

Proof. Suppose that $f(\theta)$ is regular in the interval $[-\pi - \epsilon, \pi + \epsilon]$. Write (2.9) in the form

$$Z = F(\xi) + F(\eta), \quad x = F_1(\xi) + F_1(\eta), \quad y = F_2(\xi) + F_2(\eta), \tag{4.4}$$

where

$$\begin{aligned} F(\xi) &= \frac{1}{2}\int_0^\xi f(\xi)d\xi, \\ F_1(\xi) &= \frac{1}{2}\int_0^\xi f(\xi)\cos\xi d\xi, \\ F_2(\xi) &= \frac{1}{2}\int_0^\xi f(\xi)\sin\xi d\xi. \end{aligned} \tag{4.5}$$

F, F_1, F_2 are regular on the interval $[-\pi - \epsilon, \pi + \epsilon]$. Let $\xi = \theta + \sigma$, $\eta = \theta - \sigma$, $(\pi > \sigma > 0)$. (4.4) is meaningful for $-\pi - \epsilon + \sigma < \theta < \pi + \epsilon - \sigma$ even for $\pi \geq \sigma \geq \frac{\pi}{2}$. That is to say that the time-like extension can cross the line at infinity of the (p, q)-plane. For the line $\sigma = \pi$, we have

$$
\begin{aligned}
z &= \frac{1}{2}\left(\int_0^{\theta+\pi} f(\xi)d\xi + \int_0^{\theta-\pi} f(\xi)d\xi\right) \\
&= \frac{1}{2}\left(\int_\pi^{\theta+\pi} f(\xi)d\xi + \int_\pi^{\theta+\pi} f(\xi-2\pi)d\xi\right) \\
&\quad + \frac{1}{2}\left(\int_0^{\pi} f(\xi)d\xi + \int_0^{-\pi} f(\xi)d\xi\right), \\
x &= \frac{1}{2}\left(\int_\pi^{\theta+\pi} f(\xi)\cos\xi d\xi + \int_\pi^{\theta+\pi} f(\xi-2\pi)\cos\xi d\xi\right) \\
&\quad + \frac{1}{2}\left(\int_0^{\pi} f(\xi)\cos\xi d\xi + \int_0^{-\pi} f(\xi)\cos\xi d\xi\right), \\
y &= \frac{1}{2}\left(\int_\pi^{\theta+\pi} f(\xi)\sin\xi d\xi + \int_\pi^{\theta+\pi} f(\xi-2\pi)\sin\xi d\xi\right) \\
&\quad + \frac{1}{2}\left(\int_0^{\pi} f(\xi)\sin\xi d\xi + \int_0^{-\pi} f(\xi)\sin\xi d\xi\right),
\end{aligned} \tag{4.6}
$$

where $-\epsilon < \theta < \epsilon$. It is an null arc C'. Near the arc the surface has the expression

$$z = \frac{1}{2}\left(\int_0^{\theta+\sigma} f(\xi)d\xi + \int_0^{\theta-\sigma} f(\eta-2\pi)d\eta\right), \qquad \text{etc.}$$

Hence

$$\left.\frac{\partial z}{\partial \sigma}\right|_{\sigma=0} = \frac{1}{2}\left(f(\theta+\pi) - f(\theta-\pi)\right).$$

If $f(\xi) \neq f(\xi - 2\pi)$, from the proof of Theorem 2.1 it is seen that the surface is not of C^2 along the arc. Hence the surface has a C^1 boundary.

Remark 4.1. One can construct another ESMT from C' by the time-like extension and the space-like extension. But the surface obtained is not the original one except the case that $f(\theta) = f(\theta - 2\pi)$.

Lemma 4.4. *If $f(\theta)$ has a pole at $\theta = \theta_0$, then under the mapping (2.9) the image of the characteristic $\xi = \theta_0 + \epsilon$ and $\eta = \theta_0 - \epsilon$ on the (p, q)-plane tends to infinity as $\epsilon \to 0$.*

Proof. This is a direct consequence of (2.9)

§ 5. Construction of Complete EMST

At first, we assume that $f(\theta)$ is a real analytic function defined on $-\infty < \theta < +\infty$. From Lemme 4.3, we see that $f(\theta)$ should be of period 2π. For simplicity, we assume that $f(\theta + i\sigma)$ is meromorphic on the complex plane. From Lemma 4.2, $f(\theta)$ should have no zero. If $f(\zeta)$ has no pole, then the Gauss map omits the direction $(0, 0, -1)$ and the surface consists of infinite number of sheets which are congruent via translations. Let $\xi_\alpha = \theta_\alpha + i\sigma_\alpha$ ($\sigma_\alpha < 0$, $\alpha = 1, 2, \cdots, N$) be given on the complex plane on $0 \le \theta < 2\pi$, there exists entire function $m(\zeta)$ of period 2π such that $\{\xi_\alpha + 2n\pi; n = 0, \pm 1, \cdots\}$ is the set of zeros of $m(\zeta)$.

Let $f(\zeta)=\dfrac{1}{m(\zeta)}$. We can construct a complete ESMT from $f(\zeta)$ such that its Gauss map omits the direction $\left(\dfrac{\cos\theta_\alpha}{\cos\sigma_\alpha},\dfrac{\sin\theta_\alpha}{\cos\sigma_\alpha},-1\right)$. In fact, starting from the null curve (2.17), the space-like extention gives a space-like region of an ESMT where Gauss map omits the time-like direction $\left(\dfrac{\cos\theta_\alpha}{\cos\sigma_\alpha},\dfrac{\sin\theta_\alpha}{\cos\sigma_\alpha},-1\right)$. By using the time-like extension another null curve appears with the expressions

$$\begin{aligned} z&=\frac{1}{2}\left(\int_{-\pi}^{0}f(\xi+\pi)d\xi+\int_{\pi}^{0}f(\xi-\pi)d\xi\right)\\ &\quad+\int_{0}^{\theta}f(\xi+\pi)d\xi,\\ x&=\frac{1}{2}\left(\int_{-\pi}^{0}f(\xi+\pi)\cos\xi d\xi+\int_{\pi}^{0}f(\xi-\pi)\cos\xi d\xi\right)\\ &\quad-\int_{0}^{\theta}f(\xi+\pi)\cos\xi d\xi,\\ y&=-\frac{1}{2}\left(\int_{-\pi}^{0}f(\xi+\pi)\sin\xi d\xi+\int_{\pi}^{0}f(\xi-\pi)\sin\xi d\xi\right)\\ &\quad-\int_{0}^{\theta}f(\xi+\pi)\sin\xi d\xi. \end{aligned} \tag{5.2}$$

We can make space-like extension as before, but $p=-\dfrac{\cos\theta}{\cos\sigma}$, $q=-\dfrac{\sin\theta}{\cos\sigma}$. On the other hand, the function $f(\xi+\pi)$ has poles at $\{\xi_\alpha+(2n+1)\pi; n=0,\pm1,\cdots\}$. Consequently, the Gauss map omits the directions $\left(\dfrac{\cos\theta_\alpha}{\cos\sigma_\alpha},\dfrac{\sin\theta_\alpha}{\cos\sigma_\alpha},-1\right)$ too. Thus we proved the theorem:

Theorem 5.1. *For a given set $\Sigma=\{\zeta_\alpha=\theta_\alpha+i\sigma_\alpha; 0\le\theta_\alpha<2\pi,\sigma_\alpha<0\}$ there exist complete ESMT whose Gauss maps omit the set Σ. Moreover, each of them has 2 space-like regions and 1 time-like region.*

For example, we may take

$$f(\theta)=\prod_\alpha\frac{1}{(\cos\theta-a_\alpha)^2+b_\alpha^2}\qquad(b_\alpha<0). \tag{5.3}$$

Here $a_\alpha+ib_\alpha=\cos(\theta_\alpha+i\sigma_\alpha)$. In this case, the complete ESMT can be expressed by elementary functions.

Let Ω_α be a small region of the complex plane (θ,σ) around ζ_α. Under the map (2.12), the image of $\Omega_\alpha\backslash\{\zeta_\alpha\}$ is called an end. If the integrals (2.12) along a contour Γ_α containing ζ_α inside are not zero, then the end is non-annular. In this case the map is multivalued. The image contains an infinite number of parts which are the image of $\Omega_\alpha\backslash\{\text{a cut from }\zeta_\alpha\}$ and congurent by a translation. Otherwise, the end is called annular, the image is single valued and it is homeomorphic to $S^1\times[1,0)$[12].

If we use (5.3) to construct the ESMT, then the ends are all non-simple. Instead, we can choose

$$f(\theta) = \left(\prod_{\alpha} \frac{1}{[(\cos\theta - a_\alpha)^2 + b_\alpha^2]^2} \right) e^{P(\cos\theta, \sin\theta)},$$

where P is a polynomial with real coefficients. We can find P such that the integrals (2.9) along the contour Γ_α's are all zero, i.e. the residues of the integrals of

$$f(\zeta), \qquad f(\zeta)\cos\zeta, \qquad f(\zeta)\sin\zeta$$

are real (see Appendix A). Hence all ends are annular, we have

Theorem 5.2. *There exist complete ESMT whose Gauss maps omit the given set Σ and each end is annular.*

We turn to the case that $f(\theta)$ has poles on $-\infty < \theta < \infty$. From Lemma 4.2, the distance of two neighbor poles should not be larger than 2π. Let these poles be θ_α $(\alpha = 0, \pm 1, \pm 2, \cdots)$. It is well known that there are integral functions without zero in the complex plane such that the only poles are θ_α $(\alpha = 0, \pm 1, \pm 2, \cdots)$ and they are real valued on the real axies. Let $f(\zeta)$ be one of them, we obtain complete ESMTs with time-like spans which are the images of the characteristic triangle based on the intervals $(\theta_\alpha, \theta_{\alpha+1})$ $\alpha = 0, \pm 1, \pm 2, \cdots$ on $\sigma = 0$.

We choose $f(\zeta)$ suitably such that the integrals (2.12) are of period 2π. For example, we take $0 = \theta_0 < \theta_1 < \cdots \theta_{N-1} < \theta_N = 2\pi$ and

$$f(\theta) = \frac{1}{\prod_{\alpha=0}^{N} (1 - \cos(\theta - \theta_\alpha))}. \tag{5.4}$$

By using the substitution of the variables

$$t = \tan\frac{\theta}{2},\ a_\alpha = \tan\frac{\theta_\alpha}{2},\ t_\alpha = \tan\frac{\theta - \theta_\alpha}{2} = \frac{t - a_\alpha}{1 + t a_\alpha}, \quad (\alpha = 1, 2, \cdots, N). \tag{5.5}$$

The integrands of the integrals (2.12) which are expressed as functions of t are rational functions with respect to t. Moreover, their poles are all real and the integrals (2.12) are periodic functions of θ. Thus we obtain

Theorem 5.3. *There exist complete ESMT with N time-like spans and one space-like region.*

We take

$$f(\theta) = \prod_{k=1}^{m} \frac{1}{(\cos\theta - a_k)^2 + b_k^2} \prod_{\alpha=0}^{N} \frac{1}{(1 - \cos(\theta - \theta_\alpha))}. \tag{5.6}$$

Here $a_k + i b_k = \cos(\theta_k + i\sigma_k)$ $\quad (\sigma_k < 0)$. Then the complete ESMT constructed has N time-like spans and its Gauss map omits the set of time-like directions $\left(\dfrac{\cos\theta_k}{\cos\sigma_k}, \dfrac{\sin\theta_k}{\cos\sigma_k}, -1 \right)$.

If we want to obtain annular-ends only, then we should change (5.6). For example, we take

$$f(\theta) = \left(\prod_{k=1}^{m} \frac{1}{[(\cos\theta - a_k)^2 + b_k^2]^2} \prod_{\alpha=0}^{N} \frac{1}{1 - \cos(\theta - \theta_\alpha)} \right) e^{P(\tan\frac{\theta}{2})}.$$

Here P is a suitably chosen polynomial with real coefficients. Then the integrals (2.12) are all single-valued and of period 2π (see Appendix A). We obtain

Theorem 5.4. *There exist complete ESMT with N time-like spans and m annular ends, where m and N are any given nonnegative integers and $N>0$. The surface is the immersion of $R^2\backslash\Sigma$.*

Finally we take

$$f(\theta)=\frac{1}{(1-\cos\theta)^n},\quad n=2,3,\cdots. \tag{5.7}$$

It is easily seen that the complete ESMT constructed are real algebraic rational surfaces.

In fact, we have

$$\begin{aligned} f(\theta)d\theta&=\frac{(1+t^2)^{n-1}dt}{2^{n-1}t^{2n}},\\ f(\theta)\cos\theta d\theta&=\frac{(1+t^2)^{n-2}(1-t^2)dt}{2^{n-1}t^{2n}},\\ f(\theta)\sin\theta d\theta&=\frac{(1+t^2)^{n-2}dt}{2^{n-2}t^{2n-1}}. \end{aligned} \tag{5.8}$$

Hence, the bordline has the expression

$$z=f_3(\lambda),\quad x=f_1(\lambda),\quad y=f_2(\lambda)\qquad (\lambda=\frac{1}{t}).$$

Here, f_1, f_2, f_3 are polonomials of λ. From the expression of time-like part and the remark in (2.3), we see the surface is an algebraic surface.

Theorem 5.5. *There are infinite number of algebraic rational complete ESMT.*

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Appendix A. Eliminating the Residues of the Integrals (2.12)

Let $f(\zeta)$ have simple poles ζ_α ($\zeta_\alpha=\theta_\alpha+i\sigma_\alpha$, $\sigma_\alpha\neq 0$). If the principal part of $f(\zeta)$ for the pole ζ_α is

$$\frac{a_\alpha}{\zeta-\zeta_\alpha}, \tag{A1}$$

the residues of the integrals (2.12) are

$$\mathrm{Re}(2\pi i a_\alpha),\quad \mathrm{Re}(2\pi i a_\alpha\cos\zeta_\alpha),\quad \mathrm{Re}(2\pi i a_\alpha\sin\zeta_\alpha).$$

They can not be zero simultaneously. We suppose the principal parts of an analytic function $\psi(\zeta)$ around the double pole ζ_α to be

$$\frac{b_\alpha}{(\zeta-\zeta_\alpha)^2}+\frac{a_\alpha}{\zeta-\zeta_\alpha}\quad (b_\alpha\neq 0), \tag{A2}$$

and let $f(\zeta)=\psi(\zeta)G(\zeta)$. Here, $G(\zeta)$ is analytic and in the form $e^{P(\zeta)}$. Later we will take $P(\zeta)$ to be a polynomial of $e^{i\zeta}$. In order that the integrals (2.12) are single-valued, we want to have

$$\begin{aligned}&\mathrm{Re}\{2\pi i(a_\alpha G(\zeta_\alpha)+b_\alpha G'(\zeta_\alpha))\}=0,\\&\mathrm{Re}\{2\pi i[(a_\alpha G(\zeta_\alpha)+b_\alpha G'(\zeta_\alpha))\cos\zeta_\alpha-b_\alpha G(\zeta_\alpha)\sin\zeta_\alpha]\}=0,\\&\mathrm{Re}\{2\pi i[(a_\alpha G(\zeta_\alpha)+b_\alpha G'(\zeta_\alpha))\sin\zeta_\alpha+b_\alpha G(\zeta_\alpha)\cos\zeta_\alpha]\}=0.\end{aligned}\tag{A3}$$

They are linear homogenous equations of the four real unknowns, the real parts and complex parts of $G(\zeta_\alpha)$ and $G'(\zeta_\alpha)$. There is some $\lambda_\alpha \neq 0$ and μ_α except a nonvanishing real multiplier such that (A3) are satisfied by

$$G(\zeta_\alpha)=\lambda_\alpha(\neq 0),\quad G'(\zeta_\alpha)=\mu_\alpha \ \text{ or } \ P(\zeta_\alpha)=\ln\lambda_\alpha,\quad P'(\zeta_\alpha)=\frac{\mu_\alpha}{\lambda_\alpha}.\tag{A4}$$

Choose the degree of $P(\zeta)$ sufficiently high, the existence of $P(\zeta)$ such that all (A4) are satisfied is evident. If $\psi(\zeta)$ is real when $\zeta=$ real, then $\bar\zeta_\alpha$ are also poles with principal parts

$$\frac{\bar b_\alpha}{(\zeta-\bar\zeta_\alpha)^2}+\frac{\bar a_\alpha}{\zeta-\bar\zeta_\alpha}.$$

Besides (A4), we should have

$$P(\bar\zeta_\alpha)=\ln\bar\lambda_\alpha,\quad P'(\bar\zeta_\alpha)=\ln\frac{\bar\mu_\alpha}{\bar\lambda_\alpha}.\tag{$\overline{A}4$}$$

The existence of $P(\zeta)$ which is real-valued for real ζ is evident too. If some poles have higher order, we can do this in the same way.

Appendix B. Figures of Some ESMT

Here are figures of some ESMT given in this paper. In Figure 1, 2, 5, the darker part is the space-like part of the surface. In Figure 3, 4, the null line is drawn, where the convex side of the null line is the space-like part of the surface.

Figure 1. $x = u\cos v$, $y = u\sin v$, $z = v$. $|u| < 1$ is the time-like part of the surface, $|u| > 1$ is the space-like part of the surface.

Figure 2. $x = u\cosh v$, $y = v$, $z = u\sinh v$. $|u| < 1$ is the space-like part of the surface, $|u| > 1$ is the time-like part of the surface.

Figure 3. $z = -x + \sinh^{-1}(2xe^y)$. The null lines are given by $y = -\ln|1 - x^2|$, $z = -x + \ln\left|\dfrac{1+x}{1-x}\right|$, $(x^2 \neq 1)$.

Figure 4. $z = \sinh^{-1}(e^y \sin x)$. The null lines are given by $y = -\ln|\cos x|$, $z = \ln|(\sec x + \tan x)|$, $(x \neq k\pi,\ k \in Z)$.

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Figure 5. The figure is given by $x = \dfrac{u}{2}\left(-\dfrac{u^2}{3} - v + 1\right)$, $\quad y = \dfrac{1}{2}(u^2 + v)$, $\quad z = \dfrac{u}{2}\left(-\dfrac{u^2}{3} - v - 1\right)$. $v = 0$ corresponds to the null curve. It is an algebraic ruled surface without singularities. The part where $v > 0$ is the time-like part, and $v < 0$ is the space-like part.