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PROGRESSING WAVE SOLUTIONS TO QUASI-LINEAR SYSTEMS MIXED PROBLEMS**

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Abstract

The author studies the technique of paradifferential operator defined on a space of conormal distribution with three indeces, and then use this technique to prove that a progressing wave which hits the boundary is reflected according to the usual law.

Keywords Quasi-linear system, Mixed problem, Conormal distribution, Tangential Paradifferential operator, Reflection of progressing wave.

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M. Brals and G. Métivier[3,4] have studied the reflection of transversal progressing wave for semi-linear equations in higher dimensions. However, it has never been touched for quasi-linear equations. On the other hand, in the quasi-linear case, we have two noticeable paper, one is Métivier's on interaction of two shock waves and the other is Alinhac's on interaction of two progressing waves with weak singularities[9,2]. S. X. Chen[8] discussed the interaction of a shock wave and a progressing wave with weak singularities, but only in case of one dimension. As we know, the problem of interaction of a shock wave and progressing wave with weak singularities can be transformed into the reflection of progressing wave for quasi-linear systems with free boundary. So the discussion of reflection can be regarded as a preliminary step for the interaction of a shock wave and a progressing wave with weak singularities.

In this paper we consider the propagation of regularity when the solution to the mixed problem of 2×2 quasi-linear systems is conormal with respect to a single characteristic surface in the past. We show that if this characteristic surface hits the boundary of the domain transversally, and one reflected characteristic surface issues from the intersection, then the solution will be conormal with respect to the union of these surfaces.

The main result in the paper is described in section 1. The space of conormal distributions and para-diferential operators are discussed in section 2. We study the regularity of characteristic surfaces in section 3. Finally, the proof of the main theorem is completed in section 4 and section 5.

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conjugacy function $h : V_0 \times J(f_{b_0}) \longrightarrow \overline{\mathbb{C}}$ so that for all b in V_0

(i) h_b is a conjugacy between f_{b_0} on $J(f_{b_0})$ and f_b on $J(f_b)$ and h_{b_0} is identity;

(ii) for each z, $h_b(z)$ is analytic on b;

(iii) for each b, h_b is quasiconformal.

$h : V_0 \times J(f_{b_0}) \longrightarrow \overline{\mathbb{C}}$ is admissible.

Applying the extended λ-lemma[19,20], we see that h_b is the restriction to $J(f_{b_0})$ of a quasiconformal self-map H_b of $\overline{\mathbb{C}}$. $J(f_b) = H_b(J(f_{b_0}))$. Choose $b \in V_0 - O$, $J(f_b)$ is contained in a quasidisk. Hence $J(f_{b_0})$ is contained in a quasidisk, and the measure is null.

§3. The Mapping Class Group

D. Sullivan exploits the intimate connection between the classification problem of dynamical system and the theory of moduli for Riemann surfaces. In particular, he defines the mapping class group (MCG) for rational maps, and shows that the mapping class group of a generic rational map can be built from subgroups of the mapping class groups of punctured tori.

It is natural to identify two rational maps if they are conjugate by a Moebius transformation. For a rational map f, we denote by $M(f)$ the space of PSL$(2,\mathbb{C})$ conjugcy classes of rational maps which are quasiconformally (qc) conjugate to f.

Let $Q(f)$ be the space of qc homomorphisms $h : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ for which $h \circ f \circ f \circ h^{-1}$ is again a rationnal map. Homomorphisms h_0, h_1 are eauivalent if there is a Moebius transformation q and an isotopy betwen $q \circ h_0$ and h_1 through elements of $Q(f)$. The quotient space of $Q(f)$ by this equivalence relation is the Teichmuller space of f and we denote it by Teich(f).

Equivalent homeomoephisms in $Q(f)$ conjugate f to rational maps that are PSL$(2,\mathbb{C})$ conjugate to one another. So there is a projection

$$P : \text{Teich}(f) \longrightarrow M(f)$$

given by

$$P([h]) = [h \circ f \circ h^{-1}].$$

Let $Q_0(f) \subset Q(f)$ be the subgroup of quasiconformal homeomorphisms which commute with f. The mapping class group is the quotient $Q_0(f)$ obtained by identifying homeomorphisms which are isotopic to the identity through elements of $Q_0(f)$. It is often infinite and quite complicated.

This group acts naturally on the Teichmuller space of f. It acts properly discontinuously by complex analytic bijections on Teich(f)[7]. The action is given by

$$[g] \circ [h] = [h \circ g]$$

and the orbit of any $[h] \in \text{Teich}(f)$ is precisely a fiber of P.

This gives

$$M(f) \cong \text{Teich}(f)/\text{MCG}(f).$$

We associate to each rational map f a Riemann surface $S(f)$ which classifies the large orbits of f except for those touching the closure of all periodic points (which is the Julia set union a finite set). A large orbit is an equivalence class of the relation $x \sim y$ iff $f^m(x) = f^n(y)$

§1. Main Result

Let us consider a solution u of the problem

$$Lu = \sum_{j=1}^{3} A_j(u)\partial_j u = 0, \quad x_3 > 0, \tag{1.1}$$

$$\varphi(u) = 0, \quad x_3 = 0. \tag{1.2}$$

Here $\partial_j = \partial_{x_j}$ $(j = 1,2,3)$, $u = {}^t(u_1, u_2)$. Let $\Omega\,(\subset \mathbf{R}^3)$ be an open neighborhood which contains the origin and we take Ω small enough.

$$\Omega_+ = \Omega \cup \{x_3 \geq 0\},\quad b\Omega = \Omega \cap \Sigma_0,\ \Sigma_0 = \{x_3 = 0\},$$

$A_j \in C^\infty(\mathbf{R}^2)\,(j = 1,2,3)$ are 2×2 matrices, $\varphi \in C^\infty(\mathbf{R}^2)$, and $\varphi' \neq 0$. Let $u \in H^{s+1}_{\text{loc}}(\Omega_+)$ $(s > \frac{3}{2})$ be a solution of (1.1) and (1.2).

First, assume that

(H.1) The boundary $\partial\Omega$ is non-characteristic;

(H.2) The problem (1.1) and (1.2) satisfies the uniform Lopatinski condition.

Let Σ_- be a characteristic surface for L, which intersects the boundary Σ_0 in $\mathbf{R}^3$. Because Σ_0 is not characteristic, Σ_- intersects Σ_0 transversally along a curve we shall call Γ. And we assume that Γ goes through the origin in $\mathbf{R}^3$.

Second, assume further that

(H.3) There is one (real) reflected characteristic surface Σ_+ through Γ.

We denote $\Sigma_\pm = \{x_1 = \varphi_\pm(x_2, x_3)\}$, $\Gamma = \{x_3 = 0,\ x_1 = h(x_2)\}(h(x_2) = \varphi_-(x_2, 0))$, and assume $\Gamma \subset \Omega_+ \cap \{x_1 \geq 0\}$. It is easy to see

$$\varphi_+(x_2, 0) = \varphi_-(x_2, 0), \tag{1.3}$$

$$\partial_3(\varphi_+(x_2, x_3) - \varphi_-(x_2, x_3))|_{x_3=0} \neq 0. \tag{1.4}$$

Finally, we assume

(H.4) L is strictly hyperbolic with respect to the surface

$$2x_1 - \varphi_+(x_2, x_3) - \varphi_-(x_2, x_3) = \text{constant}.$$

Let S be a smooth surface. The space of conormal distributions $H^{s,k}_{\text{loc}}(\Omega, S)$ is the set of those $u \in H^s_{\text{loc}}(\Omega)$ such that $Z^I u \in H^s_{\text{loc}}(\Omega)$ for all $Z^I = Z_{i_1} \cdots Z_{i_m}$ with $m = |I| \leq k$, where Z_j are C^∞ vector fields in $\mathbf{R}^3$ tangent to surface S. And we define $H^{s,k}_{\text{loc}}(\Omega_+, S)$ to be the set of restrictions to Ω_+ of functions in $H^{s,k}_{\text{loc}}(\Omega, S)$. With these notations we can state our main result.

Theorem 1.1. *Suppose that $u \in H^{s+1}_{\text{loc}}(\Omega_+)$ $(s > \frac{3}{2} + 4)$ is a solution of (1.1) and (1.2), Σ_+, Σ_- are characteristic surfaces of class $H^\sigma_{\text{loc}}(\Omega_+)$ $(\sigma > \frac{3}{2} + 5)$. And we assume that*

1) Σ_- is a C^∞ surface when $x_1 < 0$;

2) $u \in H^{s+1,\infty}_{\text{loc}}(\Omega_+ \cap \{x_1 < 0\}, \Sigma_0)$ (resp. $u \in H^{s+1,\infty}_{\text{loc}}(\Omega_+ \cap \{x_1 < 0\}, \Sigma_-)$ near Σ_0 (resp. Σ_-);

3) $u \in H^\infty_{\text{loc}}(\Omega_+ \cap \{x_1 < 0\})$ except on $\Sigma_0 \cup \Sigma_-$.

Then there exists a small neighborhood ω of the origin such that

1) Γ is C^∞;

2) The surface Σ_+, Σ_- are C^∞ except on Γ;

3) $u \in H^{s+1,\infty}_{\rm loc}(\omega_+,\Sigma_+)$ (*resp.* $H^{s+1,\infty}_{\rm loc}(\omega_+,\Sigma_-)$, $H^{s+1,\infty}_{\rm loc}(\omega_+,\Sigma_0)$) *near* $\Sigma_+\backslash\Gamma$ (*resp.* $\Sigma_-\backslash\Gamma$, $\Sigma_0\backslash\Gamma$), *where* $\omega_+ = \omega \cap \{x_3 \geq 0\}$;

4) $u \in H^{\infty}_{\rm loc}(\omega_+)$ *away from* $\Sigma_0 \cup \Sigma_+ \cup \Sigma_-$.

Remark. In this paper, we only consider system (1.1) in two dimensions. In general, by the same method we can prove that Theorem 1.1 is valid for

$$Lu = \sum_{j=1}^{n} A_j(u)\partial_j u = 0, \quad x_n > 0, \tag{1.1)'$$

$$\varphi(u) = 0, \quad x_n = 0. \tag{1.2)'$$

We now recall some notation which will be used in this paper.

1) In this paper, we will use the "dyadic decomposition" (see [5, 12]). Let

$$1 = \psi(\xi) + \sum_{j=0}^{\infty}\varphi(2^{-j}\xi), \quad 1 = \psi'(\xi') + \sum_{j=0}^{\infty}\varphi'(2^{-j}\xi'),$$

where $\xi = (\xi', \xi_3)$, $\xi' = (\xi_1, \xi_2)$ are respectively dual variables of $x = (x', x_3)$, $x' = (x_1, x_2)$,

$$\varphi(2^{-j}\xi) = \psi(2^{-j-1}\xi) - \psi(2^{-j}\xi),$$
$$\psi'(\xi') = \psi(\xi', 0), \quad \varphi'(\xi') = \varphi(\xi', 0),$$

$\psi(\xi) \in C_0^\infty(\mathbf{R}^3)$ and supp$\psi \subset B(1)$, $\psi|_{B(r)} = 1\,(r < 1, B(l)$ means the ball of radius l). Then we have

$$u = \sum_{p=-1}^{\infty} u_p = \sum_{p=-1}^{\infty} u'_p, \tag{1.6}$$

where $u_{-1} = S_0 u = \psi(D)u$, $u'_{-1} = S'_0 u = \psi'(D')u$,

$$u_p = \Delta_p u = \varphi(2^{-p}D)u, \quad u'_p = \Delta'_p u = \varphi'(2^{-p}D')u,$$

$$S_p u = \sum_{q=-1}^{p-1} u_q = \psi(2^{-p}D)u, \quad S'_p u = \sum_{q=-1}^{p-1} u'_q = \psi(2^{-p}D')u.$$

2) Let us note that some result, for instance, Theorem 6.2 of [5] and Lemma II 1.1 of [13], can be used in the system with diagonal principal part as follows (see [6])

$$\begin{pmatrix} P & & & \\ & \cdot & & \\ & & \cdot & \\ & & & P \end{pmatrix}\begin{pmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_n \end{pmatrix} + B\begin{pmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_n \end{pmatrix} = R, \tag{1.7}$$

where P is an $m \times m$ matric paradifferential operator of order 1, U_j are column vectors with m components, B is an $nm \times nm$ matric paradifferential operator of order 0, R is an nm vector.

§2. Paradifferential Operators on Space $H^{s,k}_{s'}(\Omega_+, \Sigma')$

For $s, s' \in \mathbf{R}$, the Hörmander space $H^s_{s'}(\mathbf{R}^3)$ is defined as

$$H^s_{s'}(\mathbf{R}^3) = \{u \in \mathcal{S}'(\mathbf{R}^3);\ (1+|\xi|^2)^s(1+|\xi'|^2)^{s'}\hat{u}(\xi) \in L^2(\mathbf{R}^3)\}.$$

When $s' = 0$, $H^s_0(\mathbf{R}^3)$ is the Sobolev space. The space $H^s_{s',\rm loc}(\Omega)$ is defined by

$$H^s_{s',\rm loc}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ \varphi u \in H^s_{s'}(\mathbf{R}^3),\ \forall\varphi \in C_0^\infty(\Omega)\}.$$

For $\Omega_+ = \Omega \cap \{x_3 \geq 0\}$, we define $H^s_{s',\text{loc}}(\Omega_+)$ to be the space consisting of the restriction to Ω_+ of functions in $H^s_{s',\text{loc}}(\Omega)$.

Let $\Sigma' = \Sigma_0 \cup \Sigma'_+ \cup \Sigma_-$ be the union of 3 smooth hypersurfaces, where $\Sigma'_\pm = \{x_1 = \pm x_3\}$, and let $\mathcal{M}$ be the set of smooth vector fields tangent to Σ'.

The spaces of conormal distributions associated to Σ' are defined by

$$H^{s,k}_{s'}(\Omega_+, \Sigma') = \{u;\ Z^I u \in H^s_{s',\text{loc}}(\Omega),\ Z \in \mathcal{M},\ |I| \leq k\}. \tag{2.1}$$

Following M. Beals and G. Métivier[3], we can easily see that the generators of $\mathcal{M}$ are

$$\begin{cases} Z_0 = x_1\partial_1 + x_3\partial_3, & Z_1 = x_3(x_1 + x_3)(\partial_1 + \partial_3), \\ Z_2 = \partial_2, & Z_3 = x_3(x_1 - x_3)(\partial_1 - \partial_3). \end{cases} \tag{2.2}$$

For the space $H^{s,k}_{s'}(\Omega_+, \Sigma')$ we have

Lemma 2.1. *For $s > \frac{1}{2}$, $s + s' > \frac{3}{2}$, $s + 2s' > \frac{1}{2}$, $k \geq 0$, $H^{s,k}_{s'}(\Omega_+, \Sigma')$ is an algebra. Moreover if $u_1, \cdots, u_m \in H^{s,k}_{s'}(\Omega_+, \Sigma')$ and f is a C^∞ function of its arguments, then $f(x, u_1(x), \cdots, u_m(x)) \in H^{s,k}_{s'}(\Omega_+, \Sigma')$.*

Proof. For $k = 0$, this lemma is a simple consequence of Proposition 1.7 and 2.4 of [12]. For $k > 0$, we can easily prove this lemma by induction on k.

Let $a(x)$ be a bounded function. The tangential paraproduct operator is defined by (see [12])

$$T'_a u = \sum_{p \geq N_0} S'_{p-N_0} a(x) \Delta'_p u(x), \tag{2.3}$$

where x_3 is a parameter. For tangential paraproduct operator, we have

Lemma 2.2. *If $a \in H^{t,k}_{t'}(\Omega_+, \Sigma')$, $u \in H^{s,k}_{s'}(\Omega_+, \Sigma')$ $(t + t' > \frac{3}{2},\ t > \frac{1}{2},\ k \geq 1,\ -t < s \leq t)$, then we have*

$$ZT'_a u = T'_a Zu + T'_{Za} u + R_1(a, u) + R_2(a, u), \tag{2.4}$$

where $Z \in \mathcal{M}$, $R_1(a, u) \in H^{s,k}_{s'+\rho}(\Omega_+, \Sigma')$, $R_2(a, u) \in H^{s,k-1}_{s'+\rho+1}(\Omega_+, \Sigma')$, $\rho = \min(t + t' - \frac{3}{2}, t - \frac{1}{2})$.

Proof. a) Let $\tilde{v}(\xi', x_3)$ denote the partial Fourier transform of $v(x', x_3)$ with respect to $x' = (x_1, x_2)$, $\mathcal{F}^{-1}\psi'$ denote the inverse Fourier transform of $\psi'(\xi')$. Let us note

$$\begin{aligned} S'_p v(x) &= \int e^{ix'\xi'} \psi'(2^{-p}\xi') \tilde{v}(\xi', x_3) d\xi' \\ &= \int 2^{2p} \mathcal{F}^{-1}\psi'(2^p(x' - y')) v(y', x_3) dy'. \end{aligned}$$

By simple calculation, we see that

$$x_1 S'_p v(x) = S'_p(x_1 v(x)) + 2^{-p} v^\#_p, \tag{2.5}$$

$$x_1 \Delta'_p v(x) = \Delta'_p(x_1 v(x)) + 2^{-(p+1)} v^\#_{p+1}(x) - 2^{-p} v^\#_p,$$

where $\tilde{v}^\#_p = i\frac{\partial\psi}{\partial\xi_1}(2^{-p}\xi')\tilde{v}(\xi', x_3)$.

b) It is clear that the lemma holds for $Z_2 = \partial_2$. For Z_0, Z_1, Z_3 (see (2.2)), from (2.5) and (2.6) it follows that

$$Z_j T'_a u = T'_{Z_j a} u + T'_a Z_j u + R_j \quad (j = 0, 1, 3),$$

where

$$R_0 = \sum_{p\ge N_0} 2^{-(p-N_0)}(\partial_1 a)^{\#}_{p-N_0}u'_p - \sum_{p\ge N_0} 2^{-(p+1)}a'_{p-N_0}(\partial_1 u)^{\#}_{p+1} - S_0(a)2^{-N_0}(\partial_1 u)^{\#}_{N_0},$$

$$R_1 = \sum_{p\ge N_0} 2^{-(p-N_0)}(Z_0 a + \partial_1(x_3 - x_1)a + a)^{\#}_{p-N_0}u'_p - \sum_{p\ge N_0} 2^{-(p+1)}a'_{p-N_0}(Z_0 u + \partial_1(x_3 - x_1)u + u)^{\#}_{p+1} - S_0(a)2^{-N_0}(Z_0 u + \partial_1(x_3 - x_1)u + u)^{\#}_{N_0},$$

$$R_3 = \sum_{p\ge N_0} 2^{-(p-N_0)}(-Z_0 a + \partial_1(x_3 + x_1)a - a)^{\#}_{p-N_0}u'_p - \sum_{p\ge N_0} 2^{-(p+1)}a'_{p-N_0}(-Z_0 u + \partial_1(x_3 + x_1)u - u)^{\#}_{p+1} - S_0(a)2^{-N_0}(-Z_0 u + \partial_1(x_3 + x_1)u - u)^{\#}_{N_0}.$$

c) We will prove that $R_j = (j = 0,1,3)$ can be writen as a sum $R_{j,1} + R_{j,2}$, such that $R_{j,1} \in H^{s,k}_{s'+\rho}(\Omega_+, \Sigma')$, $R_{j,2} \in H^{s,k-1}_{s'+\rho+1}(\Omega_+, \Sigma')$. In fact, we only need to prove that

$$J = \sum_p 2^{-pl}(\partial_1^{l_1} A)^*_{p-N_0}(\partial_1^{l_2} B)^*_p \quad (l_1 + l_2 \le l - \delta,\ l \le k,\ \delta = 0 \text{ or } 1,\ k \ge 0) \tag{2.7}$$

belongs to $H^{s,k-\delta}_{s'+\rho+\delta}(\Omega_+, \Sigma')$ for $A \in H^{t,k-\delta}_{t'}(\Omega_+, \Sigma')$, $B \in H^{s,k-\delta}_{s'}(\Omega_+, \Sigma')$, where $(w)^*_p = \psi^*(2^{-p}D')w(x)$, and in this expression $\psi^*(\xi')$ denotes any function in the class of function

$$\Psi = \{\psi^*(\xi');\ \psi^* = \partial^h_{\xi_1}\psi'(\xi'),\ \ h \ge 1\}.$$

Since $\partial_{\xi_1}\psi^*(\xi')$ also belongs to this class, we also denote it by $\psi^*(\xi')$.

We will prove that $J \in H^{s,k-\delta}_{s'+\rho+\delta}(\Omega_+, \Sigma')$ by induction on k. For $k = \delta$ (note $\delta = 0$ or 1), by the definition of space $H^s_{s'}(\Omega_+)$ there exist $\overline{A} \in H^t_{t'}(\Omega)$, $\overline{B} \in H^s_{s'}(\Omega)$, such that $\overline{A}|_{\Omega_+} = A$, $\overline{B}|_{\Omega_+} = B$. Let

$$\overline{J} = \sum_p 2^{-pl}(\partial_1^{l_1}\overline{A})^*_{p-N_0}(\partial_1^{l_2}\overline{B})^*_p. \tag{2.8}$$

Since $\psi^*(2^{-p}D')$ and ∂_1 are tangential pseudodifferential operators, we have

$$\overline{J}|_{\Omega_+} = \sum_p 2^{-pl}(\partial_1^{l_1}\overline{A}|_{\Omega_+})^*_{p-N_0}(\partial_1^{l_2}\overline{B}|_{\Omega_+})^*_p = J. \tag{2.9}$$

We rewrite (2.8) as follows

$$\begin{aligned}\overline{J} &= \sum_p 2^{-pl}\Big(\sum_q \Delta_q(\partial_1^{l_1}\overline{A})^*_{p-N_0}\Big)\Big(\sum_h \Delta_h(\partial_1^{l_2}\overline{B})^*_p\Big)\\ &= \sum_p 2^{-pl}\Big[\sum_q \Delta_q(\partial_1^{l_1}\overline{A})^*_{p-N_0}S_{p-N_0}(\partial_1^{l_2}\overline{B})^*_p + \sum_h S_{h-N_0}(\partial_1^{l_1}\overline{A})^*_{p-N_0}\Delta_h(\partial_1^{l_2}\overline{B})^*_p\\ &\quad + \sum_{|q-h|<N_0} \Delta_q(\partial_1^{l_1}\overline{A})^*_{p-N_0}\Delta_h(\partial_1^{l_2}\overline{B})^*_p\Big]\\ &= I_1 + I_2 + I_3.\end{aligned}$$

Note that the spectrum of every term in I_1, I_2 is included in a "ring with bi-indices" (see

[12]), and we have

$$2^{-pl}\|S_{h-N_0}(\partial_1^{l_1}\overline{A})^*_{q-N_0}\Delta_h(\partial_1^{l_2}\overline{B})^*_p\|_{L^2}\le C2^{-p(\rho+l-l_1-l_2+s')-hs}\varepsilon_{h,p},$$
$$2^{-pl}\|\Delta_q(\partial_1^{l_1}\overline{A})^*_{q-N_0}S_{q-N_0}(\partial_1^{l_2}\overline{B})^*_p\|_{L^2}\le C2^{-p(\rho'+l-l_1-l_2+t')-qt}\varepsilon_{q,p},$$

where $(\varepsilon_{h,p})$, $(\varepsilon_{q,p})\in l_2(\mathbf{N}^2)$, $\rho=\min(t+t'-\frac{3}{2},t-\frac{1}{2})$, $\rho'=s+s'-\frac{3}{2}$. From Lemma 1.2 of [12] it follows that $I_1\in H^s_{s'+\rho+\delta}(\Omega)$, $I_2\in H^t_{t'+\rho'+\delta}(\Omega)\subset H^s_{s'+\rho+\delta}(\Omega)$ (since $s\le t$).

For I_3, we can write

$$I_3=\sum_{|a|\le N_0}\sum_p\Big\{2^{-pl}\sum_q\Delta_q(\partial_1^{l_1}\overline{A})^*_{p-N_0}\Delta_{q+a}(\partial_1^{l_2}\overline{B})^*_p\Big\}.$$

The spectrum of every term in I_3 is included in a "ball-ring" (see [12]), and

$$2^{-pl}\|\Delta_q(\partial_1^{l_1}\overline{A})^*_{p-N_0}\Delta_{p+a}(\partial_1^{l_2}\overline{B})^*_p\|_{L^2}\le C2^{-q(\rho+s)-p(l-l_1-l_2-s')}\varepsilon_{q,p},$$

with $\varepsilon_{q,p}\in l^2(\mathbf{N}^2)$. From Lemma 1.4 of [12] it follows that $I_3\in H^{s+\rho}_{s'+\delta}(\Omega)\subset H^s_{s'+\rho+\delta}(\Omega)$.

Thus we have proved $\overline{J}\in H^s_{s'+\rho+\delta}(\Omega)$, so $J\in H^s_{s'+\rho+\delta}(\Omega_+)$.

d) Let us suppose that $J\in H^{s,k-\delta-1}_{s'+\rho+\delta}(\Omega_+,\Sigma')$ is proved for $A\in H^{t,k-\delta-1}_{t'}(\Omega_+,\Sigma')$ and $B\in H^{s,k-\delta-1}_{s'}(\Omega_+,\Sigma')$. We will prove that if $A\in H^{t,k-\delta}_{t'}(\Omega_+,\Sigma')$, $B\in H^{s,k-\delta}_{s'}(\Omega_+,\Sigma')$, then $j\in H^{s,k-\delta}_{s'+\rho+\delta}(\Omega_+,\Sigma')$.

Let us note that

$$x_1(w)^*_p=2^{-p}(w)^*_p+(x_1w)^*_p.$$

Then, it is easy to see

$$\begin{aligned}Z_1J=\sum_{p=-1}^{\infty}&2^{-pl}\Big\{(Z_1\partial_1^{l_1}\overline{A})^*_{p-N_0})(\partial_1^{l_2}\overline{B})^*_p+(\partial_1^{l_1}\overline{A})^*_{p-N_0}(Z_1\partial_1^{l_2}\overline{B})^*_p\Big\}\\&+2^{-p(l+1)}\{((Z_0+(x_3-x_1)\partial_1)\partial_1^{l_1}A)^*_{p-N_0}(\partial_1^{l_2}B)^*_p\\&+(\partial_1^{l_1}A)^*_{p-N_0}((Z_0+(x_3-x_1)\partial_1)\partial_1^{l_2}B)^*_p\}.\end{aligned}$$

And we have

$$[Z_0,\partial_1^l]=-l\partial_1^l,\ [Z_1,\partial_1^l]=l\partial_1^l(x_1-x_3)-l\partial_1^{l-1}(Z_0+1).$$

Since $Z_jA\in H^{t,k-1}_{t'}$, $Z_jB\in H^{s,k-1}_{s'}$, and every term in Z_1J has the same form as J, following the induction hypothesis, we obtain $Z_1J\in H^{s,k-1}_{s'+\rho+\delta}$. Similarly, we can prove that $Z_jJ\in H^{s,k-1}_{s'+\rho+\delta}$ $(j=0,2,3)$, this means $J\in H^{s,k}_{s'+\rho+\delta}(\Omega_+,\Sigma')$.

e) From c) and d), we have $J\in H^{s,k}_{s'+\rho+\delta}(\Omega_+,\Sigma')$ by induction. Now we consider R_j $(j=0,1,3)$. Let us note that R_j can be separated into two parts: one satisfies $\delta=l-l_1-l_2=1$, $A\in H^{s,k-1}_{s'}$, $B\in H^{s,k-1}_{s'}$, the other satisfies $\delta=0$, $A\in H^{t,k}_{t'}$, $B\in H^{s,k}_{s'}$. Thus we have $R_j=R_{j,1}+R_{j,2}$, and $R_{j,1}\in H^{s,k}_{s'+\rho}(\Omega_+,\Sigma')$, $R_{j,2}\in H^{s,k-1}_{s'+\rho+1}(\Omega_+,\Sigma')$. The lemma is proved.

Using Lemma 2.2, we can prove

Lemma 2.3. *If $a,b\in H^{t,k}_{t'}(\Omega_+,\Sigma')$ $(t>\frac{1}{2},t+t'>\frac{3}{2})$, $u\in H^{s,k}_{s'}(\Omega_+,\Sigma')$ $(-t<s\le t)$, then*

1) $T'_au\in H^{s,k}_{s'}(\Omega_+,\Sigma')$;

2) *If $\overline{T}'_a$ is a paraproduct operator defined with other choices (N_0 and the dyadic partition of unity), then $T'_a-\overline{T'}_a$ maps $H^{s,k}_{s'}(\Omega_+,\Sigma')$ into $H^{s,k}_{s'+\rho}(\Omega_+,\Sigma')$, with $\rho=\min(t+t'-\frac{3}{2},t-\frac{1}{2})$;*

3) *$R=T'_aT'_b-T'_{ab}$ maps $H^{s,k}_{s'}(\Omega_+,\Sigma')$ into $H^{s,k}_{s'+\rho}(\Omega_+,\Sigma')$.*

Proof. From Propositions 1.8, 19 and 1.10 of [12], it is easy to see that Lemma 2.3 holds for $k=0$. Using Lemma 2.2, we can prove the lemma by induction on k.

Lemma 2.4. *If* $v_1,\cdots,v_m\in H^{s,k}_{s'}(\Omega_+,\Sigma')$ $(s>\frac{1}{2}, s+s'>\frac{3}{2}, s+2s'>\frac{1}{2}, k\ge 0)$ *and* $v_j(j=1,\cdots,m)$ *are real functions and have compact support in* Ω_+, F *is a* C^∞ *function, then we have*

$$F(v_1,\cdots,v_m)=\sum_{j=1}^{m}T'_{F'_{v_j}}v_j+R. \tag{2.10}$$

Here $R\in H^{s,k}_{s'+\rho}(\Omega_+,\Sigma')$, $\rho=\min(s+s'-\frac{3}{2}, s-\frac{1}{2})$.

Proof. For $k=0$, this lemma is a simple consequence of Proposition 2.4 of [12]. Let us assume that the lemma holds for $k-1$. For $Z\in\mathcal{M}$, we have

$$ZF(v_1,\cdots,v_m)=F'_{v_1}Zv_1+\cdots+F'_{v_m}Zv_m, \tag{2.11}$$

where $F'_{v_1}Zv_1+\cdots+F'_{v_m}Zv_m$ is a C^∞ function of the variables v_j and Zv_j $(j=1,\cdots,m)$. Following the induction hypothesis, we have

$$F'_{v_1}Zv_1+\cdots+F'_{v_m}Zv_m=\sum_{j,l}T'_{F''_{v_jv_l}Zv_j}v_l+\sum_j T'_{F'_{v_j}}Zv_j+R_1, \tag{2.12}$$

with $R_1\in H^{s,k-1}_{s'+\rho}(\Omega_+,\Sigma')$. On the other hand, using Lemma 2.2 for (2.10), we have

$$ZF(v_1,\cdots,v_m)=\sum_{j,l}T'_{F''_{v_jv_l}Zv_j}v_l+\sum_j T'_{F'_{v_j}}Zv_j+ZR+\sum_j R(F'_{v_j},v_j), \tag{2.13}$$

with $R(F'_{v_j},v_j)\in H^{s,k}_{s'+\rho}(\Omega_+,\Sigma')$. Following (2.11), (2.12) and (2.13), we have

$$ZR\in H^{s,k-1}_{s'+\rho}(\Omega_+,\Sigma'),\ \text{so}\ R\in H^{s,k}_{s'+\rho}(\Omega_+,\Sigma').$$

Let $\Omega_0=\Omega_+\cap\{x_1=0\}$, $\mathcal{M}_0=\mathcal{M}|_{\Omega_0}$. Then $\mathcal{M}_0$ is the set of smooth vector fields tangent to $\Delta=\{x_1=0,x_3=0\}$ on Ω_0, the generators of $\mathcal{M}_0$ are ∂_2, $x_3\partial_3$. We now introduce the space $H^{s,k}_{s'}(\Omega_0,\Delta)$ as follows

$$H^{s,k}_{s'}(\Omega_0,\Delta)=\{u; M^I u\in H^s_{s',\mathrm{loc}}(\Omega_0),\ M\in\mathcal{M}_0,\ |I|\le k\}. \tag{2.14}$$

Because $(Zu)|_{\Omega_0}=M(u|_{\Omega_0})$ for $Z\in\mathcal{M}$, $M\in\mathcal{M}_0$, we can deduce the following results by induction on k.

Lemms 2.5. *For any* $u\in H^{s,k}(\Omega_+,\Sigma')$ $(s>\frac{1}{2},k\ge 0)$, *the trace*

$$u|_{x_1=\lambda x_3}=u(\lambda x_3,x_2,x_3)\in H^{s-\frac{1}{2},k}(\Omega_0,\Delta)\ (\lambda=-1,0,1).$$

For the space $H^{s,k}_{s'}(\Omega_0,\Delta)$, we can easily prove the following result by induction on k.

Lemma 2.6. *If* $v_1,\cdots,v_m\in H^{s,k}_{s'}(\Omega_0,\Delta)$ $(s+s'>1, s>\frac{1}{2}, k\ge 0)$, f *is a* C^∞ *function of its argument, then* $f(y,v_1(y),\cdots,v_m(y))\in H^{s,k}_{s'}(\Omega_0,\Delta)$, *where* $y=(x_2,x_3)$.

For $a\in H^{t,k}_{t'}(\Omega_0,\Delta)$, $b\in H^{s,k}_{s'}(\Omega_0,\Delta)$ $(t+t'>1, t>\frac{1}{2}, k\ge 1)$, it is easy to see

$$MT'_ab=T'_a(Mb)+T'_{Ma}b. \tag{2.15}$$

By (2.15), we know that the analogues of Lemmas 2.3 and 2.4 hold for the space $H^{s,k}_{s'}(\Omega_0,\Delta)$.

Lemma 2.7. *If* $a,b\in H^{t,k}_{t'}(\Omega_0,\Delta)$, $(t+t'>1, t>\frac{1}{2}, k\ge 1)$, $u\in H^{s,k}_{s'}(\Omega_0,\Delta)$ $(-t<s\le t)$, *then*

1) $T'_au\in H^{s,k}_{s'}(\Omega_0,\Delta)$;

2) *If* $\overline{T'}_a$ *is a paraproduct operator defined with other choices* (N_0 *and the dyadic partition of unity*), *then* $T'_a-\overline{T'}_a$ *maps* $H^{s,k}_{s'}(\Omega_0,\Delta)$ *into* $H^{s,k}_{s'+\rho}(\Omega_0,\Delta)$, *with* $\rho=\min(t-\frac{1}{2}, t+t'-1)$;

3) $R = T'_a T'_b - T'_{ab}$ *maps* $H^{s,k}_{s'}(\Omega_0, \Delta)$, *into* $H^{s,k}_{s'+\rho}(\Omega_0, \Delta)$.

Lemma 2.8 *If* $v_1, \cdots, v_m \in H^{s,k}_{s'}(\Omega_0, \Delta')$ $(s > \frac{1}{2}, s + s' > 1, s + 2s' > \frac{1}{2}, k \geq 0)$ *and* v_j $(j = 1, \cdots, m)$ *are real functions and have compact support in* Ω_0, F *is a* C^∞ *function, then we have*

$$F(v_1, \cdots, v_m) = \sum_{j=1}^{m} T'_{F'_{v_j}} v_j + R, \tag{2.16}$$

where $R \in H^{s,k}_{s'+\rho}(\Omega_0, \Delta')$, $\rho = \min(s' - \frac{1}{2}, s + s' - 1)$.

In this paper, we also use the space $H^{s,k}(b\Omega, \Delta)$ of function u such that $M^I u \in H^s_{\rm loc}(b\Omega)$ (with $|I| \leq k$, $M \in \mathcal{M}'$, $\mathcal{M}'$ is the set of smooth vector fields tangent to Δ on $b\Omega$). The analogues of Lemmas 2.6, 2.7 and 2.8 can be found in [1].

§3. Regularity of Characteristic Surface

In this section, φ denotes either φ_+ or φ_-. We assume that $\varphi \in H^\sigma(\Omega_0)$, $(\Omega_0 = \Omega_+ \cap \{x_1 = 0\}$, $\sigma > 0)$. Let $l(u(x), \xi) = \sum_{j=1}^{3} A(u(x))\xi_i$ be the symbol of operator L (see (1.1)), $\tau = \lambda_j(u(x), \xi_2, \xi_3)$ $(j = 1, 2)$ be the roots of equation $\mathrm{del}(l(u(x), \tau, \xi_2, \xi_3)) = 0$ with respect to τ. It is evident that λ_j are C^∞ with respect to ξ_2, ξ_3 and $u(x)$ $(\xi_2^2 + \xi_3^2 \neq 0)$, and are homogeneous of degree 1 with respect to ξ_2, ξ_3.

For simplicity, write $y = (x_2, x_3)$, $\eta = (\xi_2, \xi_3)$ and denote by $\lambda(u(x), y)$ either $\lambda_1(u(x), y)$ or $\lambda_2(u(x), y)$.

If λ is a proper root with respect to characteristic surface $\{x_1 = \varphi(x_2, x_3)\}$, then $\varphi(y)$ satisfies

$$-1 = \lambda(u(\varphi(y), y), \varphi_y). \tag{3.1}$$

In this statement φ_y denotes the vector ${}^t(\partial_2\varphi, \partial_3\varphi)$. By (1.4), we have a diffeomorphism of class $H^\sigma(\Omega_+)$ as follows

$$\zeta^{-1}(x_1, x_2, x_3) = (2x_1 - \varphi_+(x_2, x_3) - \varphi_-(x_2, x_3),\ x_2, \varphi_+(x_2, x_3) - \varphi_-(x_2, x_3)). \tag{3.2}$$

The inverse diffeomorphism ζ is given as follows

$$\zeta(x_1, x_2, x_3) = (\frac{1}{2}(x_1 + \varphi_+(x_2, \psi(x_2, x_3)) + \varphi_-(x_2, \psi(x_2, x_3))),\ x_2, \psi(x_2, x_3)). \tag{3.3}$$

Here ψ is the inverse function of $(\varphi_+ - \varphi_-)(x_2, x_3)$ in x_3, and satisfies

$$x_3 = (\varphi_+ - \varphi_-)(x_2, \psi(x_2, x_3)). \tag{3.4}$$

Then $\Sigma_\pm = \zeta(\Sigma'_\pm)$, $\Sigma_0 = \zeta(\Sigma_0)$, $\Gamma = \zeta(\Delta)$.

Let $\mathcal{M}_\Delta$ denote the set of smooth vector fields tangent to Δ in Ω_+. As usual we introduce

$$H^{s,k}_{s'}(\Omega_+, \Delta) = \{f; Z^l f \in H^s_{s',\rm loc}(\Omega_+),\ Z \in \mathcal{M}_\Delta,\ |I| \leq k\}.$$

Lemms 3.1. *If* $\varphi_\pm(x_2, x_3) \in H^{s+\frac{1}{2},k}(\Omega_0, \Delta)$, *then there exist functions* $G_\pm(x_1, x_2, x_3) \in H^{s+1,k}(\Omega_+, \Delta)$, *such that* $G_\pm|_{x_1=0} = \varphi_\pm(x_2, x_3)$, *and* $\partial_1 G_\pm \neq 0$.

Proof. We shall set $\widetilde{\Omega}_0 = \Omega \cap \{x_1 = 0\}$. We first note that there is a continuous extension operator E from $H^{s+\frac{1}{2},k}(\Omega_0, \Delta)$ into $H^{s+\frac{1}{2},k}(\widetilde{\Omega}_0, \Delta)$ (see Lemma 2.1 of [4]). Then we will prove that there is a function $\widetilde{G}_\pm(x_1, x_2, x_3) \in H^{s+1,k}(\Omega, \Delta)$, such that $\widetilde{G}_\pm|_{x_1=0} =$

$E\varphi_\pm(x_2,x_3),\ \partial_1\widetilde{G}_\pm \neq 0$. First, we set

$$1=\psi(\eta)+\sum_{p=-1}^{\infty}\beta(2^{-p}\eta),$$

where $\eta=(\xi_2,\xi_3)$ is the dual variable of $y=(x_2,x_3)$, $\beta(\eta)=\psi(2^{-1}\eta)-\psi(\eta)$, $\psi(\eta)\in C_0^\infty(\mathbf{R}^2)$ and $\mathrm{supp}\psi\subset B(0,1)$, $\psi|_{B(0,r)}=1$ (for a certain r, $0<r<1$, $B(0,l)$ denotes the ball centered at 0 and of radius l). Then we have $u=\sum\limits_{p=-1}^{\infty}u_p$ with $u_{-1}=\psi(-i\partial_2,-i\partial_3)u$, $u_p=\Delta_p u\,(p\geq 0)$, where $\Delta_p=\beta(-2^{-p}i\partial_2,-2^{-p}i\partial_3)$.

Next, let $a(x_2,x_3)=E\varphi_\pm(x_2,x_3)$,

$$\widetilde{G}_\pm(x_1,x_2,x_3)=\sum_{p=-1}^{\infty}\theta(2^px_1)\Delta_p a(x_2,x_3),$$

where $\theta(t)\in C_0^\infty(\mathbf{R})$, $\theta'(0)\neq 0$, $\theta(0)=1$. It is easy to see that $\widetilde{G}_\pm|_{x_1=0}=a(x_2,x_3)$ and $\partial_1\widetilde{G}_\pm|_{x_1=0}\neq 0$. Since $\theta(2^px_1)\Delta_p a(x_2,x_3)\in C^\infty(\mathbf{R}^3)$ and for any $a=(a_1,a_2,a_3)\in\mathbf{N}^3$ we have

$$\begin{aligned}\int|\partial^\alpha(\theta(2^px_1)\Delta_p a(x_2,x_3))|^2dx&=\int|\partial_1^{\alpha_1}\theta(2^px_1)|^2dx\cdot\int|\partial_2^{\alpha_2}\partial_3^{\alpha_3}\Delta_p a(x_2,x_3)|^2dx_2dx_3\\&\leq\Big(2^{2p\alpha_1-p}\int|\partial_t^{\alpha_1}\theta(t)|^2dt\Big)(C_{p,\alpha_2,\alpha_3}2^{-2p(s+\frac12-\alpha_2-\alpha_3)})\\&\leq C_{\alpha_1}C_{p,\alpha_2,\alpha_3}2^{-2p(s+1-|\alpha|)}\end{aligned}$$

with $(C_{\alpha_1}C_{p,\alpha_2,\alpha_3})_p\in l^2$, following Theorem 10.1.5 of [15] we see that $\widetilde{G}_\pm\in H^{s+1}(\Omega)$.

Finally, for any function $\rho\in C_0^\infty(\mathbf{R})$, we will prove that

$$\begin{cases}\text{if } f(x_2,x_3)\in H^{t+\frac12,l}(\Omega_0,\Delta), \text{ then}\\ \tilde{f}(x_1,x_2,x_3)=\sum\limits_{p=-1}^{\infty}\rho(2^px_1)\Delta_p f\in H^{t+1,l}(\Omega,\Delta),\end{cases}\qquad(3.5)_l$$

by induction on l. Assume that $(3.5)_{l-1}$ is valid and note

$$\mathcal{M}_\Delta=\{x_1\partial_1,x_3\partial_1,\partial_2,x_1\partial_3,x_3\partial_3\}.$$

By direct calculation we have

$$x_1\partial_1\tilde{f}=\sum_{p=-1}^{\infty}(2^px_1)\rho'(2^px_1)\Delta_p f,$$

$$x_3\partial_1\tilde{f}=\sum_{p=-1}^{\infty}\rho'(2^px_1)(2^p\Delta_p(x_3f)+\Delta_p^{\#}f),$$

$$\partial_2\tilde{f}=\sum_{p=-1}^{\infty}\rho(2^px_1)\Delta_p(\partial_2 f),$$

$$x_1\partial_3\tilde{f}=\sum_{p=-1}^{\infty}(2^px_1)\rho(2^px_1)(2^{-p}\Delta_p(\partial_3 f)),$$

$$x_3\partial_3\tilde{f}=\sum_{p=-1}^{\infty}\rho(2^px_1)(\Delta_p(x_3\partial_3 f)+2^p\Delta_p^{\#}(\partial_3 f)),$$

where $\Delta_p^{\#} = \beta^{\#}(2^p D')$, $\beta^{\#}(\eta) = i\partial_\eta \beta(\eta)$. Since

$$x_3 f \in H^{t+1+\frac{1}{2},l-1},\ x_3\partial_3 f,\ \partial_2 f \in H^{t+\frac{1}{2},l-1},\ \partial_3 f \in H^{t-\frac{1}{2},l},$$

by the induction hypothesis, $x_1\partial_1\tilde{f}$, $x_2\partial_1\tilde{f}$, $\partial_2\tilde{f}$, $x_1\partial_3\tilde{f}$ and $x_3\partial_3\tilde{f}$ belong to $H^{t+1,l-1}$, so $\tilde{f} \in H^{t+1,l}(\Omega,\Delta)$. Let $f = a, \tilde{f} = \widetilde{G}_j^{+}$ and $\rho(t) = \theta(t)$, we have $\widetilde{G}_\pm \in H^{s+1,k}(\Omega,\Delta)$. Let $G_\pm = \widetilde{G}_\pm|_{x_3 \geq 0}$, the lemma is proved.

By Lemma 3.1, we have diffeomorphism of class $H^{s+1,k}(\Omega_+,\Delta)$ as follows

$$\alpha_\pm(x_1,x_2,x_3) = (G_\pm(x_1,x_2,x_3),x_2,x_3), \tag{3.6}$$

and $\alpha_\pm(\{x_1 = 0\}) = \Sigma_\pm$, $\alpha_\pm(\Delta) = \Gamma$. Let $\alpha_\pm^{-1}$ be the inverse diffeomorphism of $\alpha_\pm$. Denote by $\Phi_\pm$ the first component of $\alpha_\pm^{-1}$. Then

$$\Phi_\pm(G_\pm(x_1,x_2,x_3),x_2,x_3) = x_1. \tag{3.7}$$

And $\Sigma_\pm$ can be written as $\{(x_1,x_2,x_3)|\Phi_\pm(x_1,x_2,x_3) = 0\}$.

Because Σ_+ intersects Σ_- transversally, without loss of generality we can assume

$$\Pi = \det\begin{pmatrix} \partial_1\Phi_+ & \partial_1\Phi_- \\ \partial_3\Phi_+ & \partial_3\Phi_- \end{pmatrix} \neq 0.$$

By $\Pi \neq 0$, we have a diffeomorphism as follows

$$\chi^{-1}(x_1,x_2,x_3) = (\Phi_+ + \Phi_-, x_2, \Phi_- - \Phi_+). \tag{3.8}$$

The inverse diffeomorphism χ is given as follows

$$\chi(x_1,x_2,x_3) = (\Psi_1,x_2,\Psi_3), \tag{3.9}$$

where Ψ_1, Ψ_3 satisfy

$$\begin{cases} x_1 = \Psi_1(\Phi_+ + \Phi_-, x_2, \Phi_- - \Phi_+), \\ x_3 = \Psi_3(\Phi_+ + \Phi_-, x_2, \Phi_- - \Phi_+), \end{cases} \tag{3.10}$$

and

$$\begin{cases} x_1 = (\Phi_+ + \Phi_-)(\Psi_1,x_2,\Psi_3), \\ x_3 = (\Phi_- - \Phi_+)(\Psi_1,x_2,\Psi_3). \end{cases} \tag{3.11}$$

Then $\chi(\Sigma_\pm) = \Sigma_\pm$, $\chi(\Delta) = \Gamma$, $\chi(\Sigma_0) = \Sigma_0$.

Lemms 3.2. *If $\varphi_\pm(x_2,x_3) \in H^{s+\frac{1}{2},k}(\Omega_0,\Delta)$, then $\Psi_j(x_1,x_2,x_3) \in H^{s+1,k}(\Omega_+,\Delta)$ $(j = 1,3)$.*

Proof. a) By Lemma 3.1, we have $G_\pm(x_1,x_2,x_3) \in H^{s+1,k}(\Omega_+,\Delta)$. From (3.7) it follows that $(\nabla\Phi_\pm)\circ\alpha_\pm \in H^{s,k}(\Omega_+,\Delta)$.

Let $\mathcal{M}_\Gamma$ be the set of vector fields tangent to Γ. It is easy to see that

$$\mathcal{M}_\Gamma = \{\Phi_-\partial_1, \Phi_-\partial_2, \Phi_-\partial_3, x_3\partial_1, x_3\partial_2, x_3\partial_3, (\partial_2\Phi_-)\partial_1 - (\partial_1\Phi_-)\partial_2\}.$$

We now introduce the space $H^{t,k}(\Omega_+,\Gamma)$, $(t \leq s)$ by induction on k. Let $H^{t,0}(\Omega_+,\Gamma) = H^t(\Omega_+)$. We assume that

$$\begin{cases} \text{The space } H^{t,k-1}(\Omega_+,\Gamma) \text{ has been defined, and if} \\ f \in H^{t,k-1}(\Omega_+,\Delta), \text{ then } f\circ\alpha_\pm^{-1} \in H^{t,k-1}(\Omega_+,\Gamma). \end{cases} \tag{3.12$_{k-1}$}$$

In particular, we have $\nabla\Phi_\pm \in H^{t,k-1}(\Omega_+,\Gamma)$ for $t \leq s$, and define $H^{t,k}(\Omega_+,\Gamma)$ to be the set of those $u \in H^{t,k-1}(\Omega_+,\Gamma)$ such that $Zu \in H^{t,k-1}(\Omega_+,\Gamma)$ for $Z \in \mathcal{M}_\Gamma$. For $f \in H^{t,k}(\Omega_+,\Delta)$,

$\widetilde{Z} \in \mathcal{M}_\Gamma$, we have

$$\widetilde{Z}(f \circ \alpha_\pm^{-1}) = (Zf) \circ \alpha_\pm^{-1}, \tag{3.13}$$

where $Z \in \mathcal{M}_\Delta$. Following the induction hypothesis and (3.13), we obtain $Z(f \circ \alpha_\pm^{-1}) \in H^{t,k-1}(\Omega_+, \Gamma)$; this means that $f \circ \alpha_\pm^{-1} \in H^{t,k}(\Omega_+, \Gamma)$. In particular, we have $\nabla\Phi_\pm \in H^{s,k}(\Omega_+, \Gamma)$, so $(3.12)_k$ is valid.

b) From (3.10) and a), we know that $(\nabla\Psi_j)\circ\chi^{-1} \in H^{s,k}(\Omega_+, \Gamma)$ and $\Psi_j \in H^{s+1}$ $(j = 1, 3)$. For $Z \in \mathcal{M}_\Delta$, we have

$$Z(\nabla\Psi_j) = (\widetilde{Z}((\nabla\Psi_j) \circ \chi^{-1})) \circ \chi. \tag{3.14}$$

Using (3.14), we can prove that $\Psi_j \in H^{s+1,k}(\Omega_+, \Delta)$ $(j = 1, 3)$ by induction on k.

Remark. Because $\mathcal{M}_\Delta \supset \mathcal{M}_{\Sigma'}$, we have $H^{s+1,k}(\Omega_+, \Delta) \subset H^{s+1,k}(\Omega_+, \Sigma')$, so Ψ_j $(j = 1, 3)$ also belong to $H^{s+1,k}(\Omega_+, \Sigma')$.

Lemma 3.3. *If $v \circ \chi \in H^{t,k}(\Omega_+, \Sigma')$ $(t \le s+1)$, $\varphi_\pm \in H^{s+\frac{1}{2},k}(\Omega_0, \Delta)$, then*

$$v_\varphi(x_1, x_2) = v(\varphi(x_2, x_3), x_2, x_3) \in H^{t-\frac{1}{2},k}(\Omega_0, \Delta).$$

Proof. Because $\zeta^{-1} \circ \chi(\Sigma') = \Sigma'$, it is easily seen that

$$\begin{aligned} &2\Psi_1(x_3, x_2, x_3) - \varphi_+(x_2, \Psi_3(x_3, x_2, x_3)) - \varphi_-(x_2, \Psi_3(x_3, x_2, x_3)) \\ &= \varphi_+(x_2, \Psi_3(x_3, x_2, x_3)) - \varphi_-(x_2, \Psi_3(x_3, x_2, x_3)). \end{aligned}$$

Then $\Psi_1(x_3, x_2, x_3) = \varphi_+(x_2, \Psi_3(x_3, x_2, x_3))$, and

$$\begin{aligned} v \circ \chi|_{x_1 = x_3} &= v(\Psi_1(x_3, x_2, x_3), x_2, \Psi_3(x_3, x_2, x_3)) \\ &= v(\varphi_+(x_2, \Psi_3(x_3, x_2, x_3)), x_2, \Psi_3(x_3, x_2, x_3)). \end{aligned} \tag{3.15}$$

We shall write $g(x_2, x_3) = \Psi_3(x_3, x_2, x_3)$, and define β maps (x_2, x_3) into $(x_2, g(x_2, x_3))$. On the other hand, from (3.10) it follows that $x_3 = \Psi_3(\Phi_-(x_3, x_2, x_3), x_2, \Phi_-(x_3, x_2, x_3))$. Writing $f(x_2, x_3) = \Phi_-(x_3, x_2, x_3)$, we know that the inverse diffeomorphism of β is given as follows

$$\beta^{-1}(x_2, x_3) = (x_2, f(x_2, x_3)). \tag{3.16}$$

With (3.15) and (3.16), we have

$$v(\varphi_+(x_2, x_3), x_2, x_3) = (v \circ \chi|_{x_1 = x_3}) \circ \beta^{-1}. \tag{3.17}$$

Since $v \circ \chi|_{x_1 = x_3} = (v \circ \chi)(x_3, x_2, x_3) \in H^{t-\frac{1}{2},k}(\Omega_0, \Delta)$, and β^{-1} is a diffeomorphism of class $H^{s+\frac{1}{2},k}(\Omega_0, \Delta)$, we have $v(\varphi_+(x_2, x_3), x_2, x_3) \in H^{t-\frac{1}{2},k}(\Omega_0, \Delta)$. For $v(\varphi_-(x_2, x_3), x_2, x_3)$, the proof is the same.

Now we shall prove the main result in this section.

Proposition 3.1. 1) *If $u \in H^{s+1}(\Omega_+)$, $\varphi \in H^\sigma(\Omega_0)$ $(\sigma > 6)$, then $\varphi \in H^{s+\frac{1}{2}}(\Omega_0)$.*

2) *If $u \circ \chi \in H^{s+1,k}(\Omega_+, \Sigma')$, $(\partial_i u) \circ \chi \in H^{s,k}(\Omega_+, \Sigma')$, $(\partial_i \partial_h u) \circ \chi \in H^{s-1,k}(\Omega_+, \Sigma')$ $(i, h = 1, 2, 3)$, then we have $\varphi \in H^{s+\frac{1}{2},k}(\Omega_0, \Delta)$.*

Proof. a) Differentiate equation (3.1) with respect to x_j and x_l $(j, l = 2, 3)$. Then with $\lambda_{\xi_3} \neq 0$ (since $\{x_3 = 0\}$ is non-characteristic for L), we have

$$(\partial_3 + (\lambda_{\xi_3})^{-1}(\lambda_{\xi_2})\partial_2)\varphi_{x_j x_l} + (\lambda_{\xi_3})^{-1}\lambda_{\eta\eta}\varphi_{y x_j}\varphi_{y x_l} + \sum_{p,q} A_{p,q}\varphi_{x_p x_q} = R_1, \tag{3.18}$$

Let $\tilde{u}=u\circ\chi$, $\tilde{v}=v\circ\chi$, $\tilde{w}=w\circ\chi$ (χ is defined in (3.9)). Then we have

$$\widetilde{L}\tilde{w}+A_0(\tilde{u},\tilde{v})\tilde{w}=g(\tilde{u},\tilde{v}),\quad x\in\Omega_+, \tag{4.5}$$

$$\varphi'(\tilde{u})\tilde{w}=\psi(\tilde{u},\tilde{v}),\quad x\in b\Omega. \tag{4.6}$$

Here

$$\begin{aligned}&\widetilde{L}=\partial_3+\widetilde{A}_1\partial_1+\widetilde{A}_2\partial_2,\quad \widetilde{A}_1=(\overline{A}_3)^{-1}\overline{A}_1,\\ &\widetilde{A}_2=(\overline{A}_3)^{-1}\overline{A}_2,\quad \overline{A}_1=\partial_1\Psi_1+A_1(\tilde{u})\partial_2\psi_1+A_2(\tilde{u})\partial_3\Psi_1,\\ &\overline{A}_2=A_2(\tilde{u}),\ \overline{A}_3=\partial_1\Psi_3+A_1(\tilde{u})\partial_2\Psi_3+A_2(\tilde{u})\partial_3\Psi_3.\end{aligned}$$

Since $\{x_3=0\}$ is non-characteristic for L, $\overline{A}_3$ is non-degenerate. Thus it makes sense to write $(\overline{A}_3)^{-1}$.

Let us suppose

$$\tilde{u}\in H^{s+1,k}(\Omega_+,\Sigma'),\quad \tilde{v}\in H^{s,k}(\Omega_+,\Sigma'),\quad \tilde{w}\in H^{s-1,k}(\Omega_+,\Sigma'), \tag*{$(4.7)_k$}$$

$$\tilde{w}|_{x_3=0}\in H^{s-1,k}(b\Omega,\Sigma'). \tag*{$(4.8)_k$}$$

We will prove that $(4.7)_{k+1}$ and $(4.8)_{k+1}$ are valid below.

From Proposition 3.1 and Lemma 3.2 it follows that $\Psi_1,\Psi_3\in H^{s+1,k}(\Omega_+,\Sigma')$. Following Lemma 2.1, we have $\widetilde{A}_j\in H^{s,k}(\Omega_+,\Sigma')\,(j=0,1,2)$, $g\in H^{s,k}(\Omega_+,\Sigma')$.

We make tangential paralinearization for (4.5). Then

$$P\tilde{w}=(\partial_3+T'_{\widetilde{A}_1}\partial_1+T'_{\widetilde{A}_2}\partial_2+T'_{\widetilde{A}_0})\tilde{w}=R_1+R_2, \tag{4.9}$$

where

$$R_1=-T'_{\partial_1\tilde{w}}\widetilde{A}_1-T'_{\partial_2\tilde{w}}\widetilde{A}_2-T'_{\tilde{w}}\widetilde{A}_0+g(\tilde{u},\tilde{v})\in H^{s,k}(\Omega_+,\Sigma'),$$

R_2 is the regularizing remainder term. Following Lemma 2.4, we have $R_2\in H_2^{s-2,k}(\Omega_+,\Sigma')$ (when $s>4+\frac{3}{2}$).

We denote by f_0 the restrictions to $\{x_3=0\}$ of function f. Then we can rewrite (4.6) as follows

$$\varphi'(\tilde{u}_0)\tilde{w}_0=\psi(\tilde{u}_0,\tilde{v}_0). \tag*{$(4.6)'$}$$

Making paralinearization for (4.6)', we have

$$T'_{\varphi'(\tilde{u}_0)}\tilde{w}_0=-T'_{\tilde{w}_0\varphi''(\tilde{u}_0)}\tilde{u}_0+T'_{\partial_{\tilde{u}_0}\psi}\tilde{u}_0+T'_{\partial_{\tilde{v}_0}\psi}\tilde{v}_0+R_3. \tag{4.10}$$

Since $\tilde{w}_0\in H^{s-1,k}$, using Lemma 2.6 of [1], we obtain $R_3\in H^{s,k}(b\Omega,\Delta)$ (when $s>4$).

Lamma 4.1. *There exists an operator T'_Q with symbol $Q(x)\in H^{s,k}(\Omega_+,\Sigma')$, such that*

$$T'_QP-\widetilde{P}T'_Q=R,$$

where $\widetilde{P}=\partial_3+T'_{B_1\partial_1}+T'_{B_2\partial_2}+T'_{B_0}$ and

$$B_1(x)=\operatorname{diag}(\lambda_1(x),\lambda_2(x)),\quad \lambda_j(x)\in H^{s,k}(\Omega_+,\Sigma')\,(j=1,2),$$

B_0, $B_2\in H^{s,k}(\Omega_+,\Sigma')$, R maps $H^{s,k-1}(\Omega_+,\Sigma')$ into $H_2^{s-2,k}(\Omega_+,\Sigma')$ (when $s>2+\frac{3}{2}$).

Proof. By hypothesis (H.4) we know that there exists an invertible matrix $Q(x)\in H^{s,k}(\Omega_+,\Sigma')$, such that $Q\widetilde{A}_1=B_1Q$ and $B_1(x)=\operatorname{diag}(\lambda_1(x),\lambda_2(x))\in H^{s,k}(\Omega_+,\Sigma')$, $\lambda_1\neq\lambda_2$.

Let $B_2(x) = Q(x)\widetilde{A}_2(x)Q^{-1}(x)$. Then we have $B_2(x) \in H^{s,k}(\Omega_+, \Sigma')$, and choose $B_0(x)$ such that

$$B_0(x)Q(x) = Q(x)\widetilde{A}_0(x) + \sum_j \partial_{\xi_j}(\xi_3 + \widetilde{A}_1\xi_1 + \widetilde{A}_2\xi_2)\left(\frac{1}{i}\right)\partial_j Q(x).$$

It is easy to see $B_0(x) \in H^{s,k-1}(\Omega_+, \Sigma')$. Using Lemma 2.3, we conclude that R maps $H^{s,k-1}(\Omega_+, \Sigma')$ into $H_2^{s-2,k}(\Omega_+, \Sigma')$ (when $s > 2 + \frac{3}{2}$).

Let $\bar{w} = T'_Q\tilde{w}$. By Lemma 4.1 and (4.9), we have

$$\widetilde{P}\bar{w} = (\partial_3 + T'_{B_1}\partial_1 + T'_{B_2}\partial_2 + T'_{B_0})\bar{w} = F, \tag{4.11}$$

where $F \in H_2^{s-2,k}(\Omega_+, \Sigma')$, $\bar{w} \in H^{s,k-1}(\Omega_+, \Sigma')$, $B_j \in H^{s,k}(\Omega_+, \Sigma')\,(j = 1,2)$, $B_0 \in H^{s,k-1}(\Omega_+, \Sigma')$.

For boundary condition (4.10), because T'_Q is a tangential paraproduct operator, we have

$$T'_Q\tilde{w}|_{x_3=0} = T'_{Q|_{x_3=0}}\tilde{w}|_{x_3=0}.$$

Denoting $T'_{Q_0} = T'_{Q|_{x_3=0}}$, $\tilde{w}_0 = \tilde{w}|_{x_3=0}$, and using Lemma 2.4 of [1], we see that (4.10) becomes

$$T'_{\varphi'(\tilde{u}_0)}\bar{w}_0 = T'_{E_1}\tilde{u}_0 + T'_{E_2}\tilde{v}_0 + G, \tag{4.12}$$

where $G \in H^{s,k}(b\Omega, \Delta)$, $E_1 \in H^{s-\frac{3}{2},k}(b\Omega, \Delta)$, $E_2 \in H^{s-\frac{1}{2},k}(b\Omega, \Delta)$.

Lemma 4.2. *For $Z \in \mathcal{M}$, $|I| \leq k+1$, we have*

$$[Z^I, \widetilde{P}] = \sum_{|J|<|I|, J\subset I} T'_{B_{j,J}} Z_j Z^J + \sum_{|J'|<|I|, J'\subset I} T'_{A_{J'}} Z^{J'}\widetilde{P} + R_I,$$

where

$$B_{j,J} \in H^{s-1,k-|I|+|J|}, \quad A_{J'} \in H^{s-1,k-|I|+|J'|},$$

R_I maps $H^{s,k-1}(\Omega_+, \Sigma')$ into $H^{s-3,k-|I|+|J|}(\Omega_+, \Sigma')$.

(The proof will be given in section 5).

For $|I| = l$, from Lemma 4.2 and (4.11) it follows that

$$\widetilde{P}Z^I\bar{w} + \sum_{|J|=|I|-1} T'_{B_{j,J}} Z_j Z^J \bar{w} = F_I, \tag{4.13}$$

where

$$F_I = -\sum_{|J'|\leq|I|-2} T'_{B_{j,J}} Z_j Z^J \bar{w} - \sum_{|J'|<|I|} T'_{A_{J'}} Z^{J'} F + Z^I F - R_I\bar{w}.$$

We use Lemma 4.2 and obtain $F_I \in H_2^{s-3,k-|I|-|J|}(\Omega_+, \Sigma')$.

Let $\mathcal{M}' = \mathcal{M}|_{x_3=0}$. Then it is easy to see $\mathcal{M}' = \{x_1\partial_1, \partial_2\}$. Aumme $M \in \mathcal{M}'$. By (4.12) we see that

$$T_{\varphi'(\tilde{u}_0)} M^I \bar{w}_0 = G_I, \tag{4.14}$$

where $G_I = [T_{\varphi'(\tilde{u}_0)}, M^I]\bar{w}_0 + M^I(T_{E_1}\tilde{u}_0 + T_{E_2}\tilde{v}_0 + G)$. Let $I = I' \cup \{n\}$, $\chi_0 = \chi|_{x_3=0}$. Writing

$$\begin{aligned} M^I T_{E_2}\tilde{v}_0 &= M^{I'} T_{E_2} M_n \tilde{v}_0 + M^{I'}[T_{E_2}, M_n]\tilde{v}_0 \\ &= M^{I'} T_{E_2}\Big(\sum_j a_{l,j}(\nabla(\Psi_1)_0)((\partial_j\partial_n u)_0)\Big)\circ\chi \\ &\quad + M^{I'}[T_{E_2}, M_n]\tilde{v}_0, \quad (j, n = 1,2), \end{aligned}$$

we have

$$G_I = [T_{\varphi'(\tilde{u}_0)}, M^I]\tilde{w}_0 + M^I(T_{E_1}\tilde{u}_0 + G)$$
$$+ M^{I'}T_{E_2}\Big(\sum_j a_{i,j}(\nabla(\Psi_1)_0)(\tilde{w}_0)\Big) + M^{I'}[T_{E_2}, M_n]\tilde{v}_0.$$

Because $\tilde{w}_0 \in H^{s-1,k}(\Omega_0,\Delta)$ (see $(4.8)_k$), $\tilde{u}_0 \in H^{s+\frac{1}{2},k}(\Omega_0,\Delta)$, $a_{l,j}$, $\tilde{v}_0 \in H^{s-\frac{1}{2},k}(\Omega_0,\Delta)$, we have $G_I \in H^{s-1,k-|I|+1}(b\Omega,\Delta)$.

Let $U_l = {}^t((Z^{I_1}\bar{w},\cdots,Z^{I_h}\bar{w})$ for $|I_j| = l\,(j=1,\cdots,h)$, $Z \in \mathcal{M}$. Then $U_l|_{x_3=0} = {}^t(M^{I_1}\bar{w}_0,\cdots,M^{I_h}\bar{w}_0)$ for $M \in \mathcal{M}'$. Then one can rewrite (4.13) and (4.14) as follows

$$\widetilde{P}U_l + T'_B U_l = F_l, \tag{4.13$'$}$$

$$T_{\varphi'(\tilde{u}_0)}(U_l|_{x_3=0}) = G_l. \tag{4.14$'$}$$

The proof of $(4.7)_{k+1}$ and $(4.8)_{k+1}$ will thus be finished once we have established the following results.

Lemma 4.3. *If u satisfies the conditions of Theorem* 1.1, *and* $(4.7)_k$ *is valid, then* $U_l \in H^{s-1}_{\rm loc}(\Omega_+ \cap \{x_1<0\})$, *with* $l \le k+1$.

Proof. Clearly, we need only to prove $\tilde{w} \in H^{s-1,k+1}(\Omega_+ \cap \{x_1<0\},\Sigma')$. For any point $x^* \in \Omega_+ \cap \{x_1<0\}$, we say that $u \in H^{s,k}(x^*,\Sigma_\#)$ if there is a function $\theta \in C_0^\infty$ and $\theta(x^*)=1$ such that $\theta u \in H^{s,k}(\Omega_+,\Sigma_\#)$, where $\Sigma_\#$ stands for Σ_0 or Σ_-. Following the conditions of Theorem 1.1, we have

$$\begin{cases} w \in H^{s-1,\infty}(x^*,\Sigma_0), \text{ for } x^* \text{ near } \Sigma_0, \\ w \in H^{s-1,\infty}(x^*,\Sigma_-), \text{ for } x^* \text{ near } \Sigma_-. \end{cases} \tag{4.15}$$

Using Lemma 3.2 and Proposition 3.1, we know from $(4.7)_k$, that $\Psi_j \in H^{s+1,k}(\Omega_+,\Sigma')$ $(j=1,3)$. Thus $(\nabla\Phi_\pm)\circ\chi \in H^{s,k}(\Omega_+,\Sigma')$ (by (3.11)).

We will prove

$$\begin{cases} \text{If the function } g \in H^{s-1,k}(x^*,\Sigma_0) \text{ (resp. } H^{s-1,k}(x^*,\Sigma_-)) \\ \text{for } x^* \text{ near } \Sigma_0 \cap \{x_1<0\} \text{ (resp.} \Sigma_- \cap \{x_1<0\}), \text{ then} \\ g\circ\chi \in H^{s-1,k}(x^0,\Sigma_0) \text{ (resp. } H^{s-1,k}(x^0,\Sigma'_-), \text{ with } x^0 = \chi^{-1}(x^*). \end{cases} \tag{4.16$_k$}$$

It is easy to see that χ is a diffeomorphism of class $H^{s+1}_{\rm loc}(\Omega_+)$, so $g\circ\chi \in H^{s-1}_{\rm loc}(\Omega_+)$. Thus $(4.16)_0$ is proved. We assume that $(4.16)_{k-1}$ is valid and note that

$$\begin{aligned} \mathcal{M}_{\Sigma_-\backslash\Gamma} &= \{\Phi_-((\partial_3\Phi_+)\partial_1 - (\partial_1\Phi_+)\partial_3), (\partial_1\Phi_-)\partial_2 - (\partial_2\Phi_-)\partial_1, \\ &\quad (\partial_1\Phi_-)\partial_3 - (\partial_3\Phi_-)\partial_1\}, \\ \mathcal{M}_{\Sigma_0} &= \{\partial_1,\partial_2,x_3\partial_3\}. \end{aligned}$$

In the expression above, $\mathcal{M}_{\Sigma_-\backslash\Gamma}$, $\mathcal{M}_{\Sigma_0}$ denote the sets of smooth vector feilds tangent to $\Sigma_-\backslash\Gamma$ and Σ_0 respectively.

For the set of smooth vector fields tangent to Σ', we have

$\mathcal{M}$ is generated by $\{\partial_1,\partial_2,x_3\partial_3\}$ near $\Sigma_0 \cap \{x_1<0\}$,

$\mathcal{M}$ is generated by $\{(x_1+x_3)\partial_1,\partial_2,\partial_3-\partial_1\}$ near $\Sigma'_- \cap \{x_1<0\}$.

If x^0 near $\Sigma_0 \cap \{x_1 < 0\}$, for any $Z \in \{\partial_1, \partial_2, x_3\partial_3\}$, by direct calculation we have

$$Z(g\circ\chi) = \sum_j a_j(\partial_i\Psi_n, x_3\partial_3\Psi_n, \partial_i\Psi_n/\Psi_n, x_3\partial_3\Psi_n/\Psi_n)(\widetilde{Z}_j g)\circ\chi, \quad i = 1,2;\ n = 1,3,$$

where a_j are C^∞ functions with respect to $\partial_i\Psi_n, \cdots, x_3\partial_3\Psi_n/\Psi_n$, $\widetilde{Z}_j \in \mathcal{M}_{\Sigma_0}$. Because $\Psi_n \in H^{s+1,k}(\Omega_+, \Sigma')$ and $\Psi_n|_{x_3=0} = 0$, we have $a_j \in H^{s,k}(\Omega_+, \Sigma')$. Following the induction hypothesis, we know that $g\circ\chi \in H^{s-1,k}(X^0, \Sigma_0 \cap \{x_1 < 0\})$.

If x^0 near $\Sigma'_- \cap \{x_1 < 0\}$, for any $Z \in \{(x_1 + x_3)\partial_1, \partial_2, \partial_3 - \partial_1\}$, by calculation and (3.11) we have

$$Z(g\circ\chi) = \sum_j b_j((\nabla\Phi_\pm)\circ\chi)(\widetilde{Z}_j g)\circ\chi,$$

where b_j are C^∞ functions with respect to $(\nabla\Phi_\pm)\circ\chi$, $\widetilde{Z}_j \in \mathcal{M}_{\Sigma_-\backslash\Gamma}$. Because $(\nabla\Phi_\pm)\circ\chi \in H^{s,k}(\Omega_+, \Sigma')$, we have $b_j \in H^{s,k}(\Omega_+, \Sigma')$. Following the induction hypothesis, we know that $g\circ\chi \in H^{s-1,k}(x^0, \Sigma'_- \cap \{x_1 < 0\})$, and $(4.16)_k$ is proved. Lemma 4.3 will be proved once we use $(4.16)_k$ for w.

Lemma 4.4. *If $F_l \in H_2^{s-3}(\Omega_+)$, $G_l \in H_{\rm loc}^{s-1}(b\Omega)$ and $U_l \in H_{\rm loc}^{s-1}(\Omega_+ \cap \{x_1 < 0\})$, then $U_l \in H_{\rm loc}^{s-1}(\Omega_+)$, $U_l|_{x_3=0} \in H_{\rm loc}^{s-1}(b\Omega)$.*

(The proof can be found in [7] anf [11]).

Proposition 4.1. *If $(4.7)_k$ is valid, $(4.7)_{k+1}$ is also true.*

Proof. Following Lemmas 4.3 and 3.4, we have

$$U_l \in H_{\rm loc}^{s-1}(\Omega_+), \quad U_l|_{x_3=0} \in H_{\rm loc}^{s-1}(b\Omega).$$

Thus

$$\widetilde{w} \in H^{s-1,k+1}(\Omega_+, \Sigma'), \ \widetilde{w}_0 \in H^{s-1,k+1}(b\Omega, \Delta).$$

Let us note that $\widetilde{w} = {}^t(\partial_1^2 u, \partial_1\partial_2 u, \partial_2^2 u)\circ\chi$. Since $\{x_3 = 0\}$ is non-chaacteristic for L (see (4.1)), differentiating the equation (4.1) with respect to x_j $(j = 1,2)$ and the diffeomorphism χ (see (3.9)), we have

$${}^t(\partial_1^2 u, \partial_1\partial_2 u, \partial_2^2 u, \partial_1\partial_3 u, \partial_2\partial_3 u)\circ\chi \in H^{s-1,k+1}(\Omega_+, \Sigma').$$

By the chain rule for differentiation, we have

$$\partial_j((\partial_l u)\circ\chi) = \sum_{h=1}^{3} a_{h,l}(\nabla\Psi_n)(\partial_h\partial_l u)\circ\chi \quad (j = 1,2,3; l = 1,2;\ n = 1,3),$$

where $a_{h,l}$ are C^∞ functions with respect to $\nabla\Psi_n$. By Proposition 3.1 and Lemma 3.3, we know that $\partial_j\Psi_n \in H^{s,k}(\Omega_+, \Sigma')$ $(j = 1,2,3;\ n = 1,3)$, so $\tilde{v} \in H^{s,k+1}(\Omega_+, \Sigma')$.

Using the same method as above, we can only prove that $\tilde{u} \in H^{s+1,k}(\Omega_+, \Sigma')$ because we only know that $\Psi_n \in H^{s+1,k}(\Omega_+, \Sigma')$. But now we have proved that $\widetilde{w} \in H^{s-1,k+1}(\Omega_+, \Sigma')$, $\tilde{v} \in H^{s,k+1}(\Omega_+, \Sigma')$, $\tilde{u} \in H^{s,k+1}(\Omega_+, \Sigma')$. Using Proposition 3.1 and Lemma 3.3 again, we have $\Psi_n \in H^{s+1,k+1}(\Omega_+, \Sigma')$. With the same method as above again, we have $\tilde{u} \in H^{s+1,k+1}(\Omega_+, \Sigma')$ and $(4.7)_{k+1}$ is proved.

Finally, we prove Theorem 1.1.

1) If $u \in H_{\rm loc}^{s+1}(\Omega_+)$, $\varphi_\pm \in H_{\rm loc}^{\sigma}(\Omega_0)$, following Proposition 3.1 1), we have $\varphi_\pm \in H_{\rm loc}^{s+1}(\Omega_0)$. By Lemma 3.2, we know that χ is a diffeomorphism of class H^{s+1}, so $(4.7)_0$ is true. Using Lemma 4.4 for (4.11) and (4.12), we conclude that $(4.8)_0$ is true.

2) By Propositions 3.1 and 4.1, we know that $(4.7)_k$, $(4.8)_k$ and $\varphi_\pm \in H^{s+\frac{1}{2},k}(\Omega_0,\Delta)$ are valid for any k. Then $\tilde{u} \in H^{s+1,\infty}(\Omega_+,\Sigma')$, $\varphi \in H^{s+\frac{1}{2},\infty}(\Omega_0,\Delta)$. Because

$$\Gamma = \{x_3 = 0, x_1 = h(x_2)\} \text{ and } h(x_2) = \varphi_-(x_2,0) = \varphi_-|_\Delta \in C^\infty,$$

Γ is a smooth curve. From $\varphi \in H^{s+\frac{1}{2},\infty}(\Omega_0,\Delta)$, it follows that Σ_+,Σ_- are C^∞ except Γ. Using the same method as in the proof of $(4.16)_k$, we can prove that

$$u \in H^{s+1,\infty}(\Omega_+,\Sigma_+) \text{ (resp. } H^{s+1,\infty}(\Omega_+,\Sigma_-),\ H^{s+1,\infty}(\Omega_+,\Sigma_0))$$

near $\Sigma_+\backslash\Gamma$ (resp. $\Sigma_-\backslash\Gamma$, $\Sigma_0\backslash\Gamma$) with Ω_+ small enough.

§5. Commutator Argument

In this section, first we prove Lemms 4.2. For Lemma 3.4, because its proof is simpler than Lemma 4.2 and the idea is the same as Lemma 4.2, we only sketch it.

Proof of Lemma 4.2. Let $\sigma_1(\widetilde{P})$ denote the pricipal symbol of operator $\widetilde{P}$. Then

$$\begin{aligned}\sigma(\widetilde{P}) &= \xi_3 + B_1\xi_1 + B_2\xi_2\\ &= \frac{1}{2}(B_1 + Id)(\xi_1+\xi_3) + \frac{1}{2}(B_1 - Id)(\xi_1 - \xi_3) + B_2\xi_2,\end{aligned}$$

where $B_j \in H^{s,k}(\Omega_+,\Sigma')$ (j=1,2) (see (4.11)). Because $\{x_1 = \pm x_3\}$ are characteristic surfaces for $\widetilde{P}$, we have

$$\mathrm{Det}(B_1 \pm Id)|_{x_1 \pm x_3 = 0} = 0.$$

So we can write

$$B_1 = \mathrm{diag}(1 + (x_1 - x_3)\mu_1(x),\ \ -1 + (x_1 + x_3)\mu_2(x)).$$

Denote

$$\Lambda_+ = \partial_3 + \partial_1 + T'_{(x_1-x_3)\mu_1(x)}\partial_1,\quad \Lambda_- = \partial_3 - \partial_1 + T'_{(x_1+x_3)\mu_2(x)}\partial_1,$$
$$\Lambda = \mathrm{diag}(\Lambda_+,\Lambda_-),\quad \mu(x) = \mathrm{diag}(\mu_1,\mu_2).$$

We have

$$\widetilde{P} = \Lambda + T'_{B_2}\partial_2 + T'_{B_0}, \tag{5.1}$$

where $\Lambda = \mathrm{diag}(\partial_3+\partial_1,\partial_3-\partial_1) + T'_\mu \mathrm{diag}((x_1-x_3)\partial_1,\ (x_1+x_3)\partial_1)$.

For generators of $\mathcal{M}$, we can easily verify

$$\begin{cases}[\partial_3 \pm \partial_1, Z_0] = \partial_3 \pm \partial_1,\quad [\partial_3 \pm \partial_1, Z_1] = (x_1 \pm 3x_3)(\partial_1 \pm \partial_3),\\ [\partial_3 \pm \partial_1, Z_2] = 0,\quad [\partial_3 \pm \partial_1, Z_3] = (x_1 \mp x_3)(\partial_1 \mp \partial_3),\end{cases} \tag{5.2}$$

and

$$\begin{cases}[(x_1 \mp x_3)\partial_1, Z_0] = 0,\quad [(x_1 \mp x_3)\partial_1, Z_1] = x_3(x_1 \mp x_3)(\partial_3 \pm \partial_1),\\ [(x_1 \mp x_3)\partial_1, Z_2] = 0,\quad [(x_1 \mp x_3)\partial_1, Z_3] = x_3(x_3 \mp x_1)(\partial_1 \pm \partial_3).\end{cases} \tag{5.3}$$

From (5.2), (5.3) and Lemma 2.2, for any $Z_l \in \mathcal{M}$ $(l = 0,1,2,3)$, we have

$$\begin{aligned}[Z_l,\Lambda] = {}& C_1(x)\mathrm{diag}(\partial_3+\partial_1,\partial_3-\partial_1)\\ &+ (T'_{Z_l\mu(x)} + R(\mu))\mathrm{diag}((x_1-x_3)\partial_1,(x_1+x_3)\partial_1)\\ &+ (T'_{Z_l\mu(x)}C_2(x)\mathrm{diag}(x_3(x_1-x_3)(\partial_1+\partial_3), x_3(x_1+x_3)(\partial_1-\partial_3)),\end{aligned} \tag{5.4}$$

where $C_1(x), C_2(x)$ are C^∞ matrices, $R(\mu)$ maps $H^{s-2,k}(\Omega_+,\Sigma')$ into $H_2^{s-2,k-1}(\Omega_+,\Sigma')$ (when $s > \frac{3}{2}+3$).

On the other hand, from (5.1) it follows that

$$\mathrm{diag}(\partial_3+\partial_1, \partial_3-\partial_1) = \widetilde{P} - T'_{\mu(x)}\mathrm{diag}((x_1-x_3)\partial_1, (x_1+x_3)\partial_1) - T'_{B_2}\partial_2 - T'_{B_0}.$$

Noting $(x_1 \pm x_3)\partial_1 = Z_0 - x_3(\partial_1 \mp \partial_3)$, and taking x_3 small enough such that $\mathrm{Det}(Id - x_3\mu(x)) \neq 0$, we have

$$\mathrm{diag}(\partial_3+\partial_1, \partial_3-\partial_1) = T'_{C_3(x)}(\widetilde{P} - T'_{\mu(x)}Z_0 - T'_{B_2}Z_2 - T'_{B_0}) + R_1, \tag{5.5}$$

where $C_3(x) = (Id - x_3\mu(x))^{-1} \in H_2^{s-1,k}(\Omega_+,\Sigma')$, $R_l = \{-T'_{C_3}T'_{x_3\mu} - T'_{C_3(x_3\mu)}\}\mathrm{diag}(\partial_3 + \partial_1, \partial_3-\partial_1)$. Following Lemma 2.3, R_1 maps $H^{s-2,k}$ into $H_2^{s-3,k}(\Omega_+,\Sigma')$ (when $s > \frac{3}{2}+3$).

Summing up (5.1), (5.4) and (5.5), and using Lemma 2.3, we know that

$$[Z_l, \widetilde{P}] = \sum_j T'_{B_{j,J}}Z_j + T'_{A_l}\widetilde{P} + R_l, \tag{5.6}$$

where $B_{j,l},\ A_l \in H^{s-1,k-1}(\Omega_+,\Sigma')$, R_l maps $H^{s-1,k}$ into $H_2^{s-2,k-1}(\Omega_+,\Sigma')$.

Assuming that Lemma 4.2 holds for k, we will prove that Lemma 4.2 also holds for $k+1$. In fact, for $I = I' \cup \{l\}$, $|I| = k+1$, following the induction hypothesis, we have

$$\begin{aligned}
[Z^I, \widetilde{P}] &= Z_l[Z^{I'}, \widetilde{P}] + [Z_l, \widetilde{P}]Z^{I'} \\
&= Z_l\Big(\sum_{|J|<|I'|} T'_{B_{j,J}}Z_jZ^J + \sum_{|J'|<|I'|} T'_{A_{J'}}Z^{J'}\widetilde{P} + R_{I'}\Big) \\
&\quad + (T'_{B_{j,J}}Z_j + T'_{A_l}\widetilde{P} + R_l)Z^{I'} \\
&= \sum T'_{B_{j,J}}Z_lZ_jZ^J + \sum T'_{Z_l(B_{j,J})}Z_jZ^J + \sum R(B_{j,J})Z_jZ^J \\
&\quad + \sum T'_{A_{J'}}Z_lZ^{J'}\widetilde{P} + \sum T'_{Z_l(A_{J'})}Z^{J'}\widetilde{P} + R(A_{J'})Z^{J'}\widetilde{P} \\
&\quad + Z_lR_{I'} + \sum T'_{B_{j,J}}Z_jZ^{I'} + T'_{A_l}Z^{I'}\widetilde{P} + T'_{A_l}[\widetilde{P}, Z^{I'}] + R_lZ^{I'}.
\end{aligned} \tag{5.7}$$

Applying the induction hypothesis to $[\widetilde{P}, Z^{I'}]$ again, we have

$$\begin{aligned}
[Z^I, \widetilde{P}] &= \Big(\sum T'_{B_{j,J}}Z_lZ_jZ^J + \sum T'_{Z_l(B_{j,J})}Z_jZ^J \\
&\quad + \sum T'_{B_{j,J}}Z_jZ^{I'} - \sum T'_{A_lB_{j,J}}Z_jZ^J\Big) \\
&\quad + \Big(\sum T'_{A_{J'}}Z_lZ^{J'} + \sum T'_{Z_l(A_{J'})}Z^{J'} \\
&\quad + T'_{A_l}Z^{I'} - \sum T'_{A_lA_{J'}}Z^{J'}\Big)\widetilde{P} + R_I,
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
R_I &= \sum R(B_{j,J})Z_jZ^J + R(A_{J'})Z^{J'}\widetilde{P} + Z_lR_{I'} \\
&\quad - T'_{A_l}R_{I'} + R_lZ^{I'} - \sum(T'_{A_l}T'_{B_{j,J}} - T'_{A_lB_{j,J}})Z_jZ^J \\
&\quad - \sum(T'_{A_l}T'_{A_{J'}} - T_{A_lA_{J'}})Z^{J'}\widetilde{P}.
\end{aligned}$$

Is it easy to see that

$$B_{j,J} \in H^{s-1,k-|I'|+|J|} = H^{s-1,k-|I|+(|J|+1)}(\Omega_+, \Sigma'),$$
$$B_{j,l} \in H^{s-1,k-1} = H^{s-1,k-|I|+|I'|}(\Omega_+, \Sigma'),$$
$$A_{J'} \in H^{s-1,k-|I'|+|J'|} = H^{s-1,k-|I|+(|J'|+1)}(\Omega_+, \Sigma'),$$
$$A_l \in H^{s-1,k-1} = H^{s-1,k-|I|+|I'|}(\Omega_+, \Sigma'),$$
$$A_l A_J \in H^{s-1,\theta_1} \subset H^{s-1,k-|I|+|J'|}(\Omega_+, \Sigma') \ (\theta_1 = \min(k-1, k-|I'|+|J'|),$$
$$A_l B_{j,J} \in H^{s-1,\theta_2} \subset H^{s-1,k-|I|+|J|}(\Omega_+, \Sigma') \ (\theta_2 = \min(k-1, k-|I'|+|J|).$$

Then regularity of coefficient are proved. Using Lemmas 2.2 and 2.3, we we see that R_l maps $H^{s-1,k}$ into $H_2^{s-3,k-|I|+1}(\Omega_+, \Sigma')$ and Lemma 4.2 is proved.

Proof of Lemma 3.4. Note that $\mathcal{M}_0 = \{\partial_2, x_3\partial_3\}$. By direct calculation, we have

$$[M_l, Q] = C_l Q + T'_{B_{j,J}} M_j + R_l, \tag{5.9}$$

where C_l is a constant, $R_{l,j} \in H^{s-\frac{1}{2},k-2}(\Omega_0, \Delta)$, R_l maps $H^{s-\frac{3}{2},k-1}$ into itself. Using the same method as in the proof of Lemma 4.2, we can prove Lemma 3.4 by induction on k.

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