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MATHER SETS FOR SUBLINEAR DUFFING EQUATIONS

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Abstract

The existence of Mather sets (generalized quasiperiodic solutions and unlinked periodic solutions) for sublinear Duffing equations is shown. Here the approach is based on the use of action-angle variables and the application of a generalized version of Aubry-Mather theorem on semi-cylinder with finite twist assumption.

Keywords Sublinear Duffing equation, Finite twist, Aubry-Mather theorem, Mather sets.

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§1. Introduction

This paper deals with the existence of Mather sets for the Duffing equation

$$\ddot{x} + g(x) = p(t), \tag{1.1}$$

with sublinear growth condition

(g1) $$\lim_{|x|\to\infty} \operatorname{sgn}(x) g(x) = +\infty \quad \text{and} \quad \lim_{|x|\to\infty} \frac{g(x)}{x} = 0.$$

where $p(t)$ is 1-periodic continuous and g satisfies some regular conditions we shall mention later.

Since the celebrated works of Aubry-Le Daeron[1] and Mather[5] about a class of important invariant sets (called Mather sets later) for area-preserving monotone twist homeomorphism of an annulus, Aubry-Mather theory has been rapidly developed in several fields such as differential geometry, dynamical systems and solid state physics (see [8], [2], [10], [6] and [7]).

In general, area-preserving maps can occur as Poincaré maps of continuous conservative systems with two degrees of freedom. And then Aubry-Mather theory has become an important tool in studying dynamics of differential equations. In [9], Moser gave some necessary conditions for the application of Aubry-Mather theory. His student, Denzler[3] proved the existence of Mather sets (i.e., certain generalized quasiperiodic solutions and unlinked periodic solutions) for periodic Hamiltonian planar systems.

Recently, M. Pei proved in his Ph.D thesis the existence of Mather sets for some superlinear Duffing equations by using a generlized Aubry-Mather theorem on infinite cylinder with infinite twist assumption (see [11], [12]).

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The essential difference between the superlinear Duffing equations and the sublinear ones is that the Poincaré maps of the latter do not satisfy the infinite twist assumption. For this reason, we first have to generalize the Aubry-Mather theorem to semi-cylinder with finite twist assumption following the idea of Katok[4]. Then, using some delicate estimates of action-angle variables, we embed the Poincaré map of the sublinear Duffing equation into an area-preserving monotone twist homeomorphism on semi-cylinder and thus obtain the Mather sets with small rotation numbers.

Our main result is the following

Theorem A. *Besides* (g1), *suppose equation* (1.1) *satisfies that*

(g2) $$\lim_{|x|\to\infty} x^{-\frac{1}{3}} g(x) = +\infty \quad or \quad \lim_{|x|\to\infty} |g(x)x^{\frac{2}{3}}||g(-x)|^{-1} = +\infty;$$

(g3) *There are positive constants* ε_0, k_0 *and* d, *such that*

$$xg(x) - x^2 g'(x) \ge \varepsilon_0 G(x), \quad for\ |x| \ge d;$$

$$G(x) \le xg(x) \le k_0 G(x), \quad for\ |x| \ge d;$$

$$|x^3 g''(x)| \le k_0 G(x), \quad for\ |x| \ge d.$$

Then there exists $\omega_0 > 0$, *such that for any* $\omega \in (0, \omega_0]$, *equation* (1.1) *possesses a solution* $z_\omega(t) = (x_\omega(t), \dot{x}_\omega(t))$ *of Mather type with rotation number* ω. *It follows that*

(i) *if* ω *is a rational number* p/q, *the solutions* $z_\omega^i(t) = z_\omega(t+i)$, $0 \le i \le q-1$ *are mutually unlinked periodic solutions of period q;*

(ii) *if* ω *is an irrational number, the solution* $z_\omega(t)$ *is either a usual quasiperiodic solution or a generalized one exhibiting a Denjoy's minimal set*

$$M_\omega = \overline{\{z_\omega(i),\ \in \mathbf{Z}\}};$$

(iii) *all these bounded solutions* $z_\omega(t)$ *are arranged in order from finity to infinity, i.e.,*

$$\inf_{t\in\mathbf{R}} (x_\omega^2(t) + \dot{x}_\omega^2(t)) \to +\infty, \quad as\ \omega \to 0.$$

Remark 1.1. If g is odd, the condition (g1) implies the condition (g2).

Remark 1.2. The typical example for Theorem A is

$$\ddot{x} + \operatorname{sgn}(x)|x|^\alpha (1+x^2)^{-1} = p(t), \quad \alpha \in (2,3),$$

where $p(t)$ is 1-periodic continuous.

§2. Area-Preserving Monotone Twist Map on Semi-Cylinder

In this section, we shall prove a generalized version of Aubry-Mather theorem on semi-cylinder.

Let $\mathbf{A} = S^1 \times [0, +\infty)$ be the standard semi-cylinder, and let $\mathbf{S} = \mathbf{R} \times [0, +\infty)$ be its universal covering. We assume that $f : \mathbf{A} \to \mathbf{A}$ is an area-preserving and orientation-preserving monotone twist homeomorphism, sometimes call it twist map for simplicity. Let F be a lift of f with the form

$$F(\theta, I) = (\theta + F_1(\theta, I), F_2(\theta, I)).$$

It is easy to see that the function $F_1(\theta, I)$ is a strictly monotone function of I.

Suppose that

(A1): F_1 is strictly decreasing with respect to I for any $\theta \in \mathbf{R}$;

(A2): F preserves the boundary $\mathbf{R} \times \{0\}$ and the infinity, which means

$$F_2(\theta, 0) = 0, \quad \text{for } \theta \in \mathbf{R}, \quad \text{and} \quad \lim_{I \to +\infty} \inf_{\theta \in \mathbf{R}} F_2(\theta, I) = +\infty;$$

(A3): F has small twist at infinity, that is

$$F_1(\theta, I) > 0, \quad \text{for } (\theta, I) \in \mathbf{S} \quad \text{and} \quad \lim_{I \to +\infty} \sup_{\theta \in \mathbf{R}} F_1(\theta, I) = 0;$$

(A4): F has an infinite graph, namely

$$\inf_{\theta \in \mathbf{R}} \int_0^{+\infty} F_1(\theta, \rho(\theta, I)) dI = +\infty,$$

where $\rho = \rho(\theta, I)$ is the inverse function of $F_2(\theta, \rho) = I$ with parameter θ.

Let $f_0 = f|_{S^1 \times \{0\}}$. Then f_0 is an orientation-preserving homeomorphism on S^1, and its lift F_0 is an order-preserving homeomorphism on $\mathbf{R}$. Denote by $\alpha(f_0)$ the Poincaré rotation number of f_0.

Recall that, for two positive integers p and q, a point $w \in \mathbf{A}$ is called a Birkhoff point of type (p, q) if for a lift u of w there exists a map

$$\eta: \quad \mathbf{Z} \to \mathbf{S}; \quad n \mapsto (\theta, I), \quad \text{with } \eta(0) = u,$$

such that θ is a strictly monotone function, and

$$(\theta(n+q), I(n+q)) = (\theta(n) + 1, I(n)),$$

$$(\theta(n+p), I(n+p)) = F(\theta(n), I(n)).$$

The orbit of a Birkhoff point of type (p, q) is called a Birkhoff periodic orbit of type (p, q).

We have the following theorem for existence of Birkhoff periodic orbits. The proof of the theorem will be given in section 4.

Theorem 2.1. *Assume that* (A1)-(A4) *hold. Then for any positive integers p and q with $p/q \in (0, \alpha(f_0))$, f admits a Birkhoff periodic orbit of type (p, q).*

In order to get the existence of Mather sets with irrational rotation number, we need some estimations for Birkhoff periodic orbits.

Denote by π_1 and π_2 the projections from $\mathbf{S}$ onto its first and second factor space respectively.

Lemma 2.1. *Assume that* (A1)-(A4) *hold. Then for $I_0 > 0$ and any positive integer N, there exists $I_1 > 0$ such that*

$$\inf_{\theta \in \mathbf{R}} \{\pi_2(F^i(\theta, I)),\ i = 0, \pm 1, \cdots, \pm N.\} \geq I_0, \quad \text{for } I \geq I_1.$$

Lemma 2.2. *Besides the assumptions of Lemma 2.1, let $p/q \geq \beta > 0$. Then there exists $K(\beta) > 0$ such that for any Birkhoff periodic orbit $\psi_{\frac{p}{q}}$ of type (p, q) the following estimation holds,*

$$\sup_{n \in \mathbf{Z}} \pi_2(\psi_{\frac{p}{q}}(n)) \leq K(\beta).$$

Proof. Let N be large enough, so that $\beta > \frac{1}{N}$. From $(A3)$ we have $I_0 > 0$, such that

$$\sup_{\theta \in \mathbf{R}} (\pi_1(F(\theta, I)) - \theta) < \frac{1}{N}, \quad \text{for } I \geq I_0.$$

Then we can choose $K(\beta) = I_1$, where I_1 is as shown in Lemma 2.1. In fact, let

$$\psi_{\frac{p}{q}}: \quad \mathbf{Z} \to \mathbf{S}; \quad n \mapsto (\theta, I),$$

be a Birkhoff periodic orbit of type (p,q). We have

$$\pi_1(F^N(\theta,I)) - \theta > 1, \quad \text{for any } (\theta,I) \in \psi_{\frac{p}{q}}(\mathbf{Z}). \tag{2.1}$$

On the other hand, if $I \geq I_1$, one gets from Lemma 2.1 that

$$\pi_2(F^i(\theta,I)) \geq I_0, \quad i = 0,1,\cdots,N-1,$$

which implies

$$\pi_1(F^{i+1}(\theta,I)) - \pi_1(F^i(\theta,I)) < \frac{1}{N}, \quad i = 0,1,\cdots,N-1.$$

Therefore

$$\pi_1(F^N(\theta,I)) - \theta = \sum_{i=0}^{N-1}[\pi_1(F^{i+1}(\theta,I))] < 1.$$

By (2.1) it follows that

$$(\theta,I) \notin \psi_{\frac{p}{q}}(\mathbf{Z}).$$

The lemma is thus proved.

Lemma 2.2 means that for any closed interval $[\beta,\alpha] \subset (0,\alpha(f_0)]$ there exists $K(\beta) > 0$ such that for any fraction $\frac{p}{q} \in [\beta,\alpha]$, the Birkhoff periodic orbits of type (p,q) are in the closed annulus $S^1 \times [0,K(\beta)]$. According to the argument in [4], we can define the twist modulus $\omega_f(\cdot)$ in $S^1 \times [0,K(\beta)]$ as follows

$$\omega_f(r) = \min_{\substack{0\leq\theta\leq 1 \\ 0\leq I\leq M(\beta)-r}} (F_1(\theta,I) - F_1(\theta,I+r), \pi_1(\widehat{F}\theta,I+r) - \pi_1(\widehat{F}(\theta,I))),$$

where $\hat{F}$ is a lift of f^{-1} and

$$M(\beta) = \sup_{I\leq K(\beta)} \max_{0\leq\theta\leq 1} \left\{F_2(\theta,I), K(\beta), \pi_2(\hat{F}(\theta,I))\right\}.$$

Thus, using the Hausdorff topology of compact metric space $X = S^1 \times [0,M(\beta)]$, we can prove the following proposition (see Propositions 1, 2, 3 in [4] for details).

Proposition 2.1. *Assume that* (A1)-(A4) *hold. Then*

(i) The set of all Mather sets for f in X is closed in Hausdorff topology.

(ii) The rotation number for Mather set is continuous in Hausdorff topology.

From Theorem 2.1 and Proposition 2.1, we obtain immediately

Theorem 2.2. *Assume that* (A1)-(A4) *hold. Then for any* $\omega \in (0,\alpha(f_0)]$, *$f$ has a Mather set M_ω with rotation number ω.*

As for application, we will need to estimate the lower boundary of a Mather set. Define the indexes of $u \in \mathbf{S}$ as following

$$\mathcal{N}_i(u) = \max\{n | \pi_1(f^n(u)) - \pi_1(u) < i\}, \ i = 1,2.$$

Lemma 2.3. *Besides* (A1)-(A4) *suppose that*

$$\mathcal{N}_i(u) < +\infty, \quad \text{for } u \in \mathbf{S}, \ i = 1,2. \tag{2.2}$$

Then there exists $\omega(I_0) > 0$, such that for any $\omega \in (0,\omega(I_0)]$ the Mather set M_ω with rotation number ω is in $S^1 \times [I_0,+\infty)$.

Proof. By the definition of index, it can be easily seen that for any u, there exists a neighborhood $U(u)$ of u, such that

$$\|\mathcal{N}_i(u) - \mathcal{N}_i(u')\| \le 1, \quad \text{for } u' \in U(u),\ i = 1, 2.$$

Then the indexes are bounded uniformly in any compact region. Therefore, for any $I_0 > 0$, there exists $\tilde{N} > 0$, such that

$$\pi_2(u) \ge I_0, \quad \text{for } u \in \mathbf{S} \text{ with } \mathcal{N}_2(u) \ge \widetilde{N}. \tag{2.3}$$

For this $\widetilde{N}$, we can choose $I(\widetilde{N}) > 0$, such that

$$\mathcal{N}_1(u') \ge \widetilde{N}, \quad \text{for } u \in \mathbf{S} \text{ with } \pi_2(u') \ge I(\widetilde{N}_2). \tag{2.4}$$

Let

$$\omega(I_0) = \inf_{\pi_2(u) \le I(\widetilde{N})} (\pi_1(F(u)) - \pi_1(u)).$$

Denote by M_ω the Mather set with rotation number $\omega \in (0, \omega(I_0)]$. From the definition, there must be a $u' \in M_\omega$, such that $\pi_2(u') \ge I(\tilde{N})$. By (2.4) it implies that $\mathcal{N}_1(u') \ge \widetilde{N}$. And then

$$\pi_1(F^{\widetilde{N}}(u')) - \pi_1(u') < 1. \tag{2.5}$$

On the other hand, for any $u \in M_\omega$, we have $j \in \mathbf{Z}$, such that

$$\pi_1(u') \le \pi_1(u) + j \le \pi_1(u') + 1. \tag{2.6}$$

Because F is order-preserving on M_ω, (2.6) implies that

$$\pi_1(F^{\widetilde{N}}(u')) \le \pi_1(F^{\widetilde{N}}(u)) + j \le \pi(F^{\widetilde{N}}(u')) + 1. \tag{2.7}$$

Thus, by (2.5)-(2.7), we know

$$\pi_1(F^{\widetilde{N}}(u)) - \pi_1(u) < 2,$$

which means $\mathbf{N}_2(u) \ge \widetilde{N}$. Consequently, $\pi_2(u) \ge I_0$ by (2.3). This completes the proof of our lemma.

§3. Mather Sets for Sublinear Duffing Equation

At first, we consider a planar Hamiltonian system

$$\dot{\theta} = f_1(\theta, I, t), \qquad \dot{I} = f_2(\theta, I, t), \tag{3.1}$$

where f_1 and f_2 are continuous and 1-periodic functions. We assume the uniqueness and existence of the solution for initial value problem of equation (3.1). Moreover, this solution has continuous derivatives with respect to initial data.

Let $(\theta(t;\theta_0, I_0), I(t;\theta_0, t_0))$ be the solution of equation (3.1) with initial condition $\theta(0) = \theta_0$ and $I(0) = I_0$. Sometimes denote it by $(\theta(t), I(t))$ simply. Let the Poincaré map P of equation (3.1) be

$$P: \ (\theta_0, I_0) \mapsto \theta(1;\theta_0, I_0), I(1;\theta_0, I_0)).$$

It is easy to see that P is an area-preserving diffeomorphism on $S^1 \times [0, +\infty)$. Set

$$\Delta_1(t) = \frac{\partial f_2}{\partial I}(\theta(t), I(t), t), \quad \Delta_2(t) = \frac{\partial f_1}{\partial \theta}(\theta(t), I(t), t),$$

$$\Delta_3(t) = \frac{\partial f_2}{\partial \theta}(\theta(t), I(t), t), \quad \Delta_4(t) = \frac{\partial f_1}{\partial I}(\theta(t), I(t), t).$$

In [11], the following results are proved.

Lemma 3.1 *Suppose that*

(1) $\Delta_i(t) \to 0$, *uniformly on* $t \in [0,1]$, *as* $I_0 \to +\infty$, $i = 1,2$;

(2) $\Delta_3(t_1) \cdot \Delta_4(t_2) \to 0$, *as* $I_0 \to +\infty$, *for* $t_1, t_2 \in [0,1]$.

Then we have

$$\frac{\partial I}{\partial I_0}(t; \theta_0, I) = 1 + o(1), \quad \frac{\partial \theta}{\partial \theta_0}(t; \theta_0, I_0) = 1 + o(1),$$

for $I_0 >> 1$ *and* $t \in [0,1]$.

In addition, if

(3) $\Delta_4(t) < 0$, *for* $I_0 >> 1$ *and* $t \in (0,1]$,

then

$$\frac{\partial \theta}{\partial I_0}(t; \theta_0, I_0) < 0, \quad \text{for } I_0 >> 1 \text{ and } t \in (0,1].$$

Lemma 3.1 shows that under the assmptions (1),(2) and (3) there exists $I_* > 0$, such that P monotonically twists in $S_1 \times [I_*, +\infty)$.

Now we turn to Duffing equation (1.1) with the necessary regular conditions as for system (3.1). It is easily seen, under the condition $(g1)$, that the orbits

$$\gamma_h : \quad \frac{1}{2}y^2 + G(x) = h$$

of the autonomous system

$$\dot{x} = y, \quad \dot{y} = -g(x) \tag{3.2}$$

are closed curves which are star-shaped with respect to the origin O for $h \geq h_0$ with some constant $h_0 > 0$. Denote by $\tau(h)$ the time-map of equation (3.2), and $I(h)$ the area bounded by the closed curve γ_h. That is,

$$\tau(h) = \sqrt{2} \int_{c_-(h)}^{c_+(h)} [h - G(s)]^{-\frac{1}{2}}\, ds,$$

$$I(h) = 2\sqrt{2} \int_{c_-(h)}^{c_+(h)} [h - G(s)]^{\frac{1}{2}}\, ds$$

with $c_+(h) > 0 > c_-(h)$ uniquely determined by

$$G(c_+(h)) = G(c_-(h)) = h.$$

It is easy to see that $\frac{dI(h)}{dh} = \tau(h)$. Denote by $T(x_0, y_0)$ the time in which the orbit γ_h comes from $(c_+(h), 0)$ to (x_0, y_0) with $h = \frac{1}{2}y_0^2 + G(x_0)$. Let

$$\theta(x_0, y_0) = \frac{T(x_0, y_0)}{\tau(h)}.$$

Then

$$\Psi : \quad S^1 \times [h_0, +\infty) \to \mathbf{R}^2 \backslash \{0\}; \quad (\theta(\mathrm{mod} 1), I(h)) \mapsto (x_0, y_0),$$

is a 1-1 map. Moreover, we know that Ψ is a symplectic transformation. Equation (3.2) and (1.1) are then transformed to

$$\dot{\theta} = \frac{1}{\tau(h(I))}, \quad \dot{I} = 0 \tag{3.3}$$

and

$$\dot{\theta} = \frac{1}{\tau(h(I))}(1 - x_h(\theta, I)p(t)), \quad \dot{I} = x_\theta(\theta, I)p(t) \tag{3.4}$$

respectively, where $h = h(I)$ is the inverse function of $I(h) = I$, and x_h, x_θ are partial derivatives of x with respect to h and θ, respectively.

In the following, let c_1, c_2 and c be some constants, and let x and h be large enough.

Lemma 3.2. *Suppose that* (g1) *and* (g3) *hold. Then we have the following estimates*

$$\tau'(h) > 0, \tag{3.5}$$

$$\frac{c_1 I(h)}{h^{i+1}} \le \tau^{(i)}(h) \le \frac{c_2 I(h)}{h^{i+1}}, \quad i = 0, 1, \tag{3.6}$$

$$|\tau''(h)| \le c\frac{I(h)}{h^3}. \tag{3.7}$$

The proof will be given in section 4.

Lemma 3.3. *Under the same assumptions of Lemma* 3.2 *we have*

$$x_h = o(1), \quad y_h = o(1); \tag{3.8}$$

$$|x_{hh}| \le c[\frac{c_+(h) + |c_-(h)|}{h^2}](1 + \frac{|c_-(h)|}{c_+(h)}). \tag{3.9}$$

Moreover

$$h|x_{hh}| = o(1) \tag{3.10}$$

provided that the assumption (g2) *holds.*

The proof is similar to that of Lemma 7 in [11].

Next we consider the solution of equation (3.4). For simplicity, let

$$h(t; \theta_0, I_0) = h(I(t; \theta_0, I_0)).$$

Lemma 3.4. *Under the same assumptions of Lemma* 3.2 *we have*

$$\frac{h(t_1; \theta_0, I_0)}{h(t_2; \theta_0, I_0)} = 1 + o(1), \quad \textit{for } I_0 >> 1 \textit{ and } t_1, t_2 \in [0, 1],$$

which implies that

$$\frac{I(t_1; \theta_0, I_0)}{I(t_2; \theta_0, I_0)} = 1 + o(1),$$

$$\frac{\tau(h(t_1; \theta_0, I_0))}{\tau(h(t_2; \theta_0, I_0))} \le c,$$

for $I_0 >> 1$ *and* $t_1, t_2 \in [0, 1]$.

Proof. Note that

$$\left|\frac{d\sqrt{h}}{dt}\right| = \left|\frac{y}{2\sqrt{h}}p(t)\right| \le E = \max_{0 \le t \le 1} |p(t)|.$$

Then

$$\left|\frac{\sqrt{h(t_1; \theta_0, I_0)}}{\sqrt{h(t_2; \theta_0, I_0)}} - 1\right| \le \frac{E|t_1 - t_2|}{\sqrt{h(t_2; \theta_0, I_0)}}. \tag{3.11}$$

Because $h(t; \theta_0, I_0) >> 1 \Leftrightarrow I_0 >> 1$, the conclusions follow from (3.11) and Lemma 3.2.

Lemma 3.5. *Suppose that* (g1)-(g3) *hold. Then there exists* $I_* > 0$, *such that the Poincaré map* P *of equation* (3.4) *is an area-preserving, order-preserving and monotone twist homeomorphism on* $S^1 \times [I_*, +\infty)$. *In addition,* P *satisfies* (A2), (A3) *and* (A4) *except the preservation of the boundary* $\mathbf{R} \times \{I_*\}$.

The proof will be given in section 4.

Now we can prove Theorem A.

Proof of Theorem A. From Theorem 2.2 and Lemma 2.3, we only need to construct a twist map $\hat{P}$ from P, such that $\widehat{P}$ satisfies the assumptions of Lemma 2.3 in $S^1 \times [I', +\infty)$ with some constant $I' > 0$. Moreover, there exists $I'' \geq I'$, such that $P \equiv \widehat{P}$ for $I \geq I''$. We divide the proof into two steps.

Step 1. Construct an area-preserving homeomorphism $\widetilde{P}$ which preserves $S^1 \times \{I'\}$ and satisfies $\widetilde{P} \equiv P$ for some constant $\widetilde{I} \geq I'$.

Consider the following Hamiltonian

$$H(\theta; I, t) = h(I) - K(I)x(\theta, I)p(t) \tag{3.12}$$

where $x(\theta, I)$ is determined by Ψ, and $K(I)$ is a C^2 function which satisfies

$$|K(I)| \leq 1, \quad |K'(I)| \leq \frac{1}{I} \text{ and } |K''(I)| \leq \frac{1}{I^2}. \tag{3.13}$$

The Hamiltonian system with Hamiltonian (3.12) is

$$\dot{\theta} = \frac{1}{\tau(h)}(1 - K'(I)x(\theta, I)\tau(h)p(t) - K(I)x_h(\theta, I)p(t)), \tag{3.14}$$

$$\dot{I} = K(I)x_\theta(\theta, I)p(t).$$

Denote the Poincaré map of equation (3.14) by

$$\tilde{P}: \quad (\theta, I) \mapsto (\theta + \phi_1(\theta, I), I + \psi_1(\theta, I)).$$

Note that

$$\left.\frac{dh}{dt}\right|_{(3.14)} = K(I)yp(t).$$

Thus similar to the proof of Lemma 3.4, we can get

$$|\psi_1(\theta, I)| \leq \frac{I}{2}, \quad \text{for } I \text{ large enough.}$$

On the other hand, for any $K(I)$ satisfying (3.13), we can prove that

$$K'(I)x_\theta \cdot p(t) + \frac{K(I)x_{\theta h} \cdot p(t)}{\tau(h)} \to 0, \quad \text{for } I_0 \to +\infty \quad \text{and} \quad t \in [0,1]; \tag{3.15}$$

$$\left[-\frac{\tau'(h)}{\tau^3(h)}(1 - K(I)x_h \cdot p(t) - K''(I) \cdot x \cdot p(t) - \frac{2K'(I) \cdot x_h \cdot p(t)}{\tau(h)} + \frac{K(I) \cdot x_{hh} \cdot p(t)}{\tau^2(h)}\right]\Bigg|_{t=t_1} \cdot$$

$$\cdot [K(I)x_{\theta\theta} \cdot p(t)]\Big|_{t=t_2} \to 0, \quad \text{for } I_0 \to +\infty \quad \text{and} \quad t_1, t_2 \in [0,1]. \tag{3.16}$$

Hence, by using Lemma 3.1, we get $I' > 0$, such that

$$\left|\frac{\partial \psi_1}{\partial I}(\theta, I)\right| \leq \frac{1}{2}, \quad |\psi_1(\theta, I)| \leq \frac{I}{2}, \quad \text{for } I \geq I'.$$

Remark that, besides the restriction (3.13), I' is independent of $K(I)$. Then we can set

$$K(I) = 0, \quad \text{for } I \le I', \quad \text{and} \quad K'(I) = \frac{1}{I}, \quad \text{for } I_1 \le I \le I_2,$$

with some suitable constants I_1 and I_2. It follows that

$$K(I) = K(I_1) + \int_{I_1}^{I} K'(I)\, dI \to +\infty, \quad \text{as } I \to +\infty.$$

So there exists $I_2' > I_1$, such that $K(I_2') = 1$. Let

$$K(I) = 1, \quad \text{for } I \ge I_2'.$$

Then for $I \ge I_2'$ the Hamiltonian (3.12) is the Hamiltonian for (3.4). And therefore $\widetilde{P}$ is exactly the map as we describe at the beginning.

Step 2. Choose a non-negative and smooth function $\alpha(I)$, such that

$$\alpha(I) = 0, \quad \text{for } I \ge 2\widetilde{I};$$

$$\alpha(I) > \max_{\substack{0 \le \theta \le 1 \\ \frac{I}{2} \le s \le 2I}} |\phi_1(\theta, s)|, \quad \text{for } I \in \left(\frac{I_1}{2}, \frac{3\widetilde{I}}{2}\right);$$

$$\alpha'(I) < -2 \max_{\substack{0 \le \theta \le 1 \\ \frac{I}{2} \le s \le 2I}} \left|\frac{\partial \phi_1}{\partial I}(\theta, s)\right|, \quad \text{for } I \in \left(\frac{I_1}{2}, \frac{3\widetilde{I}}{2}\right);$$

$$\alpha'(I) \le 0, \quad \text{for } I \in \left(\frac{3\widetilde{I}}{2}, 2\widetilde{I}\right).$$

Define $\widehat{P}$ by

$$\widehat{P}: \quad (\theta, I) \mapsto (\theta + \phi_1 + \alpha(I + \psi_1), I + \psi_1).$$

It can be easily shown that

(1) $\widehat{P}$ is an area-preserving map on $S^1 \times [I', +\infty)$ which preserves $S^1 \times \{I'\}$;

(2) $\widehat{P} \equiv \widetilde{P} \equiv P$ for $I \ge I'' = 2\widetilde{I}$;

(3) $\frac{\partial \phi_1}{\partial I} + \alpha'(I + \psi_1)(1 + \frac{\partial \psi_1}{\partial I}) < 0, \quad$ for $I \ge I'$;

(4) $\phi_1 + \alpha(I + \psi_1) > 0, \quad$ for $I \ge I'$.

Thus $\hat{P}$ satisfies (A1)-(A4).

Next by the elementary phase-analysis for equation (1.1) and equation (3.2), we can prove that P satisfies the condition (2.2). Because we have done only some changes in finite I-interval, $\widehat{P}$ also satisfies condition (2.2). This means that $\widehat{P}$ is a twist map which satisfies all assumptions of Lemma 2.3.

Hence, by using Theorem 2.2 for any $\omega \in (0, \alpha(\widetilde{P}|_{I=I'})]$, $\widehat{P}$ has a Mather set M_ω with rotation number ω. Moreover, for I'' we have $\omega_0 = \omega(I'')$ as shown in Lemma 2.3, such that $M_\omega \subset S^1 \times [I'', +\infty)$ for $\omega \in (0, \omega_0]$. Therefore, M_ω is a Mather set for P. Using standard arguments (see [3], [11] or [12]) we shall obtain all conclusions of Theorem A.

§4. Proofs of Theorem 2.1, Lemma 3.2 and Lemma 3.4

4.1. Proof of Theorem 2.1

We shall work with universal covering $\mathbf{S}$. Define the space $\Phi_{p,q}$ by

$$\Phi_{p,q} = \{ \ \phi : \mathbf{Z} \to \mathbf{R} \ \mid \ \phi \text{ is nondecreasing: } \phi(n+q) = \phi(n) + 1;$$

$$0 < \delta \leq \phi(n+p) - \phi(n) \leq F_0(\phi(n)) - \phi(n) \quad \},$$

where δ will be determined later.

At first, we shall prove that $\Phi_{p,q}$ is nonempty for δ small enough.

Since $\alpha(f_0) > \frac{p}{q}$, we can easily see that

$$F_0^q(\theta_0) - \theta_0 > p, \quad \text{for some fixed } \theta_0 \in \mathbf{R}. \tag{4.1}$$

Define $F_s(\theta) = s\theta + (1-s)F_0\theta$ for any $\theta \in \mathbf{R}$, where $s \in [0,1]$ is a parmeter. By (4.1) there exists $t \in [0,1]$ such that $F_t^q(\theta_0) - \theta_0 = p$. Moreover, F_t is an order-preserving homeomorphism on $\mathbf{R}$ and $F_t(\theta+1) = F_t(\theta) + 1$.

Note that there are q points $F_t(\theta_0), \cdots, F_t^q(\theta_0)$ in the interval $(\theta_0, \theta_0 + p]$ and $F_t^q(\theta_0 + 1) - (\theta_0 + 1) = p$. Therefore, by order-preservation of F_t, we have

$$\# \mid \{F_t^r(\theta_0 + 1), r \in \mathbf{Z}\} \cap (\theta_0, \theta_0 + p] \mid = q,$$

where $\#|B|$ denotes the algebraic number of points contained in set B.

Similarly

$$\# \mid \{F_t^r(\theta_0 + k), r \in \mathbf{Z}\} \cap (\theta_0, \theta_0 + p] \mid = q, \ k = 0, 1, \cdots, p-1.$$

Moreover, $\#|E| = pq$ with $E = \{F_t^r(\theta_0 + k), r \in \mathbf{Z}, k = 0, 1, \cdots, p-1\} \cap (\theta_0, \theta_0 + p]$.

Now we are going to define a map $\phi : \ \mathbf{Z} \to \mathbf{R}$. Let

$$\phi(0) = \theta_0, \quad \phi(i) \in E, \ i = 0, 1, \cdots, pq,$$

$$\phi(i) \leq \phi(i') \quad \text{if and only if} \quad i \leq i'.$$

Then extend it to whole $\mathbf{Z}$ by the recurrence formula

$$\phi(ipq + j) = \phi(j) + ip, \quad \text{for } i \in \mathbf{Z} \quad \text{and } j = 0, 1, \cdots, pq-1.$$

So ϕ is a nondecreasing function.

To prove $\phi(n+q) = \phi(n) + 1$ we remark that

$$F_t^r(\phi_0 + k) = F_t^r(\theta_0) + k, \quad k = 0, 1, \cdots, p-1.$$

Then the numbers of points in $(\theta + k, \theta_0 + k + 1]$, $k = 0, 1, \cdots, p-1$, are the same. From this we deduce that

$$\phi(kq) = \theta_0 + k, \quad k = 0, 1, \cdots, p-1.$$

Moreover

$$F_t^r(\phi(n) + k) = F_t^r(\phi(n)) + k, \quad \text{for any } \phi(n) \in \phi(\mathbf{Z}).$$

This implies that

$$\phi(n + kq) = \phi(n) + k, \quad k = 0, 1, \cdots, p-1. \tag{4.2}$$

By putting $k = 1$ in (4.2) we come to the conclusion $\phi(n+q) = \phi(n) + 1$.

Next suppose that there are p' points $\theta_1, \cdots, \theta_{p'}$ in $E \cup (\theta_0, F_t(\theta_0)]$. Then

$$F_t^i(\theta_0) < F_t^i(\theta_1), \cdots, F_t^i(\theta_{p'}) \leq F_t^{i+1}(\theta_0), \quad i = 1, 2, \cdots, q-1, \tag{4.3}$$

by the order-preservation of $F_t(\theta)$. In addition, any point u in E has the form

$$u = F_t^i(\theta_j), \quad \text{for some } i \in \{1, 2, \cdots, q-1\} \text{ and } j \in \{0, 1, \cdots, p'\}.$$

Consequently, there are $p'q$ points in E which implies that $p' = p$, namely $\phi(p) = F_t(\theta_0)$. Similarly $\phi(n+p) = F_t(\phi(n))$. And then

$$\phi(n+p) - \phi(n) = F_t(\phi(n)) - \phi(n) \le F_0(\phi(n)) - \phi(n).$$

In order to get the lower boundary of δ, such that $\phi \in \Phi_{p,q}$, we need to estimate

$$\inf_{n\in\mathbf{Z}} (\phi(n+p) - \phi(n)).$$

By using $F_t^q(\theta_0) = \theta_0 + p$, we obtain a $\theta_* \in \mathbf{R}$, such that

$$F_t(\theta_*) - \theta_* \ge \frac{p}{q},$$

which means

$$F_t(\theta_*) - \theta_* = (1-t)(F_0(\theta_*) - \theta_*) \ge \frac{p}{q}. \tag{4.4}$$

On the other hand

$$0 < F_m = \inf_{\theta\in\mathbf{R}} (F_0(\theta) - \theta) \le \sup_{\theta\in\mathbf{R}} (F_0(\theta) - \theta) = F_M < +\infty.$$

Thus, by (4.4), we have

$$\begin{aligned} \inf_{n\in\mathbf{Z}} (\phi(n+p) - \phi(n)) &\ge \inf_{n\in\mathbf{Z}} (F_t(\phi(n)) - \phi(n)) \\ &\ge (1-t)F_m \ge \frac{pF_m}{qF_M}. \end{aligned}$$

Therefore, $\delta < \frac{pF_m}{qF_M}$ implies that $\phi \in \Phi_{p,q}$. In the following, we write ϕ instead of ϕ_0 as shown above .

Define an operator $L_{p,q}$ on $\Phi_{p,q}$ by

$$L_{p,q}(\phi) = \sum_{n=0}^{q-1} \mu(T(\phi(n), \phi(n+p))),$$

where $T(\theta, \theta')$ is the region bounded by $F(\{\theta\} \times [0, +\infty)), \{\theta'\} \times [0, +\infty)$ and $\mathbf{R} \times \{0\}$; and $\mu(T(\theta, \theta'))$ denotes the area of $T(\theta, \theta')$.

From (A1) and (A3) we know that $T(\theta, \theta')$ is detemined uniquely by θ and θ'. Moreover, $\mu(T(\theta, \theta'))$ is monotonic decreasing with respect to θ' for any fixed θ.

It is easy to show that

$$\sup_{\theta'-\theta \ge \frac{pF_m}{qF_M}} \mu(T(\theta, \theta')) = T^* < +\infty.$$

Then

$$L_{p,q}(\phi_0) \le qT^*.$$

In addition, by (A4), there exists $\delta_0 > 0$, such that

$$\mu(T(\theta, \theta')) > 2qT^*, \quad \text{for } 0 < \theta' - \theta \le \delta_0. \tag{4.5}$$

We want ϕ_0 is not in the "boundary" of $\Phi_{p,q}$. So we require that

$$\delta < \min\Big\{\frac{pF_m}{qF_M}, \delta_0\Big\}.$$

We also denote by $\Phi_{p,q}$ the extended space.

Given natural induced topology to $\Phi_{p,q}$, then $\Phi_{p,q}$ is a compact and nonempty space. In fact, the embedding

$$\zeta: \quad \Phi_{p,q} \to \mathbf{R}^q/\mathbf{Z}; \quad \phi \mapsto (\phi(0), \cdots, \phi(q-1))$$

is closed. Thus $\Phi_{p,q}$ is a closed subspace of the compact space $\mathbf{R}^q/\mathbf{Z}$. Consequently, $\Phi_{p,q}$ is compact. And then the continuous function $L_{p,q}$ has minimal value on $\Phi_{p,q}$.

Now let ϕ_1 be a minimal point and

$$I(\theta,\theta') = \pi_2(\{\theta'\} \times [0,+\infty) \cup F(\{\theta\} \times [0,+\infty)).$$

Define $\psi_1(n)$ by

$$\psi_1(n) = (\phi_1(n), I(\phi_1(n-p), \phi_1(n))).$$

From (4.5) and

$$L_{p,q}(\phi_1) \le L_{p,q}(\phi_0) \le qT^*,$$

we obtain

$$\inf_{n\in\mathbf{Z}} (\phi_1(n+p) - \phi_1(n)) > \delta. \tag{4.6}$$

Then we can prove that $\psi_1: \mathbf{Z} \to \mathbf{S}$ defines a Birkhoff periodic orbit of type (p,q) for f. Otherwise, according to the method used in [4], we can choose ϵ small enough and contruct $\phi_\epsilon \in \Phi_{p,q}$, such that

$$L_{p,q}(\phi_\epsilon) < L_{p,q}(\phi_1).$$

This is a contradition.

Our proof is thus complete.

4.2. Proof of Lemma 3.2

Using the representation of polar coordinates $x = r\cos\sigma$, $y = r\sin\sigma$, we have

$$I(h) = \frac{1}{2}\oint r^2 d\sigma \quad \text{and} \quad \tau(h) = \int_0^{2\pi} \frac{r^2}{xg(x)+y^2} d\sigma.$$

Moreover, γ_h is star-like with respect to the origin O, so we can consider r as the function of h and σ. As in [11], we know that

$$\tau'(h) = \int_0^{2\pi} \frac{xg(x) - x^2g'(x)}{(xg(x)+y^2)^3} r^2 d\sigma, \tag{4.7}$$

$$\tau''(h) = \int_0^{2\pi} \frac{3(x^2g'(x) - xg(x))(x^2g'(x) + y^2) - (xg(x)+y^2)(x^3g''(x))}{(xg(x)+y^2)^5} r^2 d\sigma. \tag{4.8}$$

Then the estimates (3.5), (3.7) and the right hand of (3.6) come from (4.7), (4.8), assumptions (g1) and (g2).

We are now going to find the lower boundary of $\tau'(h)$. Let us work on the first quadrant.

Denote that

$$\tilde{\tau}'_+(h) = \int_0^{\frac{\pi}{2}} \frac{xg(x) - x^2g'(x)}{(xg(x)+y^2)^3} r^2\, d\sigma,$$
$$\tilde{I}_+ = \int_0^{\frac{\pi}{2}} r^2\, d\sigma,$$
$$I(\delta_+) = \int_0^{\frac{\pi}{2}-\delta_+} r^2\, d\sigma, \tag{4.9}$$

where δ_+ is the angle between positive Y-axis and the line segment $\overline{OQ}$ with $Q = (c_+(\frac{h}{2}), \sqrt{h})$. We see that

$$\tilde{I}_+ \le c_+(h)\cdot\sqrt{2h} \quad \text{and} \quad I(\delta_+) \ge \frac{1}{2}c_+\big(\frac{h}{2}\big)\cdot\sqrt{h}.$$

Moreover, $G(x) \ge \frac{h}{2}$ for $x \ge c_+(\frac{h}{2})$ since $G(x)$ is nondecreasing. Hence, for h large enough, one gets

$$\begin{aligned}\tilde{\tau}'_+(h) &\ge \int_0^{\frac{\pi}{2}-\delta_+} \frac{xg(x) - x^2g'(x)}{(xg(x)+y^2)^3} r^2\, d\sigma \\ &\ge \int_0^{\frac{\pi}{2}-\delta_+} \frac{\epsilon_0 G(x)}{(xg(x)+y^2)^3} r^2\, d\sigma \ge c\frac{I(\delta_+)}{h^2}.\end{aligned} \tag{4.10}$$

On the other hand, we remark that

$$\Big(\frac{G(x)}{x}\Big)' = \frac{xg(x) - G(x)}{x^2} \ge 0, \quad \text{for } |x| > d,$$

which implies that

$$c_+(h) \le 2c_+\big(\frac{h}{2}\big). \tag{4.11}$$

It follows from (4.9)-(4.11) that

$$\tau'_+(h) \ge \frac{c}{h^2}\int_0^{\frac{\pi}{2}} r^2\, d\sigma.$$

In other cases, similar discussion shows that

$$\tau'(h) \ge \frac{c}{h^2} I(h).$$

This completes our proof.

4.3. Proof of Lemma 3.5

Remark that for equation (3.2), the conditions for Lemma 3.1 can be deduced by the following conditions

$$\frac{x_{\theta h}(\theta(t), I(t), p(t))}{\tau(h(t))} \to 0, \quad \text{for } I_0 \to +\infty; \tag{4.12}$$

$$-\frac{\tau'(h(t))}{\tau^3(h(t))}\cdot x_{\theta\theta}(\theta(t_2), I(t_2)) \to 0, \quad \text{for } I_0 \to +\infty \quad \text{and } t_1, t_2 \in [0,1]; \tag{4.13}$$

$$\tau'(h(t)) > 0, \quad \text{for } I_0 >> 1, \quad \text{and } \frac{\tau(h(t))}{\tau'(h(t))} x_{\theta\theta}(\theta(t), I(t)) \to 0, \quad \text{for } I_0 \to +\infty. \tag{4.14}$$

On the other hand, we know that

$$x_\theta(\theta, I) = \tau(h)y, \qquad y_\theta(\theta, I) = -\tau(h)g(x),$$

$$x_{\theta\theta}(\theta, I) = -\tau^2(h)g(x), \qquad x_{\theta h}(\theta, I) = \tau'(h)y + y_h\tau(h).$$

Thus by Lemma 3.2–Lemma 3.4 and some elementary computations, we easily obtain (4.12)-(4.14). Therefore, P is an area-preserving monotone twist homormeophism on $S^1 \times [I_*, +\infty)$ except the preservation of boundaries, where I_* is a suitable positive constant.

Using the following well-known fact for sublinear Duffing equation

$$\tau(h) \to +\infty, \quad \text{as } h \to +\infty,$$

by (3.6), (3.8) and Lemma 3.4, we can easily show that P is an orientation-preserving map satisfying (A3) and preserving the infinity.

Now let $\pi_1(P(\theta_0, I_0)) = \theta_1, \quad \pi_2(P(\theta_0, I_0)) = I_1$. By the definition, we have

$$I(0;\theta_0, I_0) = I_0 = I(0;\theta_1, I_0).$$

Thus from Lemma 3.4, there exist positive constants c_3 and c_4, such that

$$c_3 \le \frac{\tau(h(I(t;I_1,\theta_0)))}{\tau(h(I_0))} \le c_4,$$

which implies that

$$c_5 \le \frac{\tau(h(I(t_1;I_1,\theta_0)))}{\tau(h(I(t_2;I_0,\theta_0)))} \le c_6,$$

for suitable positive constants c_5 and c_6.

Therefore, by (3.8), we have

$$\int_{I_{**}}^{+\infty} F_1(\theta_0, I_0)\, dI_1 \ge \frac{c_5}{2} \int_{I_{**}}^{+\infty} \frac{dI_1}{\tau(h(I_1))} \ge \int_{h(I_{**})}^{+\infty} \frac{\tau(h(I_1))}{\tau(h(I_1))}\, dh(I_1) = +\infty,$$

where $I_{**} \ge I_*$ is a suitable constant.

Note that the above estimation holds uniformly for $\theta \in S^1$, which means that P satisfies (A4). The proof is thus completed.

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