Chin. Ann. of Math.
15B: 4(1994),443-452.

PRIMITIVITY OF SMASH PRODUCT $C_q \# B_q$

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Abstract

It is proved that the Smash product $C_q\#B_q$ is a primitive algebra, where B_q is the Hopf-algebra corresponding to the compact quantum group $S_qU(2)$ and C_q is a Hopf-subalgebra of the topological dual B'_q.

Keywords q-binomal coefficient, Smash product, Quantum group.
1991 MR Subject Classification 05A30, 17B37.

§0. Introduction

The quantum Lorentz group $S_qL(2,\mathbb{C})$ was introduced by Podlés and Woronowicz in 1990. This quantum group is combined with the double group $S_qU(2)$ through the Iwasawa decomposition[3]. Recently, Takeuchi developed a $*$-Hopf algebraic version of their work and described explicitly all finite dimensional representations of the quantum Lorentz group $S_qL(2,\mathbb{C})$[4].

In Takeuchi's work, the $*$-Hopf algebra B_q, which is corresponding to the compact quantum group $S_qU(2)$, was defined as the $\mathbb{C}$-algebra generated by a, b, c, d with the following relations:

$$\left.\begin{array}{ll} b\,a = q\,a\,b, \quad c\,a = q\,a\,c, & d\,b = q\,b\,d, \quad d\,c = q\,c\,d, \\ c\,b = b\,c, \quad a\,d - q^{-1}b\,c = d\,a - q\,b\,c, & \end{array}\right\} \tag{0.1}$$

where q is a real parameter $\neq 0, \pm 1$. The algebra B_q has the following $*$-Hopf algebra structure:

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes 1 & b\otimes 1 \\ c\otimes 1 & d\otimes 1 \end{pmatrix}\cdot\begin{pmatrix} 1\otimes a & 1\otimes b \\ 1\otimes c & 1\otimes d \end{pmatrix}, \tag{0.2}$$

$$\mathcal{E}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{0.3}$$

$$\mathcal{S}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}. \tag{0.4}$$

Let B'_q =$\mathrm{Hom}_{\mathbb{C}}(B_q, \mathbb{C})$, the topological dual of B_q. [4] defined C_q to be a $*$-subalgebra of B'_q generated by the following three elements p, p^{-1} and n, where p was defined as an algebra

Manuscript received July 2, 1992.
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map $B_q \longrightarrow \mathbb{C}$ decided by

$$p: \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix},$$

p^{-1} was the inverse of p in the sense of the convolution product, and n was induced by an opposite algebra map $\pi : B_q \longrightarrow M_2(\mathbb{C})$:

$$a \mapsto \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$c \mapsto \begin{pmatrix} 0 & q^{\frac{1}{2}}(1-q^{-2}) \\ 0 & 0 \end{pmatrix}, \qquad d \mapsto \begin{pmatrix} q^{-\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}.$$

In general,

$$\pi(x) = \begin{pmatrix} p(x) & n(x) \\ 0 & p^{-1}(x) \end{pmatrix}, \quad \text{for any } x \in B_q. \tag{0.5}$$

Then $n \in B_q'$ (but not an algebra map). It was proved that p, p^{-1} and n satisfy the following relations:

$$\left.\begin{aligned} &p^* p = p p^*, \quad n p = q p n, \quad n p^* = q p^* n, \\ &[n^*, n] = (1-q^{-2})(p p^* - p^{-1}(p^{-1})^*). \end{aligned}\right\} \tag{0.6}$$

Moveover, [4] showed that C_q is a $*$-Hopf subalgebra of B_q'.

In this paper, we first prove that C_q is dense in B_q'. Next, we give a representation of a Smash product; the result may be viewed as a development of Xu's theory of the complete ring of linear transformations[7–8]. Finally, we use this technique to show that the Smash product $C_q \# B_q$ is primitive.

§1 q-Binomial Coefficient and the Density of C_q

We start with a definition of q-binomial coefficient.

Let s and t be two non-negative integers. We define inductively so-called "q-binomial coefficient" $\begin{pmatrix} s \\ t \end{pmatrix}_q$ as follows:

$$\begin{pmatrix} s \\ 0 \end{pmatrix}_q = \begin{pmatrix} s \\ s \end{pmatrix}_q = 1,$$
$$\begin{pmatrix} s+1 \\ t \end{pmatrix}_q = \begin{pmatrix} s \\ t-1 \end{pmatrix}_q + \begin{pmatrix} s \\ t \end{pmatrix}_q \cdot q^{2t}, \quad \text{for } 0 < t \le s. \tag{1.1}$$

It is well-defined, and we agree on $\begin{pmatrix} s \\ t \end{pmatrix}_q = 0$ for any $s < t$. Note that our q-binomial coefficient is just usual one when $q = 1$. It is easy to check that

$$\begin{pmatrix} s \\ 1 \end{pmatrix}_q = \begin{pmatrix} s \\ s-1 \end{pmatrix}_q = \sum_{i=0}^{s-1} q^{2i}. \tag{1.2}$$

In the following, we always let e, f, g, h and i, j, k, l be non-negative integers with $0 \le i \le e,\ 0 \le j \le f,\ 0 \le k \le g,\ 0 \le l \le h$ and denote

$$\begin{pmatrix} e & f & g & h \\ i & j & k & l \end{pmatrix}_q = \begin{pmatrix} e \\ i \end{pmatrix}_q \cdot \begin{pmatrix} f \\ j \end{pmatrix}_q \cdot \begin{pmatrix} g \\ k \end{pmatrix}_q \cdot \begin{pmatrix} h \\ l \end{pmatrix}_q.$$

For the sake of convenience, we omit the lower index q in the sequal.

Proposition 1.1. *For any positive integers e, f, g, h, we have*

$$\Delta(a^e) = \sum_{i=0}^{e} \binom{e}{i} a^i b^{e-i} \otimes a^i c^{e-i},$$

$$\Delta(b^f) = \sum_{j=0}^{f} \binom{f}{j} a^j b^{f-j} \otimes b^j d^{f-j},$$

$$\Delta(c^g) = \sum_{k=0}^{g} \binom{g}{k} c^k d^{g-k} \otimes a^k c^{g-k},$$

$$\Delta(d^h) = \sum_{l=0}^{h} \binom{h}{l} c^l d^{h-l} \otimes b^l d^{h-l}.$$

Proof. We prove only the first statement by induction, the others can be proved similarly.

If $e = 1$, the result is clear from (0.2). Now let $e > 1$ and assume that the statement for e is correct. Then

$$\begin{aligned}\Delta(a^{e+1}) &= \Delta(a) \cdot \Delta(a^e)\\ &= (a \otimes a + b \otimes c)\Big(\sum_{i=0}^{e} \binom{e}{i} a^i b^{e-i} \otimes a^i c^{e-i}\Big)\\ &= \sum_{i=1}^{e+1} \binom{e}{i-1} a^i b^{e+1-i} \otimes a^i c^{e+1-i} + \sum_{i=0}^{e} \binom{e}{i} q^{2i} a^i b^{e+1-i} \otimes a^i c^{e+1-i}\\ &= \sum_{i=0}^{e+1} \binom{e+1}{i} a^i b^{e+1-i} \otimes a^i c^{e+1-i}.\end{aligned}$$

Here we have used (1.1). The statement follows by induction.

From the proposition above, one can compute easily

$$\begin{aligned}\Delta(a^e b^f c^g d^h) &= \Delta(a^e)\Delta(b^f)\Delta(c^g)\Delta(d^h)\\ &= \sum_{i,j,k,l} \begin{pmatrix} e & f & g & h \\ i & j & k & l \end{pmatrix} q^{j(e-i)+l(g-k)+(f-j-k)(g-k+l)}\\ &\quad \cdot a^{i+j} b^{e+f-i-j} c^{k+l} d^{g+h-k-l} \otimes a^i b^{j+l} c^{e+g-i-k} d^{f-j} a^k d^{h-l}.\end{aligned}$$

Denote

$$\begin{bmatrix} e & f & g & h \\ i & j & k & l \end{bmatrix} = \begin{pmatrix} e & f & g & h \\ i & j & k & l \end{pmatrix} q^{j(e-i)+l(g-k)+(f-j-k)(g-k+l)}.$$

We need to notice that

$$\begin{bmatrix} e & f & g & h \\ i & j & k & l \end{bmatrix} \neq 0. \tag{1.3}$$

Proposition 1.2. *For any non-negative integers e, f, g, h, we have*

$$\Delta(a^e b^f c^g d^h) = \sum_{i,j,k,l} \begin{bmatrix} e & f & g & h \\ i & j & k & l \end{bmatrix} a^{i+j} b^{e+f-i-j} c^{k+l} d^{g+h-k-l} \otimes a^i b^{j+l} c^{e+g-i-k} d^{f-j} a^k d^{h-l}. \tag{1.4}$$

Proposition 1.3. *For any positive integer s, we have*

$$n^s(a^eb^fc^gd^h)\neq 0 \quad \textit{iff} \quad f=0 \textit{ and } g=s.$$
$$n^{*s}(a^eb^fc^gd^h)\neq 0 \quad \textit{iff} \quad f=s \textit{ and } g=0.$$

Proof. We use induction for s to show the first statement. In the case of $s=1$, from (0.5), we have

$$\begin{aligned}&\pi(a^eb^fc^gd^h)\\ &=\pi(d^h)\cdot\pi(c^g)\cdot\pi(b^f)\cdot\pi(a^e)\\ &=\begin{pmatrix} q^{-\frac12} & 0\\ 0 & q^{\frac12}\end{pmatrix}^h\cdot\begin{pmatrix} 0 & q^{\frac12}(1-q^{-2})\\ 0 & 0\end{pmatrix}^g\cdot\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}^f\cdot\begin{pmatrix} q^{\frac12} & 0\\ 0 & q^{-\frac12}\end{pmatrix}^e\\ &=\begin{pmatrix} p(a^eb^fc^gd^h) & n(a^eb^fc^gd^h)\\ 0 & p^{-1}(a^eb^fc^gd^h)\end{pmatrix},\end{aligned}$$

which implies that $n(a^eb^fc^gd^h)\neq 0$ iff $f=0$ and $g=1$. If $s>1$ and the result for s is correct, then

$$\begin{aligned}&n^{s+1}(a^eb^fc^gd^h)\\ &=(n^s\cdot n)(a^eb^fc^gd^h)\\ &=\sum_{i,j,k,l}\begin{bmatrix} e & f & g & h\\ i & j & k & l\end{bmatrix} n^s\big(a^{i+j}b^{e+f-i-j}c^{k+l}d^{g+h-k-l}\big)\cdot\\ &\quad\cdot n\big(a^ib^{j+l}c^{e+g-i-k}d^{f-j}a^kd^{h-l}\big).\end{aligned}$$

The non-zero term (i,j,k,l) in the summation above occurs only when (i,j,k,l) satisfies the following equations:

$$\begin{cases} e+f-i-j=0, & k+l=s,\\ j+l=0, & e+g-i-k=1.\end{cases}$$

Noting that $0\le i\le e$, we get a unique solution $(i,j,k,l)=(e,0,s,0)$ and $f=0,\ g=s+1$. Hence, $n^{s+1}(a^eb^fc^gd^h)\neq 0$ iff $f=0, g=s+1$. The first statement then follows by induction.

Next, note that $(*\circ\mathcal{S})\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} a & c\\ b & d\end{pmatrix}$ from (0.4), and the definition of $*$-structure on B_q' is

$$\alpha^*(x)=\overline{\alpha(\mathcal{S}(x)^*)},\quad x\in B_q,\quad \alpha\in B_q';$$

the second statement follows immediately.

Remark. It is easy to see that whether $n^i(a^eb^fc^gd^ha^sd^t)$ equals zero or not has nothing to do with s and t.

Proposition 1.4. *Suppose that $s\le f$ or $t\le g$. Then we have*

$$(n^{*s}\cdot n^t)(a^eb^fc^gd^h)\neq 0 \textit{ iff } f=s \textit{ and } g=t\ .$$

Proof.

$$
\begin{aligned}
&(n^{*s}\cdot n^t)(a^eb^fc^gd^h)\\
=&\sum_{i,j,k,l}\begin{bmatrix} e & f & g & h\\ i & j & k & l\end{bmatrix} n^{*s}\big(a^{i+j}b^{e+f-i-j}c^{k+l}d^{g+h-k-l}\big)\cdot\\
&\cdot n^t\big(a^ib^{j+l}c^{e+g-i-k}d^{f-j}a^kd^{h-l}\big).
\end{aligned}
$$

The non-zero term (i,j,k,l) in the summation above occurs only when (i,j,k,l) satisfies the following equations:

$$
\begin{cases} e+f-i-j=s, & k+l=0,\\ e+g-i-k=t, & j+l=0.\end{cases}
$$

Noting that $i\le e$ and $s\le f$ or $t\le g$, we get a unique solution $(i,j,k,l)=(e,0,0,0)$ and $f=s,\ g=t$.

Proposition 1.5. *Suppose that $s\le f$ or $t\le g$. Then*

$$(p^m\cdot n^{*s}\cdot n^t)(a^eb^fc^gd^h)\neq 0 \text{ iff } f=s \text{ and } g=t.$$

Proof.

$$
\begin{aligned}
&(p^m\cdot n^{*s}\cdot n^t)(a^eb^fc^gd^h)\\
=&\sum_{i,j,k,l}\begin{bmatrix} e & f & g & h\\ i & j & k & l\end{bmatrix} p^m\big(a^{i+j}b^{e+f-i-j}c^{k+l}d^{g+h-k-l}\big)\cdot\\
&\cdot(n^{*s}\cdot n^t)\big(a^ib^{j+l}c^{e+g-k-i}d^{f-j}a^kd^{h-l}\big).
\end{aligned}
$$

Since $p^m(a^eb^fc^gd^h)\neq 0$ iff $f=g=0$, the non-zero term (i,j,k,l) occurs in the following case:

$$
\begin{cases} e+f-i-j=0, & k+l=0,\\ j+l=s, & e+g-i-k=t.\end{cases}
$$

We get a unique solution $(i,j,k,l)=(e,s,0,0)$ and $f=s,\ g=t$.

Denote the non-zero element $\mu(efgh)=(n^{*f}\cdot n^g)(a^eb^fc^gd^h)$. Then the following result is obtained easily

$$(p^m\cdot n^{*f}\cdot n^g)(a^eb^fc^gd^h)=\mu(efgh)\,q^{\frac{m}{2}(e+f-g-h)}. \tag{1.5}$$

Theorem 1.1. *C_q is dense in B_q'.*

Proof. We have to prove that for any $\lambda\in B_q\,(\lambda\neq 0)$ there exists $x\in C_q$ such that $x(\lambda)\neq 0$.

From [6], we can write

$$\lambda=\sum_{\substack{e,f,g,h\\ eh=0}}\alpha_{efgh}a^eb^fc^gd^h,\qquad \alpha_{efgh}\in\mathbb{C}.$$

Take

$$s=\min\{f\mid \alpha_{efgh}\neq 0\},\quad t=\min\{g\mid f=s\}.$$

For any $m = 1, 2, \cdots$, we have

$$(p^m \cdot n^{*s} \cdot n^t)(\lambda) = \sum_{\substack{e,h \\ eh=0}} \alpha_{efgh}\mu(esth)q^{\frac{m}{2}(e+s-t-h)}. \tag{1.6}$$

Take

$$-N = \min\{e+s-t-h, \quad 0\}.$$

We see that the polynomial

$$\sum_{\substack{e,h \\ eh=0}} \alpha_{esth}\mu(esth)\, X^{N+e+s-t-h} \tag{1.7}$$

is non-zero, since

$$N+e+s-t-h = N+e'+s-t-h'$$

forces $(e,h) = (e',h')$ in the condition $eh = e'h' = 0$.

On the other hand, if (1.6) keeps zero for any m, then Equation (1.7) has infinite number of solutions, since q is not a root of unity. Hence there exist m, s and t such that

$$(p^m \cdot n^{*s} \cdot n^t)(\lambda) \neq 0.$$

This ends the proof.

§2. A Lemma

To prove the main result, we introduce first an isomorphism theorem for Smash products.

Let H be a Hopf-algebra over a field K. Then $\mathrm{End}_K(H)$ is the quantum double of H in the sense of [3] and [4]. Here, the multiplication as quantum double is the convolution product induced by the Hopf-algebra structure of H. In the following, notations "$\cdot$" and "$*$" represent the composite product and the convolution product of linear transformations of H, respectively. It is easy to see that the dual algebra $H' = \mathrm{Hom}_K(H,K)$ can be embeded in $(\mathrm{End}_K(H), *)$ in a natual way.

We write a linear transformation of H on the right. In particular, we write also $xu = \langle x, u\rangle$ for any $u \in H', x \in H$.

Let A be a sub-bialgebra of H'. For any $x \in H, u \in A$, we define

$$x \rightharpoonup u = \sum_{(u)} u_{(1)}\langle x, u_{(2)}\rangle. \tag{2.1}$$

Then A becomes an H-module algebra. Therefore, we can form a Smash product $A\#H$.

Let H^σ be the algebra of all right multiplications of H. Then (2.1) can be expressed as

$$x \rightharpoonup u = x^\sigma \cdot u\ .$$

In fact, for any $h \in H$,

$$h\Big(\sum_{(u)} u_{(1)}\langle x, u_{(2)}\rangle\Big) = \big\langle h, \sum_{(u)} u_{(1)}\langle x, u_{(2)}\rangle\big\rangle$$
$$= \sum_{(u)}\langle h, u_{(1)}\rangle\langle x, u_{(2)}\rangle$$
$$= \langle hx, u\rangle = (hx)u$$
$$= h(x^\sigma \cdot u),$$

which implies

$$\sum_{(u)} u_{(1)}\langle x, u_{(2)}\rangle = x^\sigma \cdot u.$$

Lemma 2.1. *Let H be a Hopf-algebra over a field K, A a sub-bialgebra of H'. Then the Smash product $A\#H$ is isomorphic to the subalgebra $A * H^\sigma$ of* $(\mathrm{End}_K(H), \cdot)$.

Proof. We first prove that

$$A * H^\sigma = \{u * x^\sigma \mid u \in A,\ x \in H\}$$

is a subalgebra of $(\mathrm{End}_K(H),\ \cdot\ )$. Because of the identity map on H, and

$$I = \mathcal{E} * 1_H^\sigma \in A * H^\sigma,$$

it is enough to show that $A * H^\sigma$ is closed related to the composite product. Indeed, for any

$$u * x^\sigma, v * y^\sigma \in A * H^\sigma,$$

and $h \in H$, we have

$$\begin{aligned} h((u * x^\sigma) \cdot (v * y^\sigma)) &= (h(u * x^\sigma))(v * y^\sigma) \\ &= \Big(\sum_{(h)}(h_{(1)}u)(h_{(2)}x^\sigma)\Big)(v * y^\sigma) \\ &= \Big(\sum_{(h)}\langle h_{(1)}, u\rangle h_{(2)}x\Big)(v * y^\sigma) \\ &= \sum_{(h),(x)}\langle h_{(1)}, u\rangle\langle h_{(2)}x_{(1)}, v\rangle(h_{(3)}x_{(2)})y^\sigma \\ &= \sum_{(h),(x)}\langle h_{(1)}, u\rangle\langle h_{(2)}, x^\sigma_{(1)} \cdot v\rangle h_{(3)}(x^\sigma_{(2)} \cdot y^\sigma) \\ &= h\big(\sum_{(x)}(u * x^\sigma_{(1)} \cdot v) * x^\sigma_{(2)} \cdot y^\sigma\big). \end{aligned}$$

Therefore,

$$(u * x^\sigma) \cdot (v * y^\sigma) = \sum_{(x)}(u * x^\sigma_{(1)} \cdot v) * (x^\sigma_{(2)} \cdot y^\sigma). \tag{2.2}$$

Noting that

$$u * x^\sigma_{(1)} \cdot v \in A,\ x^\sigma_{(2)} \cdot y^\sigma = (x_{(2)}y)^\sigma \in H^\sigma,$$

we complete the proof that $A * H^\sigma$ is a subalgebra of $(\mathrm{End}_K(H),\ \cdot\ )$.

Now, set a map $\varphi : A\#H \longrightarrow A * H^\sigma$ as follows :

$$\varphi : u\#x \longmapsto u * x^\sigma, \quad u \in A, \quad x \in H.$$

To prove that φ is an algebra isomorphism, it is enough to show that φ remains the multilication operations.

$$\begin{aligned}
\varphi((u\#x)(v\#y)) &= \varphi(\sum_{(x)}(u * (x_{(1)} \rightharpoonup v))\#(x_{(2)}y)) \\
&= \varphi\big(\sum_{(x)}(u * x^\sigma_{(1)} \cdot v)\#(x_{(2)}y^\sigma)\big) \\
&= \sum_{(x)}(u * x^\sigma_{(1)} \cdot v) * (x_{(2)}y^\sigma)^\sigma \\
&= \sum_{(x)}(u * x^\sigma_{(1)} \cdot v) * (x^\sigma_{(2)} \cdot y^\sigma) \\
&= (u * x^\sigma) \cdot (v * y^\sigma) \\
&= \varphi(u\#x) \cdot \varphi(v\#y).
\end{aligned}$$

Thus, we have proved that $(A\#H, \#)$ is isomorphic to $(A * H^\sigma, \cdot)$ as algebras.

§3. Main Theorem

Now we are in a position to give our main result.

Theorem 3.1. *$C_q\#B_q$ is a primitive algebra.*

Proof. By Lemma 2.1, $(C_q\#B_q, \#)$ is isomorphic to the subalgebra $(C_q * B_q^\sigma, \cdot)$ of $(\mathrm{End}_{\mathbb{C}}(B_q), \cdot)$. Thus, it is equivalent to prove that $C_q * B_q^\sigma$ is primitive for our purpose. The adventage of such doing is that we get naturally a faithful (right) $C_q * B_q^\sigma$-module B_q. The remaining thing is to show that B_q is irreducible as $C_q * B_q^\sigma$-module, that is, $\lambda(C_q * B_q^\sigma) = B_q$ for any non-zero element $\lambda \in B_q$.

Firstly, suppose $\lambda = 1$, the identity of B_q. In this case, it is clear that

$$\tilde{\lambda} = 1(\epsilon * \tilde{\lambda}^\sigma)$$

for any $\tilde{\lambda} \in B_q$, which shows $1(C_q * B_q^\sigma) = B_q$.

Secondly, suppose that λ has the following form

$$\lambda = \lambda_m = \sum_{e=0}^{m} \alpha_e\, a^e b^{m-e}, \qquad \alpha_e \in \mathbb{C}.$$

In this case, we shall prove $1 \in \lambda(C_q * B_q^\sigma)$ by induction. Then the case is reduced to the above one.

If $m = 1$, then $\lambda = \alpha a + \beta b$. Taking

$$k = \frac{p(a)}{n^*(b)} q^{-1},$$

we get

$$\lambda(p * d^\sigma - kn^* * b^\sigma) = \alpha p(a) 1;$$

hence $1 \in \lambda(C_q * B_q^\sigma)$ in the case of $\alpha \neq 0$. Otherwise, taking

$$k = \frac{n^*(b)}{p(a)} q,$$

we have

$$\lambda(n^* * a^\sigma - kp * c^\sigma) = \beta n^*(b)1;$$

hence

$$1 \in \lambda(C_q * B_q^\sigma).$$

Suppose $m > 1$, and

$$1 \in \lambda_{m-1}(C_q * B_q^\sigma).$$

Then in the case of $\lambda = \lambda_m$, take

$$k = \frac{n^*(a^{m-1}b)}{p(a^m)} \binom{m}{1} q.$$

Thus,

$$
\begin{aligned}
&\lambda(n^* * b^\sigma - kp * d^\sigma)\\
&= \sum_{e=0}^{m} \alpha_e \Big\{ \sum_{i,j} \begin{bmatrix} e & m-e \\ i & j \end{bmatrix} n^*(a^{i+j} b^{m-i-j}) a^i b^j c^{e-i} d^{m-e-j} b \\
&\quad - k \sum_{i,j} \begin{bmatrix} e & m-e \\ i & j \end{bmatrix} p(a^{i+j} b^{m-i-j}) a^i b^j c^{e-i} d^{m-e-j} d \Big\}\\
&= \alpha_0 \Big\{ \begin{bmatrix} 0 & m \\ 0 & m-1 \end{bmatrix} n^*(a^{m-1} b) b^{m-1} db - kp(a^m) b^m d \Big\}\\
&\quad + \sum_{e=1}^{m} \alpha_e \Big\{ \Big(\begin{bmatrix} e & m-e \\ e & m-e-1 \end{bmatrix} n^*(a^{m-1} b) a^e b^{m-e-1} d \\
&\quad + \begin{bmatrix} e & m-e \\ e-1 & m-e \end{bmatrix} n^*(a^{m-1} b) a^{e-1} b^{m-e} c \Big) b - kp(a^m) a^e b^{m-e} d \Big\}\\
&= \sum_{e=1}^{m} \alpha_e a^{e-1} b^{m-e} \Big\{ \binom{m-e}{1} q^{-(m-e-1)} n^*(a^{m-1} b) ad \\
&\quad + \binom{e}{1} q^{m-e} n^*(a^{m-1} b) bc - kp(a^m) q^{-(m-e)} ad \Big\}\\
&= \sum_{e=1}^{m} \alpha_e a^{e-1} b^{m-e} \Big\{ \Big[\Big(\binom{m-e}{1} q^{-(m-e-1)} + \binom{e}{1} q^{m-e+1} \Big) n^*(a^{m-1} b) \\
&\quad - kp(a^m) q^{-(m-e)} \Big] ad - \binom{e}{1} q^{m-e+1} n^*(a^{m-1} b) \Big\}\\
&= \sum_{e=1}^{m} (-\alpha_e) q^{m-e+1} \binom{e}{1} n^*(a^{m-1} b) a^{e-1} b^{m-e}\\
&\quad + \sum_{e=1}^{m} \alpha_e a^{e-1} b^{m-e} \Big\{ \binom{m}{1} q^{-(m-e-1)} n^*(a^{m-1} b) - kp(a^m) q^{-(m-e)} \Big\} ad\\
&= \sum_{e=1}^{m} (-\alpha_e) q^{m-e+1} \binom{e}{1} n^*(a^{m-1} b) a^{e-1} b^{m-e}\\
&= \sum_{e=0}^{m-1} \alpha'_e a^e b^{m-1-e}\\
&= \lambda_{m-1},
\end{aligned}
$$

where

$$\alpha'_e = -\alpha_{e+1} q^{m-e} \binom{e+1}{1} n^*(a^{m-1} b), \quad e = 0, 1, \cdots, m-1.$$

There exists an element x' in $C_q * B_q^\sigma$ such that $\lambda_{m-1} x' = 1$ by the inductive assumption. Take $x = (n^* * b^\sigma - kp * d^\sigma) x'$. Then $1 = \lambda x \in \lambda(C_q * B_q^\sigma)$.

Finally, λ has the following form in the general case

$$\lambda = \sum_{\substack{e,f,g,h \\ eh=0}} \alpha_{efgh} a^e b^f c^g d^h, \qquad \alpha_{efgh} \in \mathbb{C}.$$

Take $T=\max\{e+f \mid \alpha_{efgh}\neq 0\}$. If $T=0$, then

$$\lambda=\sum_{g,h}\alpha_{gh}c^g d^h.$$

If $T>0$, then

$$\begin{aligned}&\lambda(n^{*T}*1^\sigma)\\ &=\sum_{\substack{e,f,g,h\\ eh=0}}\alpha_{efgh}\sum_{i,j,k,l}\begin{bmatrix}e & f & g & h\\ i & j & k & l\end{bmatrix}n^{*T}\big(a^{i+j}b^{e+f-i-j}c^{k+l}d^{g+h-k-l}\big)\cdot\\ &\cdot a^i b^{j+l}c^{e+g-i-k}d^{f-j}a^k d^{h-l}\\ &=\sum_{\substack{e+f=T\\ eh=0}}\alpha_{efgh}n^{*T}(b^T d^{g+h})c^{e+g}d^{f+h}.\end{aligned}$$

There is no similar term to be merged in the condition of $e+f=T$ and $eh=0$, this shows that $\lambda(n^{*T}*1^\sigma)$ is non-zero. Thus, without loss of the generality we can assume that

$$\lambda=\sum_{g,h}\alpha_{gh}c^g d^h,\qquad \alpha_{gh}\in\mathbb{C}.$$

Take $T=\max\{g+h \mid \alpha_{gh}\neq 0\}$. Then

$$\begin{aligned}&0\neq\lambda(n^T*1^\sigma)\\ &=\sum_{g,h}\alpha_{gh}\sum_{k,l}\begin{bmatrix}g & h\\ k & l\end{bmatrix}' n^T\big(c^{k+l}d^{g+h-k-l}\big)b^l c^{g-k}a^k d^{h-l}\\ &=\sum_{g+h=T}\alpha_{gh}n^T(c^T)b^h a^g.\end{aligned}$$

So the case is reduced to the above ones.

This completes the proof of Theorem 3.1.

Remark. We have denoted

$$\begin{bmatrix}e & f\\ i & j\end{bmatrix}=\begin{bmatrix}e & f & 0 & 0\\ i & j & 0 & 0\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}g & h\\ k & l\end{bmatrix}'=\begin{bmatrix}0 & 0 & g & h\\ 0 & 0 & k & l\end{bmatrix}$$

in the proof of Theorem 3.1.

Acknowledgement. I would like to show my gratitute to my advisor Professor Xu Yonghua for his guidance.

References

[1] Drinfeld, V. G., Quantum groups, Proceedings ICM 1986, 798–820.
[2] Lu Diming, A note on bialgebras (preprint).
[3] Podlés, P. & Woronowicz, S. L., Quantum deformation of Lorentz group, *Comm. Math. Phys.,* **130** (1990), 381–431.
[4] Takeuchi, M., Finite dimensional representations of quantum Lorentz group, *Comm. Math. Phys.,* **144:** 3, (1992), 557-580.
[5] Woronowicz, S. L., Compact matrix pseudogroups, *Comm. Math. Phys.,* **111** (1987), 613–665.
[6] Woronowicz, S. L., Twisted $SU(2)$ group: an example of a noncommutative differential calculus, *Publ. RIMS. Kyoto Uni.,* **23** (1987), 117–181.
[7] Xu Yonghua, A theory of rings that are isomorphic to the complete rings of linear transformations I, *Acta. Math. Sinica,* **22** (1979), 204–218.
[8] Xu Yonghua, On the tensor products of algebras, *J. Alg.,* **113** (1988), 1–69.