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DISTRIBUTION OF THE $(0, \infty)$ ACCUMULATIVE LINES OF MEROMORPHIC FUNCTIONS

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Abstract

Suppose that $f(z)$ is a meromorphic function of order $\lambda\,(0 < \lambda < +\infty)$ and of lower order μ in the plane. Let ρ be a positive number such that $\mu \le \rho \le \lambda$.

(1) If $f^{(l)}(z)\,(0 \le l < +\infty)$ has $p\,(1 \le p < +\infty)$ finite nonzero deficient values $a_i\,(i = 1, \cdots, p)$ with deficiencies $\delta(a_i, f^{(l)})$, then $f(z)$ has a $(0, \infty)$ accumulative line of order $\ge \rho$ in any angular domain whose vertex is at the origin and whose magnitude is larger than

$$\max\left(\frac{\pi}{\rho}, 2\pi - \frac{4}{\rho}\sum_{i=1}^{p} \arcsin\sqrt{\frac{\delta(a_i, f^{(l)})}{2}}\right).$$

(2) If $f(z)$ has only $p\,(0 < p < +\infty)(0, \infty)$ accumulative lines of order $\ge \rho : \arg z = \theta_k\,(0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 2\pi, \theta_{p+1} = \theta_1 + 2\pi)$, then $\lambda \le \frac{\pi}{\omega}$, where $\omega = \min\limits_{1\le k\le p}(\theta_{k+1} - \theta_k)$, provided that $f^{(l)}(z)\,(0 \le l < +\infty)$ has a finite nonzero deficient value.

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§1. Introduction

Let $f(z)$ be a meromorphic function, ρ a finite nonnegative number and $a_j\,(j = 1, 2, \cdots k, 0 < k < +\infty, a_j$ may be $\infty)$ be k distinct complex numbers. A ray $\arg z = \theta_0$ is said to be an $(a_1, a_2, \cdots, a_k)$ accumulative line of order $\ge \rho$ of $f(z)$ if for any $\varepsilon > 0$

$$\varlimsup_{r\to\infty} \frac{\log^+\left\{\sum\limits_{j=1}^{k} n(\Omega(\theta_0 - \varepsilon, \theta_0 + \varepsilon, r), f = a_j)\right\}}{\log r} \ge \rho.$$

Here and below, we shall employ the usual notation of Nevanlinna theory as given in [1], [2] and [5].

The angular distribution theory of meromorphic functions, which was found by G. Julia[1], has tremendously developed. The most important result which was due to G. Valiron[1] is the existence of Borel direction of meromorphic functions. Roughly speaking, if we consider a Julia direction as a Borel direction of order zero, almost all of the study of singular directions of meromorphic functions has something related to the Borel directions.

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In the language of the accumulative lines $\arg z = \theta_0$ is said to be a Borel direction of order $\geq \rho$ of $f(z)$ if, for any three distinct numbers $a_j\,(j=1,2,3)$, $\arg z = \theta_0$ is the (a_1, a_2, a_3) accumulative line of order $\geq \rho$ of $f(z)$. From this point of view, the study of Borel directions may be considered as a special case of the study of accumulative lines. For example, the excellent work on the distribution of Borel directions, which was due to Yang Lo and Zhang Guang-hou[1, p.188], is based on the distribution of the $(0,\infty)$ accumulative lines. For this reason, special attention is paid to the $(0,\infty)$ accumulative lines. There are many results on this subject with some different forms.

In this paper, some results on the distribution of the $(0,\infty)$ accumulative lines of meromorphic functions are proved. From them, many known results can be simply derived.

§2. A Lemma

The main tool for our investigation is the Nevanlinna angular function. Refer [3] for the basic properties of these functions. In this section, we shall prove a lemma which will be used later.

Lemma 2.1. *Let $f(z)$ be a meromorphic function of order $\lambda\,(0<\lambda<+\infty)$ and of lower order $\mu\,(0\leq\mu<+\infty)$ and ρ be a positive number such that $\mu\leq\rho\leq\lambda$. Suppose that (r_n) is the sequence of Pólya peaks of order ρ.*

Suppose further that there are no $(0,\infty)$ accumulative lines of order $\geq\rho$ of $f(z)$ in $\Omega(\theta_1,\theta_2)(0\leq\theta_1<\theta_2<2\pi+\theta_1)$. If there is a complex number $a\,(\neq 0,\infty)$ such that for every $\varepsilon>0$ the inequality

$$\operatorname{mes}E\left(\theta;\ \theta_1<\theta<\theta_2,\ \log\frac{1}{|f(r_ne^{i\theta})-a|}>r_n^{\rho-\varepsilon}\right)>K \tag{2.1}$$

holds as n is sufficiently large, where K is a positive number not depending on n and ε, then we have $\theta_2-\theta_1\leq\frac{\pi}{\rho}$.

Proof. If Lemma 2.1 is not true, then we will derive a contradiction from $\theta_2-\theta_1>\frac{\pi}{\rho}$. We first take a fixed number $\alpha_0\,(>0)$ such that $\theta_2-\theta_1-6\alpha_0>\frac{\pi}{\rho}$ and

$$\operatorname{mes}E\left(\theta;\ \theta_1+\alpha_0<\theta<\theta_2-\alpha_0,\ \log\frac{1}{|f(r_ne^{i\theta})-a|}>r_n^{\rho-\varepsilon}\right)>\frac{K}{2} \tag{2.2}$$

for every sufficiently small $\varepsilon>0$.

Since there are no $(0,\infty)$ accumulative lines of order $\geq\rho$ of $f(z)$ in $\Omega(\theta_1,\theta_2)$, there obviously exists a real number τ such that $\tau<\rho$ and

$$\varlimsup_{r\to\infty}\frac{\log(n(\bar\Omega(\theta_1+\alpha_0,\theta_2-\alpha_0;r),f=0)+n(\bar\Omega(\theta_1+\alpha_0,\theta_2-\alpha_0;r),f=\infty))}{\log r}\leq\tau. \tag{2.3}$$

Taking a fixed number $\eta_0\,(>0)$ such that $\tau+4\eta_0<\rho-2\varepsilon$, we have

$$\lim_{n\to\infty}(r_n^{\tau+2\eta_0}\log r_n)r_n^{-\rho+\varepsilon}=0. \tag{2.4}$$

By using Lemma 3.13 in [5, p.252], if n is sufficiently large, the inequality

$$\log\frac{1}{|f(z)-a|}>A(K,\alpha_0,\theta_2-\theta_1,2)r_n^{\rho-\varepsilon}\equiv Ar_n^{\rho-\varepsilon} \tag{2.5}$$

holds for all $z \in \bar{\Omega}(\theta_1 + \alpha_0, \theta_2 - \alpha_0, \frac{1}{2}r_n, 2r_n)$ except some complex numbers which can be enclosed in a set of circles (γ) with finite total number and total sum of the radii not exceeding $\frac{1}{8}\alpha_0 r_n$. Hereinafter, A stands for various positive constants not depending on n.

Denoting by E_n the set of values of r which satisfies $(|z| = r) \cap (\gamma) = \phi$ and $\frac{1}{2}r_n \le r \le 2r_n$, we have

$$\text{mes}E_n \ge \frac{3}{2}r_n - \frac{1}{4}\alpha_0 r_n > r_n. \tag{2.6}$$

Therefore

$$\frac{\text{mes}E_n}{2r_n} \ge \frac{1}{2}. \tag{2.7}$$

From (5), we obtain for $r \in E_n$

$$\begin{aligned} m_{\alpha\beta}\big(r, \frac{1}{f-a}\big) &= \frac{1}{2\pi}\int_\alpha^\beta \log^+ \frac{1}{|f(re^{i\theta}) - a|}d\theta \\ &\ge \frac{A(\beta-\alpha)}{2\pi} r_n^{\rho-\varepsilon} \\ &= A r_n^{\rho-\varepsilon}, \end{aligned} \tag{2.8}$$

where $\alpha = \theta_1 + \alpha_0$ and $\beta = \theta_2 - \alpha_0$.

Since, by the definition of $m_{\alpha\beta}(r, f)$,

$$\begin{aligned} &m_{\alpha\beta}\big(r, \frac{1}{f}\big) + m_{\alpha\beta}\big(r, \frac{1}{f-a}\big) \\ &\le m_{\alpha\beta}\big(r, \frac{f'}{f}\big) + m_{\alpha\beta}\big(r, \frac{f'}{f-a}\big) + m_{\alpha\beta}\big(r, \frac{1}{f'}\big) + O(1) \\ &\le m_{\alpha\beta}\big(r, \frac{f}{f'}\big) + m_{\alpha\beta}\big(r, \frac{1}{f}\big) + O(\log r), \end{aligned}$$

we have

$$m_{\alpha\beta}\big(r, \frac{1}{f-a}\big) \le m_{\alpha\beta}\big(r, \frac{f}{f'}\big) + O(\log r). \tag{2.9}$$

Now we apply the lemma in [2, p.363] to $\frac{f}{f'}$ and $\Omega\,(\theta_1 + \alpha_0, \theta_2 - \alpha_0)$. We conclude that, for every ε' $(0 < \varepsilon' < \frac{1}{8})$ and every $d(> 1)$,

$$m_{\alpha\beta}\big(r, \frac{f}{f'}\big) \le A r^{\frac{\pi}{\beta-\alpha}} \left(S_{\alpha\beta}\big(r, \frac{f}{f'}\big) + 1\right)^d \tag{2.10}$$

for all r except possibly a set $E_{\alpha\beta}$ of values of r with $\overline{\text{dens}}E_{\alpha\beta} < \varepsilon'$.

By using the Theorem in [2, p.137], we deduce that

$$\begin{aligned} S_{\alpha\beta}\big(r, \frac{f}{f'}\big) &= S_{\alpha\beta}\big(r, \frac{f'}{f}\big) + O(1) \\ &= C_{\alpha\beta}\big(r, \frac{f'}{f}\big) + O(1) \\ &\le \bar{C}_{\alpha\beta}(r, f) + \bar{C}_{\alpha\beta}\big(r, \frac{1}{f}\big) + O(1). \end{aligned} \tag{2.11}$$

Suppose that $d_v = |d_v|e^{i\beta_v}$ $(v = 1, 2, \cdots)$ are the distinct zeros and poles of $f(z)$ in $\Omega(\alpha, \beta)$.

From (2.3), we deduce for all sufficiently large r that

$$\begin{aligned} S_{\alpha\beta}\big(r, \frac{f}{f'}\big) &\leq \bar{C}_{\alpha\beta}(r, f) + \bar{C}_{\alpha\beta}\big(r, \frac{1}{f}\big) + O(1) \\ &= 2 \sum_{1<|d_v|<r} \left(\frac{1}{|d_v|^k} - \frac{|d_v|^k}{r^{2k}}\right) \sin k(\beta_v - \alpha) + O(1) \\ &\leq 2 \sum_{1<|d_v|<r} \frac{1}{|d_v|^k} + O(1) \\ &\leq 2k \int_1^r \frac{1}{t^{1+k}} \{\bar{n}(\Omega(\alpha, \beta, t), f = 0) + \bar{n}(\Omega(\alpha, \beta, t), f = \infty\} dt \\ &\quad + \frac{1}{r^k} \{\bar{n}(\Omega(\alpha, \beta, r), f = 0) + \bar{n}(\Omega(\alpha, \beta, r), f = \infty\} + O(1) \\ &\leq r^{\tau + 2\eta_0 - k}, \end{aligned} \tag{2.12}$$

where $k = \frac{\pi}{\theta_2 - \theta_1 - 2\alpha_0}$.

Since, for sufficiently large n,

$$\begin{aligned} \frac{\text{mes}(E_n - E_{\alpha\beta})}{2r_n} &\geq \frac{\text{mes}E_n}{2r_n} - \frac{\text{mes}E_{\alpha\beta} \cap [1, 2r_n]}{2r_n} \\ &\geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \end{aligned} \tag{2.13}$$

the set $E_n \backslash E_{\alpha\beta}$ is not empty. Combining (2.8) and (2.12), we obtain

$$\begin{aligned} Ar_n^{\rho-\varepsilon} &\leq m_{\alpha\beta}\big(r, \frac{1}{f-a}\big) \\ &\leq m_{\alpha\beta}\big(r, \frac{f}{f'}\big) + O(1) \\ &\leq r^k \left\{ S_{\alpha\beta}\big(r, \frac{f}{f'}\big) + 1 \right\}^d \\ &\leq Ar^k \left\{ r^{\tau + 3\eta_0 - k} + 1 \right\}^d \\ &\leq Ar_n^k \left\{ r_n^{\tau + 3\eta_0 - k} + 1 \right\}^d, \end{aligned} \tag{2.14}$$

where $r \in E_n - E_{\alpha\beta}$.

Letting $n \to \infty, \varepsilon \to 0$ and $d \to 1$ in (14), we deduce that

$$\rho \leq \max(k, \tau + 3\eta_0) = \max\left(\frac{\pi}{\theta_2 - \theta_1 - 2\alpha_0}, \tau + 3\eta_0\right). \tag{2.15}$$

This contradiction proves the lemma.

§3. The Distribution of the $(0, \infty)$ Accumulative Lines of Meromorphic Functions

In the general case, a meromorphic function $f(z)$ may have no $(0, \infty)$ accumulative lines. This can be simply illustrated by the functions of the form $e^{g(z)}$, where $g(z)$ is a nonconstant entire function. The situation can be very different if $f(z)$ has a finite nonzero deficient value. In fact, by using a manner similar to the proof of Theorem 1 in [5], we have

Theorem 3.1 *Suppose that $f(z)$ is a meromorphic function of order $\lambda\,(0<\lambda<+\infty)$ and of lower order $\mu\,(0\le\mu<+\infty)$ in the plane. Let ρ be a positive number such that $\mu\le\rho\le\lambda$. If $f(z)$ has $p\,(1\le p<+\infty)$ finite nonzero deficient values $a_i\,(i=1,2,\cdots p)$ with deficiencies $\delta(a_i,f)$, then $f(z)$ has a $(0,\infty)$ accumulative line of order $\ge\rho$ in any angular domain whose vertex is at the origin and whose magnitude is larger than*

$$\max\left(\frac{\pi}{\rho},2\pi-\frac{4}{\rho}\sum_{i=1}^{p}\arcsin\sqrt{\frac{\delta(a_i,f)}{2}}\right). \tag{3.1}$$

From Theorem 3.1, we have

Corollary 3.1. *With the hypotheses on λ,μ and ρ for $f(z)$ in Theorem 3.1, suppose that $f(z)$ has a deficient value a_0 (may be ∞) with deficiency $\delta(a_0,f)$. Then for every angular domain $\Omega(\alpha,\beta)$, the magnitude of which is larger than*

$$\max\left(\frac{\pi}{\rho},2\pi-\frac{4}{\rho}\sum_{i=1}^{p}\arcsin\sqrt{\frac{\delta(a_i,f)}{2}}\right),$$

the inequality

$$\overline{\lim_{r\to\infty}}\frac{\log n(\Omega(\alpha,\beta;r),f=a)}{\log r}\ge\rho \tag{3.2}$$

holds for all $a\ne a_0$ except possibly one complex number.

Proof. Suppose for the contrary there exist two distinct complex numbers a_1 and a_2 such that

$$\overline{\lim_{r\to\infty}}\frac{\log^+\{\sum_{i=1}^{2}n(\Omega(\alpha,\beta;r),f=a_i)\}}{\log r}<\rho. \tag{3.3}$$

Without loss of generality we suppose $a_0\ne\infty$. We set

$$F(z)=\frac{f(z)-a_1}{f(z)-a_2}$$

when $a_i\ne\infty\,(i=1,2)$. It is easily seen that

$$\delta(a_0,f)=\delta\Big(\frac{a_0-a_1}{a_0-a_2},F\Big).$$

If one of the two numbers is ∞, say a_1, we set $F(z)=f(z)-a_2$ and we also have $\delta(a_0,f)=\delta(a_0-a_2,F)$. Appling Theorem 3.1 to $F(z)$, we can easily obtain a contradiction which implies the correctness of the corollary.

It is worth noting that the result of Corollary 2 is slightly stronger than the corresponding one (Corollary 1) in [5]. From Corollary 2 in [5],it is clear that if $f(z)$ has two exceptional values a_1 and a_2 in the sense of (3.3) in $\Omega(\alpha,\beta)$, then one of them must be the deficient value a_0.

Theorem 3.1 implies that if $\rho>\frac{1}{2}$, then $f(z)$ has at least two $(0,\infty)$ accumulative lines of order $\ge\rho$ provided that $f(z)$ satisfies the conditions of Theorem 3.1. As a complement of Theorem 3.1, we give

Theorem 3.2. *With the hypothesis on $f(z),\lambda,\mu$ and ρ in Theorem 3.1, if $f(z)$ has a finite nonzero deficient value, then $f(z)$ has at least one $(0,\infty)$ accumulative line of order ρ.*

We next turn to the discussion of the relation between the distribution of the zeros and poles of meromorphic functions and the growth of their Nevanlinna's characteristics.

Theorem 3.3. *Suppose that $f(z)$ is a meromorphic function of order $\lambda\,(0<\lambda<+\infty)$ and of lower order $\mu\,(0\le\mu<+\infty)$ in the plane. Let ρ be a number such that $\mu\le\rho\le\lambda$. If $f(z)$ has a finite nonzero deficient value a_0 and only $p\,(0<p<+\infty)$ $(0,\infty)$ accumulative lines of order $\ge\rho$:* $\arg z=\theta_k\,(0\le\theta_1<\theta_2<\cdots<\theta_p<2\pi,\theta_{p+1}=\theta_1+2\pi)$*, then we have*

(i) $\lambda\le\frac{1}{2}$, *if* $p=1$,

(ii) $\lambda\le\frac{\pi}{\omega}$, *if* $p>1$,

where $\omega=\min\limits_{1\le k\le p}(\theta_{k+1}-\theta_k)$.

Proof. For the contrary, if $\lambda>\frac{\pi}{\omega}$, we can take a sufficiently small number α_0 such that

$$\lambda>\max_{1\le k\le p}\Big(\frac{\pi}{\theta_{k+1}-\theta_k-2\alpha_0}\Big)\quad\text{and}\quad\alpha_0<\frac{1}{4p}\min\Big(2\pi,\frac{4}{\lambda}\arcsin\sqrt{\frac{\delta(a_0,f)}{2}}\Big).$$

Let (r_n) be a sequence of Pólya peaks of order λ. Then there exists a subsequence (r_{n_j}) of (r_n) such that for a fixed angular domain $\Omega(\theta_{k_0}+\alpha_0,\theta_{k_0+1}-\alpha_0)$, the inequality

$$\text{mes}E\left(\theta;\theta_{k_0}+\alpha_0<\theta<\theta_{k_0+1}-\alpha_0,\log\frac{1}{|f(r_{n_j}e^{i\theta})-a_0|}>r_{n_j}^{\lambda-\varepsilon}\right)>\frac{\alpha_0}{2p}$$

holds for every $\varepsilon>0$.

According to the assumption of the theorem, there is no $(0,\infty)$ accumulative line of order λ in $\Omega(\theta_{k_0}+\alpha_0,\theta_{k_0+1}-\alpha_0)$. Lemma 2.1 implies that $\theta_{k_0+1}-\theta_{k_0}-2\alpha_0\le\frac{\pi}{\lambda}$, which contradicts the definition of α_0. Theorem 3.3 is proved.

The following theorem is the equivalent form of Theorem 3.3.

Theorem 3.3$'$. *Suppose that $f(z)$ is a meromorphic function of order $\lambda\,(\frac{1}{2}<\lambda<+\infty)$. If $f(z)$ has a deficient value $a_0\,(\ne 0,\infty)$, then there must exist two $(0,\infty)$ accumulative lines of order λ of $f(z)$ such that the magnitude of the angle between these two lines is less than or equal to $\frac{\pi}{\lambda}$.*

By the above theorems, many known results on Borel directions can be easily deduced.

According to Theorem 3.3, we have

Corollary 3.2. *Suppose that λ,μ and ρ satisfy the assumptions for $f(z)$ in Theorem 3.3. If $f(z)$ has only $p\,(0<p<+\infty)$ Borel directions of order $\ge\rho$:* $\arg z=\theta_k\,(0\le\theta_1<\theta_2<\cdots<\theta_p<2\pi;\theta_{p+1}=\theta_1+2\pi)$*, then*

(i) $\lambda\le\frac{1}{2}$, *if* $p=1$;

(ii) $\lambda\le\frac{\pi}{\omega}$, *if* $p>1$,

where $\omega=\min\limits_{1\le k\le p}(\theta_{k+1}-\theta_k)$, *provided that $f(z)$ has a deficient value a_0 (finite or not).*

Proof. By a basic result due to G. Valiron [5, p. 126] and the Heine-Borel Theorem, there exist two distinct complex numbers a_1 and a_2 such that $a_i\ne a_0,\infty\,(i=1,2)$ and

$$\varlimsup_{r\to\infty}\frac{\log^+ n\{\bigcup\limits_{k=1}^{p}\Omega(\theta_k+\varepsilon,\theta_{k+1}-\varepsilon;r),f=a_i\}}{\log r}\le\rho'<\rho\quad(i=1,2)$$

for every $\varepsilon>0$. This implies that the set of the $(0,\infty)$ accumulative lines of order $\ge\rho$ of $F(z)=\frac{f(z)-a_1}{f(z)-a_2}$ must be contained in the set of the Borel directions of order $\ge\rho$ of $f(z)$.

Noting that $\frac{a_0-a_1}{a_0-a_2}$ is a nonzero finite deficient value of $F(z)$, we obtain the corollary by applying Theorem 3.3 to $F(z)$.

The equivalent form of Corollary 3.2 gives the main result in [6].

Corollary 3.3.[6] *Let $f(z)$ be a meromorphic function of order $\lambda\,(0<\lambda<+\infty)$. Suppose that $f(z)$ has a deficient value. If $\lambda>\frac{1}{2}$, then there exist two Borel directions of order λ of $f(z)$ such that the magnitude of the angle between them is at most $\frac{\pi}{\lambda}$.*

Corresponding to Theorem 3.1, we have

Corollary 3.4.[4] *Suppose that λ,μ and ρ satisty the assumptions for $f(z)$ in Theorem 3.1. If $f(z)$ has $p\,(1\le p<+\infty)$ deficient value $a_i\,(i=1,2,\cdots p)$ with deficiencies $\delta(a_i,f)$, then $f(z)$ has a Borel direction of order $\ge\rho$ in any angular domain, the magnitude of which is larger than*

$$\max\left(\frac{\pi}{\rho},2\pi-\frac{4}{\rho}\sum_{i=1}^{p}\arcsin\sqrt{\frac{\delta(a_i,f)}{2}}\right).$$

§4. The $(0,\infty)$ Accumulative Lines of Meromorphic Functions and Their Derivatives

In stead of giving direct generalizations of the results in §3 , we prove a general result.

Theorem 4.1. *Suppose that λ,μ and ρ satisfy the hypotheses for $f(z)$ in Theorem 3.1. Let $\Omega(\alpha,\beta)$ be an arbitrary angular domain whose magnitude is larger than $\frac{\pi}{\rho}$. Then $f(z)$ has at least a $(0,\infty)$ accumulative line of order $\ge\rho$ in $\Omega(\alpha,\beta)$ provided that $f'(z)$ has such a line in $\Omega(\alpha,\beta)$.*

Proof. The proof of the theorem is based on the comparison between the angular counting function of $f(z)$ and that of $f'(z)$.

If $f(z)$ has no $(0,\infty)$ accumulative line of order $\ge\rho$ in $\Omega(\alpha,\beta)$, then for every $\eta>0$ there exists a number τ such that $\tau<\rho$ and

$$\overline{\lim_{r\to\infty}}\frac{\log^+(n(\Omega(\alpha+\eta,\beta-\eta;r),f=0)+n(\Omega(\alpha+\eta,\beta-\eta;r),f=\infty)}{\log r}\le\tau. \tag{4.1}$$

On the other hand, since $f'(z)$ has a $(0,\infty)$ accumulative line of order $\ge\rho$ in $\Omega(\alpha,\beta)$, there exists a ray $\arg z=\theta_0$ such that $\alpha<\theta_0<\beta$ and

$$\overline{\lim_{r\to\infty}}\frac{\log\{n(\Omega(\theta_0-\varepsilon,\theta_0+\varepsilon;r),f'=0)+n(\Omega(\theta_0-\varepsilon,\theta_0+\varepsilon;r),f'=\infty)}{\log r}\ge\rho \tag{4.2}$$

for every $\varepsilon>0$.

We may take a fixed number η_0 such that

$$\beta-\alpha-6\eta_0>\frac{\pi}{\rho}\ \text{ and }\ \alpha+3\eta_0<\theta_0<\beta-3\eta_0.$$

From (4.2), it is easy to see that for every $\varepsilon'>0$ there exists a sequence $r_n\to\infty\,(n\to\infty)$ such that

$$n(\Omega(\alpha+3\eta_0,\beta-3\eta_0;r_n),f=0)+n(\Omega(\alpha+3\eta_0,\beta-3\eta_0;r_n),f=\infty)\ge r_n^{\rho-\varepsilon'} \tag{4.3}$$

for every sufficiently large n.

Denoting the zeros and poles of $f(z)$ by

$$d_v=|d_v|e^{i\beta_v}\ (v=1,2,\cdots)$$

and the zeros and poles of $f'(z)$ by

$$d'_v = |d'_v|e^{i\beta'_v} \ (v = 1, 2, \cdots)$$

and setting

$$k = \frac{\pi}{\beta - \alpha - 2\eta_0},$$

we have from (4.1)

$$\begin{aligned}
&C_{\alpha+\eta_0,\beta-\eta_0}(r, f) + C_{\alpha+\eta_0,\beta-\eta_0}\big(r, \frac{1}{f}\big)\\
&= 2 \sum_{\substack{1<|d_v|<r\\ \alpha+\eta_0<\beta_v<\beta-\eta_0}} \left(\frac{1}{|d_v|^k} - \frac{|d_v|^k}{r^{2k}}\right) \sin k(\beta_v - \alpha - \eta_0)\\
&\le 2 \sum_{\substack{1<|d_v|<r\\ \alpha+\eta_0<\beta_v<\beta-\eta_0}} \frac{1}{|d_v|^k}\\
&= 2k \int_1^r \frac{1}{t^{1+k}} \{n(\Omega(\alpha+\eta_0, \beta-\eta_0, t), f = 0) + n(\Omega(\alpha+\eta_0, \beta-\eta_0, t), f = \infty)\} dt\\
&\quad + \frac{1}{r^k}\{n(\Omega(\alpha+\eta_0, \beta-\eta_0, r), f = 0) + n(\Omega(\alpha+\eta_0, \beta-\eta_0, r), f = \infty)\}\\
&\le r^{\tau+2\varepsilon'-k} + O(1).
\end{aligned} \tag{4.4}$$

On the other hand we have

$$\begin{aligned}
&C_{\alpha+\eta_0,\beta-\eta_0}(2r, f') + C_{\alpha+\eta_0,\beta-\eta_0}\big(2r, \frac{1}{f'}\big)\\
&= 2 \sum_{\substack{1<|d'_v|<2r\\ \alpha+\eta_0<\beta'_v<\beta-\eta_0}} \left(\frac{1}{|d'_v|^k} - \frac{|d'_v|^k}{(2r)^{2k}}\right) \sin k(\beta'_v - \alpha - \eta_0)\\
&\ge 2 \sum_{\substack{1<|d'_v|<r\\ \alpha+2\eta_0<\beta'_v<\beta-2\eta_0}} \left(\frac{1}{|d'_v|^k} - \frac{|d'_v|^k}{(2r)^{2k}}\right) \sin k(\beta'_v - \alpha - \eta_0)\\
&\ge 2 \sin k\eta_0 \Big\{k \int_1^r \frac{1}{t^{1+k}} (n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, t), f' = 0)\\
&\quad + n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, t), f' = \infty)) dt\\
&\quad + \frac{1}{r^k}(n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, r), f' = 0) + n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, r), f' = \infty))\\
&\quad - \frac{r^k}{(2r)^{2k}}(n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, r), f' = 0) + n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, r), f' = \infty))\\
&\quad + \frac{k}{(2r)^{2k}} \int_1^r \frac{1}{t^{1-k}} (n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, t), f' = 0)\\
&\quad + n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, t), f' = \infty)) dt\Big\}\\
&\ge (1 - \frac{1}{2^{2k}}) \frac{\sin k\eta_0}{r^k} \{n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, r), f' = 0)\\
&\quad + n(\Omega(\alpha+2\eta_0, \beta-2\eta_0, r), f' = \infty))\}.
\end{aligned} \tag{4.5}$$

Since

$$C_{\alpha\beta}(r, f') = C_{\alpha\beta}(r, f) + \bar{C}_{\alpha\beta}(r, f)$$

and

$$\begin{aligned} C_{\alpha\beta}\big(r, \frac{1}{f'}\big) &\le C_{\alpha\beta}\big(r, \frac{f}{f'}\big) + C_{\alpha\beta}\big(r, \frac{1}{f}\big) \\ &\le S_{\alpha\beta}\big(r, \frac{f'}{f}\big) + C_{\alpha\beta}\big(r, \frac{1}{f}\big) + O(1) \\ &\le \bar{C}_{\alpha\beta}(r, f) + 2C_{\alpha\beta}\big(r, \frac{1}{f}\big) + O(1) \end{aligned}$$

for any angular domain $\Omega(\alpha, \beta)$, we have

$$\begin{aligned} &C_{\alpha+\eta_0,\beta-\eta_0}\big(r, f'\big) + C_{\alpha+\eta_0,\beta-\eta_0}\big(r, \frac{1}{f'}\big) \\ \le\; &3\left(C_{\alpha+\eta_0,\beta-\eta_0}(r, f) + C_{\alpha+\eta_0,\beta-\eta_0}\big(r, \frac{1}{f}\big)\right) + O(1). \end{aligned} \tag{4.6}$$

Substituting (4.4) and (4.5) in (4.6), we deduce by simple calculations that

$$\begin{aligned} r_n^{\rho-\varepsilon'-k} &\le C_{\alpha+\eta_0,\beta-\eta_0}(2r_n, f') + C_{\alpha+\eta_0,\beta-\eta_0}\big(2r_n, \frac{1}{f'}\big) \\ &\le 3\left(C_{\alpha+\eta_0,\beta-\eta_0}(2r_n, f) + C_{\alpha+\eta_0,\beta-\eta_0}\big(2r_n, \frac{1}{f}\big)\right) + O(1) \\ &\le A r_n^{\tau+\varepsilon'-k} + O(1) \end{aligned} \tag{4.7}$$

for a sequence (r_n) which satisfies (4.1).

Noting that $\rho > k$ and ε' can be arbitrary small, we let n tend to the infinity and obtain $\rho \le \tau$. This contradicts our assumption on τ and the theorem is proved.

By Theorem 4.1 the following theorems can be easily deduced.

Theorem 4.2. *Suppose that $f(z)$ is a meromorphic function of order $\lambda\,(0 < \lambda < +\infty)$ and of lower order $\mu\,(0 \le \mu < +\infty)$ in the plane. Let ρ be a number such that $\mu \le \rho \le \lambda$. If $f(z)$ has only $p\,(0 < p < +\infty)$ $(0, \infty)$ accumulative lines of order $\ge \rho$: $\arg z = \theta_k\ (0 \le \theta_1 < \theta_2 < \cdots < \theta_p < 2\pi, \theta_{p+1} = \theta_1 + 2\pi)$, then*

(i) $\lambda \le \frac{1}{2}$, *if* $p = 1$,

(ii) $\lambda \le \frac{\pi}{\omega}$, *if* $p > 1$,

where

$$\omega = \min_{1 \le k \le p} (\theta_{k+1} - \theta_k),$$

provided that $f^{(l)}(z)\,(l \ge 0)$ has a deficient value $a_0\,(\ne 0, \infty)$.

Theorem 4.3. *Suppose that λ, μ and ρ satisty the hypotheses for $f(z)$ in Theorem 4.2. If $f^{(k)}(z)$ has $p\,(1 \le p < +\infty)$ finite nonzero deficient values $a_i\,(i = 1, 2, \cdots p)$ with deficiencies $\delta(a_i, f^{(k)})$, then $f(z)$ has a $(0, \infty)$ accumulative line of order $\ge \rho$ in any angular domain whose vertex is at the origin and whose magnitude is larger than*

$$\max\left(\frac{\pi}{\rho}, 2\pi - \frac{4}{\rho}\sum_{i=1}^{p} \arcsin\sqrt{\frac{\delta(a_i, f)}{2}}\right).$$

Remark. As for the results concerning Borel directions in this paper, by using Nevanlinna's angular characteristic, they can be easily generalized by proving a result corresponding to Theorem 4.1.

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