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NOETHERIAN GR-REGULAR RINGS ARE REGULAR**

LI HUISHI*

Abstract

It is proved that for a left Noetherian $\mathbb{Z}$-graded ring A, if every finitely generated graded A-module has finite projective dimension (i.e., A is gr-regular) then every finitely generated A-module has finite projective dimension (i.e., A is regular). Some applications of this result to filtered rings and some classical cases are also given.

Keywords Graded ring, Regular ring, Projective dimension.
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§0. Introduction

In the study of graded ring theory many ungraded natures have been derived by using the graded techniques. For instance, let A be a $\mathbb{Z}$-graded ring, then A is graded Noetherian if and only if A is Noetherian; A has finite graded (homological) global dimension if and only if A has finite (homological) global dimension; A is a graded maximal order if and only if A is a maximal order, etc., (see e.g. [7], [2]). However, the question we raise here seems to be missing in the literature: For a $\mathbb{Z}$-graded ring A, if every finitely generated graded A-module has finite projective dimension (in this case A is called a left gr-regular ring), does it follow that every finitely generated A-module has finite projective dimension (i.e., A is a left regular ring)? In this note we will give a positive answer to this question when the ring considered is left Noetherian, and some applications will also be given.

§1. Preliminaries

For a general theory of graded rings we refer to [7].

Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded ring, where $\mathbb{Z}$ is the additive group of integers, and A-gr the category of graded left A-modules, where the morphisms in A-gr are graded morphisms of degree zero. Recall that an object $P \in A$-gr is gr-projective if and only if P is a gr-direct summand of a gr-free object in A-gr; the graded Jacobson radical of A, denoted by $J^g(A)$, is the largest proper graded ideal of A such that its intersection with A_0 is in the Jacobson radical of A_0.

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*Department of Mathematics, Shaanxi Normal Univrsity Xi'an 710062, China.
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Lemma 1.1.[7] *Let $P \in A$-gr. Then P is a projective object in A-gr if and only if P is projective as an A-module. Hence for any $M \in A$-gr, the graded projective dimension, denoted by gr.p.dim$_A M$, is equal to p.dim$_A M$, the usual projective dimension of M as an A-module.*

The following theorem is well known.

Theorem 1.1 (First "Change of Rings" Theorem). *Let X be a regular non-invertible homogeneous normalizing element of A (i.e., $XA = AX$). Put $A^* = A/XA$. If M is an A^*-module such that p.dim$_{A^*} M = n < \infty$, then p.dim$_A M = n + 1$. In particular gl.dim$A \geq 1+$ gl.dimA^* provided that gl.dimA^* is finite.*

Theorem 1.2[4] (Third "Change of Rings" Theorem for graded rings). *Let A, X and A^* be as in Theorem 1.1 above and suppose now that A is left Noetherian and $X \in J^g(A)$. If M is a finitely generated X-torsionfree graded A-module, then*

$$p.\dim_{A^*}(M/XM) = p.\dim_A M.$$

We also recall some notions on ($\mathbb{Z}$-) filtered rings. For some details concerning Noetherian filtered rings we refer to [1], [3] and [4]. Let R be a filtered ring with an increasing filtration

$$FR = \{F_n R,\ F_n R \subseteq F_{n+1} R,\ F_n R F_m R \subseteq F_{n+m} R,\ n, m \in \mathbb{Z}\}$$

consisting of additive subgroups of R such that

$$R = \bigcup_{n \in \mathbb{Z}} F_n R$$

(i.e., R is exhaustive). Then there are two graded rings associated with FR: the Rees ring

$$\widetilde{R} = \bigoplus_{n \in \mathbb{Z}} F_n R$$

and the associated graded ring

$$G(R) = \bigoplus_{n \in \mathbb{Z}} F_n R / F_{n-1} R.$$

Since $1 \in F_0 R \subseteq F_1 R$ we write X to be the homogeneous element of degree 1 in $\widetilde{R}_1 = F_1 R$ represented by 1. Then X is a central regular element in $\widetilde{R}$ such that

$$\widetilde{R}/X\widetilde{R} \cong G(R)$$

as graded rings,

$$\widetilde{R}/(1-X)\widetilde{R} \cong R \quad \text{and} \quad \widetilde{R}_{(X)} \cong R[t, t^{-1}],$$

where $\widetilde{R}_{(X)}$ is the localization of $\widetilde{R}$ at the mutiplicatively closed subset $\{1,\ X,\ X^2,\ \cdots\}$ and $R[t, t^{-1}]$ is the usual ring of finite Laurent series over R in the commuting variable t.

§2. Main Result

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded ring. Consider the polynomial ring $A[t]$ over A in a commuting variable t. $A[t]$ has the "mixed" $\mathbb{Z}$-gradation defined by

$$A[t]_n = \Big\{ \sum_{i+j=n} a_i t^j,\ a_i \in A_i \Big\},\quad n \in \mathbb{Z}.$$

Obviously, A is a graded subring of $A[t]$ with respect to the "mixed" gradation on $A[t]$.

Lemma 2.1. *Let A and $A[t]$ be as above. Suppose that A is left Noetherian and left gr-regular. Then for any finitely generated graded left $A[t]$-module M, $p.dim_A M < \infty$; and moreover $A[t]$ is left gr-regular with respect to the "mixed" gradation.*

Proof. Let M be any finitely generated graded left $A[t]$-module. Then $M \in A$-gr. Let $M^{(0)}$ be a finitely generated graded A-submodule of M such that $M = A[t]M^{(0)}$. Put

$$M^{(n)} = \sum_{i=0}^{n} t^i M^{(0)}$$

for positive $n \in \mathbb{Z}$. Then obviously $M^{(n)}$, and hence $M^{(n)}/M^{(n-1)}$, is finitely generated in A-gr. Moreover, since

$$M^{(n+1)} = tM^{(n)} + M^{(n)},$$

left multiplication by t gives sequence of graded A-module surjections

$$M^{(0)} \to M^{(1)}/M^{(0)} \to M^{(2)}/M^{(1)} \to \cdots .$$

Let K_n be the kernel of the resulting map

$$M^{(0)} \to M^{(n)}/M^{(n-1)}.$$

Then K_n is a graded A-submodule of $M^{(0)}$ and $\{K_i\}$ is an incraesing chain. Hence $K_{n+l} = K_n$ for some n and all l. Consequently

$$M^{(n+l+1)}/M^{(n+l)} \cong M^{(n)}/M^{(n-1)}$$

as graded A-modules and

$$p.\dim_A(M^{(n+l+1)}/M^{(n+l)}) = p.\dim_A(M^{(n)}/M^{(n-1)}).$$

Since A is graded regular by assumption, it is then well known that

$$\begin{aligned} p.\dim_A M^{(n+l)} &\le \sup\{p.\dim_A M^{(0)}, p.\dim_A(M^{(1)}/M^{(0)}), \cdots, p.\dim_A(M^{(n)}/M^{(n-1)})\} \\ &= w, \end{aligned}$$

say, hence

$$p.\dim_A\big(\overset{\infty}{\underset{n=0}{\oplus}} M^{(n)}\big) = w.$$

However, there is an exact sequece

$$0 \to \oplus M^{(n)} \xrightarrow{\varepsilon} \oplus M^{(n)} \xrightarrow{\pi} M \to 0,$$

where ε: $(m^{(n)}) \mapsto (m'^{(n)})$ with $m'^{(n)} = m^{(n)} - m^{(n-1)}$ and π: $(m^{(n)}) \mapsto \sum m^{(n)}$. Therefore

$$p.\dim_A M \le w + 1 < \infty.$$

Now, consider the following exact sequence of $A[t]$-modules

$$0 \to M[t] \to M[t] \xrightarrow{e} M \to 0, \tag{2.1}$$

where

$$M[t] = A[t] \otimes_A M, \quad e: \ t^i \otimes m \mapsto t^i m.$$

Then it is well known (cf. [8] Lemma 9.27.) that

$$p.\dim_A M = p.\dim_{A[t]} M[t]$$

and the exactness of (2.1) yields

$$p.\dim_{A[t]}M \leq 1 + p.\dim_{A[t]}M[t] \leq 1 + p.\dim_A M \leq \infty.$$

This proves the graded regularity of $A[t]$.

Theorem 2.1. *Let A be a left Noetherian $\mathbb{Z}$-graded ring. If A is gr-regular then A is regular.*

Proof. Once again let $A[t]$ be the polynomial ring with the "mixed" gradation. Consider the (graded) localization $A[t]_{(t)}$ of $A[t]$ at the multiplicatively closed subset $\{1,\ t,\ t^2,\ \cdots\}$, then $A[t]_{(t)} \cong A[t,t^{-1}]$ as graded rings, where $A[t,t^{-1}]$ also has the "mixed" gradation:

$$A[t,t^{-1}]_n = \Big\{ \sum_{i+j=n} a_i j^j,\ a_i \in A_i \Big\},\quad n \in \mathbb{Z},$$

in particular

$$A[t,t^{-1}]_0 = \sum_{i+j=0} A_i t^j \cong A.$$

If M is any finitely generated graded $A[t,t^{-1}]$-module, say

$$M = \sum_{i=1}^{s} A[t,t^{-1}]\xi_i,$$

where all ξ_i are homogeneous elements of M, then

$$M_0 = \sum_{i=1}^{s} A[t]\xi_i$$

is a finitely generated graded $A[t]$-module such that

$$A[t,t^{-1}] \otimes_{A[t]} M_0 = M.$$

It follows from Lemma 2.1 that

$$p.\dim_{A[t,t^{-1}]}M < \infty$$

and hence $A[t,t^{-1}]$ is gr-regular. Now the equivalence of categories[1]:

$$A[t,t^{-1}]_0\text{−mod} \ \leftrightarrow\ A[t,t^{-1}]\text{−gr}$$

gives the regularity of A.

Question. Is it possible to drop the Noetherian condition in the theorem?

§3. Some Applications

In this section we prove the following theorems.

Theorem 3.1. *Let A be a $\mathbb{Z}$-graded ring and X a regular non-invertible homogeneous normalizing element in A (i.e., $XA = AX$). Put $A^* = A/XA$. Suppose that A is left Noetherian and $X \in J^g(A)$. If A^* is left gr-regular (hence left regular by the foregoing theorem) then A is left regular.*

Proof. In view of Theorem 2.1. it suffices to prove that A is left gr-regular. Let M be any finitely generated graded left A-module and put

$$t(M) = \{m \in M,\ X^p m = 0 \text{ for some integer } \ p > 0\}.$$

The sequence

$$0 \to t(M) \to M \to M/t(M) \to 0 \tag{3.1}$$

is exact in A-gr. Moreover $M/t(M)$ is X-torsion free and $X^k t(M) = 0$ for some integer $k > 0$ since $t(M)$ is finitely generated too. Hence an easy induction on k together with Theorem 1.1. yields: $p.\dim_A t(M) < \infty$. On the other hand, Theorem 1.2. entails

$$p.\dim_A(M/t(M)) < \infty.$$

Hence it follows from the exactness of (3.1) that $p.\dim_A M < \infty$. This shows that A is gr-regular as desired.

With notations as given in section 1, recall from [3] that a filtered ring R with filtration FR is called a left Zariski ring if the Rees ring $\widetilde{R}$ of R is left Noetherian and $X \in J^g(\widetilde{R})$ (or equivalently, if $F_{-1}R$ is contained in the Jacobson radical of F_0R). In [4] it has been proved that if $G(R)$ has finite global dimension, then

$$\text{gl.dim}\widetilde{R} = 1 + \text{gl.dim}G(R).$$

Now, with satisfaction we mention the following

Theorem 3.2. *Let R be a left Zariski ring with filtration FR. Suppose that $G(R)$ is left gr-regular* (*hence left regular*) *then $\widetilde{R}$ and R are left regular.*

Proof. Since $\widetilde{R}/X\widetilde{R} \cong G(R)$, it follows immediately from Theorem 3.1 that $\widetilde{R}$ is left regular. The fact that R is left regular has been proved in [1, Corollary 5.8], but here we may obtain this result directly from the graded ring isomorphism

$$\widetilde{R}_{(X)} \cong R[t, t^{-1}],$$

where the latter one is strongly $\mathbb{Z}$-graded with the natural gradation:

$$R[t, t^{-1}]_n = Rt^n, \quad n \in \mathbb{Z}.$$

By using Theorem 3.2 the condition gl.dim$G(R) < \infty$, mentioned in [5, Theorem 2.4.], can be replaced by the condition that $G(R)$ is left gr-regular, because in this case $\widetilde{R}$ will be left regular and consequently the localization sequence in [5] works. To be precise, this result may be rementioned as follows.

Theorem 3.3 ([5, Theorem 2.4.]). *Let R be a left Zariski ring with filtration FR such that $G(R)$ is left gr-regular. Then there is an injection*

$$K_0(R) \hookrightarrow K_0(G(R))$$

mapping $[R]$ to $[G(R)]$, where $K_0(-)$ denotes the K_0-group in the sense of algebraic K-theory.

Remark 3.1. By using Theorem 3.1, some classical results may be easily recaptured. For example, let A be a left Noetherian regular ring and σ an automorphism, then the skew polynomial ring $R = A[t, \sigma]$, regarded as a graded ring with gradation

$$R_n = At^n, \quad n \in \mathbb{Z},$$

satisfies the conditions of Theorem 3.1 by putting $X = t$ and hence is left regular. It follows that each of the following rings is left regular.

(1) $A[t, \sigma, \delta]$, where δ is a σ-derivation of A;

(2) $A[t, t^{-1}, \sigma]$;

(3) The crossed product $A * G$ of A by G, where G is a poly-infinite cyclic group;

(4) The crossed product $A*U(\mathbf{g})$ of A by $U(\mathbf{g})$, where R is a k-algebra over a commutative ring k and $\mathbf{g}$ a k-Lie algebra of finite dimension.

We refer to [6] for some details about these rings.

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