Chin. Ann. of Math.
15B: 4(1994),469-488.

FOURIER PSEUDOSPECTRAL-FINITE DIFFERENCE METHOD FOR TWO-DIMENSIONAL VORTICITY EQUATION

Guo Benyu* Xiong Yueshan*

Abstract

A Fourier pseudospectral-finite difference sheme is proposed for solving two-dimensional vorticity equations. The generalized stability and the convergence are proved. The numerical results are given.

Keywords Vorticity equation, Fourier pseudospectral-finite difference method, Generalized stability, Convergence.

1991 MR Subject Classification 65M06, 65M70.

§1. Introduction

In the past twenty years, spectral and pseudospectral methods developed rapidly. Since both of them have the accuracy of "infinite" order, they have been widely applied to computational fluid dynamics (see [1-6]). On the other hand, some authors used various filtering techniques to weaken the nonlinear instability in computation (see [5,7-10]).

In studying the flow in tub and other related problems we meet unilaterally periodical boundary conditons (see [11-15]). Such problems could be solved numerically by spectral-finite difference method or Fourier-Chebyshev spectral method as in [11-13, 16-18], provided that the domain is rectangular. Otherwise it is better to use spectral-finite element method (see [19-21]).

As we know, pseudospectral approximation can be performed more easily than spectral one. In particular, it is easier to deal with nonlinear terms. Clearly, if the domain is rectangular and the boundary condition is unilaterally periodical, then it is natural to apply pseudospectral-finite difference method. But when the viscosity is small, the computation is less stable than spectral-finite difference method. Thus it is reasonable to adopt the filtering technique as in [22].

Now, let $\xi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ be the vorticity function and the stream function respectively. $\nu > 0$ is the coefficient of viscosity. $f_1(x_1, x_2, t)$ and $\xi_0(x_1, x_2)$ are given. Let

$$I = \{x_1 | 0 < x_1 < 1\}, \quad \tilde{I} = \{x_2 | 0 < x_2 < 2\pi\}, \quad \Omega = I \times \tilde{I}.$$

Manuscript received May 11, 1992. Revised June 2, 1993.
*Department of Mathematics, Shanghai University of Science and Technology, Shanghai 201800, China

We consider the two-dimensional vorticity equation

$$\begin{cases} \frac{\partial\xi}{\partial t}+\frac{\partial\psi}{\partial x_2}\frac{\partial\xi}{\partial x_1}-\frac{\partial\psi}{\partial x_1}\frac{\partial\xi}{\partial x_2}-\nu(\frac{\partial^2\xi}{\partial x_1^2}+\frac{\partial^2\xi}{\partial x_2^2})=f_1, & \text{in} \quad \Omega\times(0,T], \\ -\frac{\partial^2\psi}{\partial x_1^2}-\frac{\partial^2\psi}{\partial x_2^2}=\xi, & \text{in} \quad \Omega\times[0,T], \\ \xi(x_1,x_2,0)=\xi_0(x_1,x_2), & \text{in} \quad \overline{\Omega}. \end{cases} \tag{1.1}$$

Suppose that all funcitons have the period 2π for the variable x_2, but they are not periodical for the variable x_1. We propose a Fourier pseudospectral-finite difference scheme with the filtering technique in Section 2. The key point is the use of skew symmetric decomposition of nonlinear convection terms. In this case, the numerical solution keeps the semi-discrete energy unchanged. This is a reasonable analogy of the conservation in continuous model. Moreover such decomposition assures that the main nonlinear error terms vanish. In Section 3, we present the numerical results. We list some lemmas in Section 4 and then prove the error estimations in Sections 5-6. In particular, all estimations include the errors on the boundary, which effect the accuracy seriously in practical problems. But most of papers in this field neglected this factor. In the final section, we consider the steady problem.

§2. The Scheme and the Conservations

Hereafter all functions have the period 2π for the variable x_2. Let $h=\frac{1}{M}$ be the mesh spacing, M being a positive integer and

$$I_h=\{x_1=jh|1\le j\le M-1\},\quad \Omega_h=I_h\times\tilde{I}.$$

Let τ be the mesh spacing of t, and

$$S_\tau=\{t=k\tau|k=0,1,2,\cdots.\}.$$

Define

$$\begin{aligned} u_{x_1}(x_1,x_2,t)&=\frac{1}{h}(u(x_1+h,x_2,t)-u(x_1,x_2,t)),\\ u_{\overline{x}_1}(x_1,x_2,t)&=u_{x_1}(x_1-h,x_2,t),\\ u_{\hat{x}_1}(x_1,x_2,t)&=\frac{1}{2}(u_{x_1}(x_1,x_2,t)+u_{\overline{x}_1}(x_1,x_2,t)),\\ \Delta u(x_1,x_2,t)&=u_{x_1\overline{x}_1}(x_1,x_2,t)+\frac{\partial^2u}{\partial x_2^2}(x_1,x_2,t),\\ u_t(x_1,x_2,t)&=\frac{1}{\tau}(u(x_1,x_2,t+\tau)-u(x_1,x_2,t)). \end{aligned}$$

Let

$$V_N=\text{span}\{e^{inx_2}\big||n|\le N\}$$

and P_N be the orthogonal projection from $L^2(\tilde{I})$ onto V_N, i.e., for any $u\in H^1(\tilde{I})$,

$$\int_{\tilde{I}}(P_Nu-u)\overline{v}dx_2=0,\quad \forall v\in V_N.$$

For the pseudospectral approximation, we consider the nodes

$$x_2^{(j)}=\frac{2\pi j}{2N+1},\quad 0\le j\le 2N$$

and let $\tilde{P}_c$ be an interpolation operator such that for $u \in C(\widetilde{I})$,

$$\widetilde{P}_c u(x_2^{(j)}) = u(x_2^{(j)}), \quad 0 \le j \le 2N.$$

Define $P_c = P_N \widetilde{P}_c$. We also use the filtering operator R_r with $r \ge 1$ (see [5]). If $u(x_2) \in V_N$ has the Fourier coefficients $\{u_n\}$, then

$$R_r u(x_2) = \sum_{|n| \le N} \Big(1 - \big|\frac{n}{N}\big|^r\Big) u_n e^{inx_2}. \tag{2.1}$$

The key point for constructing the scheme is to simulate the conservations. Indeed the solution of (1.1) possesses several conservations (see [17]). To simulate them, let $\alpha = (\alpha_1, \alpha_2, \alpha_3), \alpha_l \ge 0, \alpha_1 + \alpha_2 + \alpha_3 = 1$ and define

$$\begin{aligned} J_{1,c}(u,w) &= P_c\Big(\frac{\partial w}{\partial x_2} u_{\hat{x}_1} - w_{\hat{x}_1} \frac{\partial u}{\partial x_2}\Big), \\ J_{2,c}(u,w) &= \Big[P_c\Big(\frac{\partial w}{\partial x_2} u\Big)\Big]_{\hat{x}_1} - \frac{\partial}{\partial x_2}\Big[P_c\Big(w_{\hat{x}_1} u\Big)\Big], \\ J_{3,c}(u,w) &= \frac{\partial}{\partial x_2}[P_c(w u_{\hat{x}_1})] - \Big[P_c\Big(w \frac{\partial u}{\partial x_2}\Big)\Big]_{\hat{x}_1}, \end{aligned}$$

and

$$J_c^{(\alpha)}(u,w) = \sum_{l=1}^{3} \alpha_l J_{l,c}(u,w). \tag{2.2}$$

Now, let $\eta^{(N)}$ and $\varphi^{(N)}$ be the approximations to ξ and ψ respectively,

$$\eta^{(N)}(x_1,x_2,t) = \sum_{|n| \le N} \eta_n^{(N)}(x_1,t) e^{inx_2}, \qquad \varphi^{(N)}(x_1,x_2,t) = \sum_{|n| \le N} \varphi_n^{(N)}(x_1,t) e^{inx_2}.$$

The pseudospectral-finite difference scheme for solving (1.1) is the following

$$\begin{cases} \eta_t^{(N)} + R_r J_c^{(\alpha)}(R_r(\eta^{(N)} + \delta\tau\eta_t^{(N)}), R_r\varphi^{(N)}) \\ -\nu\Delta(\eta^{(N)} + \sigma\tau\eta_t^{(N)}) = P_c f_1, \\ -\Delta\varphi^{(N)} = \eta^{(N)}, \\ \eta^{(N)}(0) = P_c \xi_0, \end{cases} \tag{2.3}$$

where δ and σ are parameters and $0 \le \delta, \sigma \le 1$. When $\delta = \sigma = 0$, this is an explicit scheme. But even if $\delta = 0$ and $\sigma \ne 0$, the coefficients of $\eta^{(N)}(x,t)$ and $\varphi^{(N)}(x,t)$ are still separated from the others. Thus we only have to solve linear algebraic equations with three diagonal matrix.

We next check the conservations. We introduce the following scalar products and norms

$$\begin{aligned}
(u(x_1,t),v(x_1,t)) &= \frac{1}{2\pi}\int_{\widetilde{I}} u(x_1,x_2,t)\overline{v}(x_1,x_2,t)dx_2,\\
\|u(x_1,t)\|^2_{\widetilde{I}} &= (u(x_1,t),u(x_1,t))_{\widetilde{I}},\\
(u(x_2,t),v(x_2,t))_{I_h} &= h\sum_{x_1\in I_h} u(x_1,x_2,t)\overline{v}(x_1,x_2,t),\\
\|u(x_2,t)\|^2_{I_h} &= (u(x_2,t),u(x_2,t))_{I_h},\\
(u(t),v(t)) &= h\sum_{x_1\in I_h}(u(x_1,t),v(x_1,t))_{\widetilde{I}},\quad \|u(t)\|^2=(u(t),u(t)),\\
|u(t)|_1^2 &= \frac{1}{2}\|u_{x_1}(t)\|^2+\frac{1}{2}\|u_{\overline{x}_1}(t)\|^2+\|\frac{\partial u}{\partial x_2}(t)\|^2,\\
|u(t)|_2^2 &= \frac{1}{2}\|u_{x_1\overline{x}_1}(t)\|^2+\|\frac{\partial}{\partial x_2}u_{x_1}(t)\|^2+\|\frac{\partial}{\partial x_2}u_{\overline{x}_1}(t)\|^2+\|\frac{\partial^2 u}{\partial x_2^2(t)}\|^2\\
&\quad+\frac{h}{4}\sum_{h\le x_1\le 1-2h}\|u_{x_1x_1}(x_1,t)\|^2_{\widetilde{I}}+\frac{h}{4}\sum_{2h\le x_1\le 1-h}\|u_{\overline{x}_1\overline{x}_1}(x_1,t)\|^2_{\widetilde{I}}.
\end{aligned}$$

We have

$$(u_{\hat{x}_1},v)+(v_{\hat{x}_1},u)=\widetilde{A}_1(u,v) \tag{2.4}$$

where

$$\widetilde{A}_1(u,v)=\frac{1}{2}[(u(1),v(1-h))_{\widetilde{I}}+(u(1-h),v(1))_{\widetilde{I}}-(u(h),v(0))_{\widetilde{I}}-(u(0),v(h))_{\widetilde{I}}].$$

Assume that $u,v,w\in V_N$. Then (see [7])

$$(P_c(uv)(t),w(t))=(u(t),P_c(vw)(t)). \tag{2.5}$$

By (2.4) and (2.5), we obtain

$$(J_{1,c}(u,w),1)=\Big(\frac{\partial w}{\partial x_2},u_{\hat{x}_1}\Big)+\Big(\Big(\frac{\partial w}{\partial x_2}\Big)_{\hat{x}_1},u\Big)=A_1(u,w), \tag{2.6}$$

where $A_1(u,w)=\widetilde{A}_1(u,\frac{\partial w}{\partial x_2})$. Similarly, we have

$$(J_{2,c}(u,w),1)=\Big(P_c\Big(\frac{\partial w}{\partial x_2}u\Big)_{\hat{x}_1},1\Big)=A_2(u,w) \tag{2.7}$$

with

$$\begin{aligned}
&A_2(u,w)\\
&=\frac{1}{2}\Big[\Big(u(1),\frac{\partial w}{\partial x_2}(1)\Big)_{\widetilde{I}}+\Big(u(1-h),\frac{\partial w}{\partial x_2}(1-h)\Big)_{\widetilde{I}}-\Big(u(h),\frac{\partial w}{\partial x_2}(h)\Big)_{\widetilde{I}}-\Big(u(0),\frac{\partial w}{\partial x_2}(0)\Big)_{\widetilde{I}}\Big].
\end{aligned}$$

Also

$$(J_{3,c}(u,w),1)=-\widetilde{A}_1\Big(w\frac{\partial u}{\partial x_2},1\Big)=-A_2(w,u)=A_2(u,w). \tag{2.8}$$

Therefore

$$(J^{(\alpha)}(u,w),1)=\alpha_1A_1(u,w)+(\alpha_2+\alpha_3)A_2(u,w). \tag{2.9}$$

On the other hand, (2.4) and (2.5) lead to

$$\Big(P_c\Big(\frac{\partial w}{\partial x_2}u_{\hat{x}_1}\Big),v\Big)+\Big(\Big(P_c\Big(\frac{\partial w}{\partial x_2}v\Big)\Big)_{\hat{x}_1},u\Big)=A_3(u,v,w),$$

where $A_3(u,v,w)=\widetilde{A}_1(u,P_c(\frac{\partial w}{\partial x_2}v))$. Thus

$$(J_{1,c}(u,w),v)+(J_{2,c}(v,w),u)=A_3(u,v,w). \tag{2.10}$$

Similarly, from (2.4) and (2.5) we have

$$-\Big(P_c\Big(w\frac{\partial u}{\partial x_2}\Big)_{\hat{x}_1},v\Big)-\Big(\frac{\partial u}{\partial x_2},P_c(wu_{\hat{x}_1})\Big)=A_4(u,v,w),$$

where

$$A_4(u,v,w)=-\widetilde{A}_1\Big(P_c\Big(\frac{\partial u}{\partial x_2}w\Big),v\Big). \tag{2.11}$$

Hence

$$(J_{3,c}(u,w),v)+\Big(\frac{\partial v}{\partial x_2},P_c(wu_{\hat{x}_1})\Big)-\Big(\frac{\partial u}{\partial x_2},P_c(wu_{\hat{x}_1})\Big)=A_4(u,v,w). \tag{2.12}$$

The combination of (2.10) with (2.12) tells us that for $\alpha_1=\alpha_2$,

$$\begin{aligned}&(J_c^{(\alpha)}(u,w),v)+(J_c^{(\alpha)}(v,w),u)\\&=\alpha_1A_3(u,v,w)+\alpha_1A_3(v,u,w)+\alpha_3A_4(u,v,w)+\alpha_3A_4(v,u,w).\end{aligned} \tag{2.13}$$

In particular, for $\alpha_1=\alpha_2$,

$$(J_c^{(\alpha)}(u,w),u)=\alpha_1A_3(u,u,w)+\alpha_3A_4(u,u,w). \tag{2.14}$$

It is easy to show that

$$(u,\Delta v)+\frac{1}{2}(u_{x_1},v_{x_1})+\frac{1}{2}(u_{\bar{x}_1},v_{\bar{x}_1})+\Big(\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_2}\Big)=B(u,v) \tag{2.15}$$

where

$$B(u,v)=\frac{1}{2}(u(1)+u(1-h),v_{\bar{x}_1}(1))_{\widetilde{I}}-\frac{1}{2}(u(h)+u(0),v_{x_1}(0))_{\widetilde{I}}.$$

In particular,

$$(u,\Delta u)+|u|_1^2=B(u,u). \tag{2.16}$$

We now check the conservations. Firstly, from (2.9) and (2.15) we have

$$\begin{aligned}&(\eta^{(N)}(t),1)+\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}\Big[\alpha_1A_1(R_r(\eta^{(N)}(y)+\delta\tau\eta_t^{(N)}(y)),R_r\varphi^{(N)}(y))\\&+(\alpha_2+\alpha_3)A_2(R_r(\eta^{(N)}(y)+\delta\tau\eta_t^{(N)}(y),R_r\varphi^{(N)}(y))-\nu B(1,\eta^{(N)}(y)+\sigma\tau\eta_t^{(N)}(y))\Big]\\&=(\eta^{(N)}(0),1)+\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(P_cf_1(t),1).\end{aligned}$$

Secondly by putting $\delta=\sigma=\frac{1}{2},\alpha_1=\alpha_2$ and $\hat{\eta}^{(N)}(t)=\frac{1}{2}(\eta^{(N)}(t)+\eta^{(N)}(t+\tau))$, we get from (2.14) and (2.15)

$$\begin{aligned}&\|\eta^{(N)}(t)\|^2+2\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}\Big[\nu|\hat{\eta}^{(N)}(y)|_1^2+\alpha_1A_3(R_r\hat{\eta}^{(N)}(y),R_r\hat{\eta}^{(N)}(y),R_r\varphi^{(N)}(y))\\&+\alpha_3A_4(R_r\hat{\eta}^{(N)}(y),R_r\hat{\eta}^{(N)}(y),R_r\varphi^{(N)}(y))-\nu B(\hat{\eta}^{(N)}(y),\hat{\eta}^{(N)}(y))\Big]\\&=\|\eta^{(N)}(0)\|^2+2\tau\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(P_cf_1(y),\hat{\eta}^{(N)}(y)).\end{aligned}$$

The above two equalities are reasonable analogies of the corresponding conservations in continuous model.

§3. Numerical Results

In this section, we present numerical results. For describing the errors, we introduce

$$\widetilde{I}_h = \{x_2 | x_2 = \frac{2\pi j}{2N+1}, 0 \le j \le 2N\},$$
$$E(t) = \Big(\frac{h}{2N+1} \sum_{(x_1,x_2)\in I_h\times \widetilde{I}_h} |\xi(x_1,x_2,t) - \eta(x_1,x_2,t)|^2\Big)^{\frac{1}{2}}$$

where η is the PS-D (pseudospectral-finite difference) or difference approximation to ξ. All computations are carried out by explicit scheme, i.e., $\delta = \sigma = 0$. In practical problems, we only know the values of ψ and $\frac{\partial \psi}{\partial n}$ for $x_1 = 0, 1$. Thus we have to use some methods to calculate the approximate values of ξ for $x_1 = 0, 1$. They induce the errors. But for simplicity of mathematical analysis, some authors considered the model with the given values of ξ on the boundary (see problem 6.3 of [23]). Here we follow this approximate model.

Example 3.1. Let

$$\xi(x_1,x_2,t) = A\exp\{B\sin(Cx_1 + x_2) + \omega t\},$$
$$\psi(x_1,x_2,t) = A\exp\{\omega t\}\sin Cx_1 \sin x_2.$$

Table 3.1 shows the results of scheme (2.3) with $A = B = \omega = 0.1, C = 3.0, \nu = 10^{-3}, \tau = 0.05$, $h = 0.1, N = 4$ and $r = 1$. Obviously, the choice $\alpha_1 = \alpha_2$ gives better numerical results.

Table 3.1. Errors of Scheme (2.3)

$(\alpha_1,\alpha_2,\alpha_3)$	$(\frac{1}{2},\frac{1}{2},0)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	(1,0,0)
E(1)	0.9098E-02	0.9084E-02	0.9184E-02
E(3)	0.2990E-01	0.2979E-01	0.3041E-01
E(5)	0.4891E-01	0.4845E-01	0.5133E-01

Table 3.2 is for the numerical results of scheme (2.3) with $A = B = \omega = 0.1, C = 0.5$, $\nu = 10^{-6}, \tau = 10^{-3}, h = 0.1, N = 4$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$. Clearly the filtering technique provides better numerical results.

Table 3.2. Errors for PS-D Scheme (2.3)

	$r=1$	$r=5$	$r=\infty$
E(0.2)	0.1935E-03	0.3008E-03	0.3199E-03
E(0.6)	0.6046E-03	0.9372E-03	0.9953E-03
E(1.0)	0.1050E-02	0.1617E-02	0.1715E-02

Example 3.2. Let

$$\xi(x_1,x_2,t) = A\exp\{B\sin(Cx_1+x_2)+\omega t\},$$
$$\psi(x_1,x_2,t) = A\exp\{\omega t\}(Cx_1+\sin x_2).$$

For comparison of pseudospectral-finite difference scheme with full finite difference scheme, let $\overline{h}=\frac{2\pi}{2N+1}$ and Ω'_h be the set of mesh points in Ω. We define

$$u_{x_2}(x_1,x_2,t) = \frac{1}{h}(u(x_1,x_2+\overline{h},t)-u(x_1,x_2,t)),$$
$$u_{\overline{x}_2}(x_1,x_2,t) = u_{x_2}(x_1,x_2-\overline{h},t),$$
$$u_{\hat{x}_2}(x_1,x_2,t) = \frac{1}{2}(u_{x_2}(x_1,x_2,t)+u_{\overline{x}_2}(x_1,x_2,t)),$$
$$\Delta_{\overline{h}}u(x_1,x_2,t) = u_{x_1\overline{x}_1}(x_1,x_2,t)+u_{x_2\overline{x}_2}(x_1,x_2,t)$$

and

$$J_{1,\overline{h}}(u,w) = w_{\hat{x}_2}u_{\hat{x}_1}-w_{\hat{x}_1}u_{\hat{x}_2},$$
$$J_{2,\overline{h}}(u,w) = (w_{\hat{x}_2}u)_{\hat{x}_1}-(w_{\hat{x}_1}u)_{\hat{x}_2},$$
$$J_{3,\overline{h}}(u,w) = (wu_{\hat{x}_1})_{\hat{x}_2}-(wu_{\hat{x}_2})_{\hat{x}_1},$$
$$J^{(\alpha)}_{\overline{h}}(u,w) = \sum_{l=1}^{3}\alpha_l J_{l,\overline{h}}(u,w).$$

Let $\eta^{\overline{h}}$ and $\varphi^{\overline{h}}$ be the finite difference approximations to ξ and ψ respectively. The difference scheme is (see [24])

$$\begin{cases} \eta^{\overline{h}}_t(t)+J^{(\alpha)}_{\overline{h}}(\eta^{\overline{h}}+\delta\tau\eta^{\overline{h}}_t,\varphi^{\overline{h}})-\nu\Delta_{\overline{h}}(\eta^{\overline{h}}+\sigma\tau\eta^{\overline{h}}_t)=f^{\overline{h}}_1, & \text{in } \Omega'_h\times S_\tau, \\ -\Delta_{\overline{h}}\varphi^{\overline{h}}=\eta^{\overline{h}}+f^{\overline{h}}_2, & \text{in } \Omega'_h\times S_\tau. \end{cases} \tag{3.1}$$

The numerical experiments are for $A=B=0.1, C=0.2, \omega=0.3, \nu=10^{-6}, \tau=0.005$, $h=0.1, \overline{h}=0.25, N=4, \alpha_1=\alpha_2=\frac{1}{2}$ and $r=1$. Table 3.3 shows that scheme (2.3) gives better numerical results than scheme (3.1).

Table 3.3. Errors of Scheme (2.3) and (3.1)

	scheme (2.3)	scheme (3.1)
E(1)	0.1755E-02	0.1897E-02
E(3)	0.9015E-02	0.9338E-02
E(5)	0.2713E-01	0.3457E-01

§4. Some Lemmas

In order to estimate the errors, we need some lemmas.

Lemma 4.1. *For all $u(x_1,x_2,t)$, we have*

$$2(u(x_1,t),u_t(x_1,t))_{\widetilde{I}} = (\|u(x_1,t)\|^2_{\widetilde{I}})_t-\tau\|u_t(x_1,t)\|^2_{\widetilde{I}},$$
$$2(u(t),u_t(t)) = \|u(t)\|^2_t-\tau\|u_t(t)\|^2.$$

Lemma 4.2. *If* $u(x_1, x_2 + 2\pi, t) = u(x_1, x_2, t)$, *then*

$$2(u_t(t), \Delta u(t)) + (|u(t)|_1^2)_t - \tau |u_t(t)|_1^2 = 2B(u_t(t), u(t)),$$
$$2(u(t), \Delta u_t(t)) + (|u(t)|_1^2)_t - \tau |u_t(t)|_1^2 = 2B(u(t), u_t(t)).$$

Lemma 4.3. *If* $u(x_1, x_2, t) \in V_N$ *for all* $x_1 \in I_h$ *and* $t \in S_\tau$, *then*

$$\|\frac{\partial u}{\partial x_2}(t)\|^2 \le N^2 \|u(t)\|^2.$$

Lemma 4.4. *For all* $u(x_1, x_2)$, *we have*

$$\|u_{\bar{x}_1}\|^2 \le \frac{4}{h^2}\|u\|^2 + h\|u_{x_1}(0)\|_{\widetilde{I}}^2, \qquad \|u_{x_1}\|^2 \le \frac{4}{h^2}\|u\|^2 + h\|u_{\bar{x}_1}(1)\|_{\widetilde{I}}^2,$$

and

$$\|u_{\bar{x}_1}\|^2 \le \frac{4}{h^2}\|u\|^2 + \frac{2}{h}\|u(0)\|_{\widetilde{I}}^2, \qquad \|u_{x_1}\|^2 \le \frac{4}{h^2}\|u\|^2 + \frac{2}{h}\|u(1)\|_{\widetilde{I}}^2.$$

Lemma 4.5. *If* $u(x_1, x_2) \in V_N$ *for all* $x_1 \in I_h$ *and* $u(0, x_2) = u(1, x_2) = 0$, *then*

$$\|u\|^2 \le C_1[|u|_1^2 + S(u)],$$

where C_1 *is a positive constant depending only on the domain* Ω_h, *and*

$$S(u) = S(u, u) = \frac{1}{2h}(u(h), u(h))_{\widetilde{I}} + \frac{1}{2h}(u(1-h), u(1-h))_{\widetilde{I}}.$$

Lemma 4.6. *If* $u(x_1, x_2) \in V_N$ *for all* $x_1 \in \overline{I}_h$ *and* $u(0, x_2) = u(1, x_2) = 0$, *then*

$$\|\Delta u\|^2 = |u|_2^2 + \frac{h}{4}(\|u_{x_1x_1}(0)\|_{\widetilde{I}}^2 + \|u_{\bar{x}_1\bar{x}_1}(1)\|_{\widetilde{I}}^2) + \frac{1}{h}\Big(\|\frac{\partial u}{\partial x_2}(h)\|_{\widetilde{I}}^2 + \|\frac{\partial u}{\partial x_2}(1-h)\|_{\widetilde{I}}^2\Big).$$

Lemma 4.7. *If* $h < 2\varepsilon$, *then for all* $x_1^* \in \overline{I}_h$,

$$\|u(x_1^*)\|_{\widetilde{I}}^2 \le \varepsilon(\|u_{x_1}\|^2 + \|u_{\bar{x}_1}\|^2) + C_0(\varepsilon)\|u\|^2$$

where $C_0(\varepsilon)$ *is a positive constant depending only on* ε *and the domain* Ω_h.

Lemma 4.8. *If* $u(x_1, x_2), v(x_1, x_2) \in V_N$ *for all* $x_1 \in \overline{I}_h$, *then*

$$\|u(x_1)v(x_1)\|_{\widetilde{I}}^2 \le (2N+1)\|u(x_1)\|_{\widetilde{I}}^2 \|v(x_1)\|_{\widetilde{I}}^2,$$
$$\|u(x_2)v(x_2)\|_{I_h}^2 \le \frac{1}{h}\|u(x_2)\|_{I_h}^2 \|v(x_2)\|_{I_h}^2,$$
$$\|uv\|^2 \le \frac{2N+1}{h}\|u\|^2\|v\|^2.$$

Lemma 4.9. *If* $u(x_1, x_2) \in H^\beta(\widetilde{I})$ *and* $v(x_1, x_2) \in V_N$ *for all* $x_1 \in \overline{I}_h$, *then*

$$\|P_N u(x_1) - u(x_1)\|_{H^s(\widetilde{I})} \le C_2 N^{s-\beta}\|u(x_1)\|_{H^\beta(\widetilde{I})}, \quad 0 \le s \le \beta,$$
$$\|P_c u(x_1) - u(x_1)\|_{H^s(\widetilde{I})} \le C_3 N^{s-\beta}\|u(x_1)\|_{H^\beta(\widetilde{I})}, \quad 0 \le s \le \beta, \beta > \tfrac{1}{2},$$
$$\|R_r v(x_1) - v(x_1)\|_{H^s(\widetilde{I})} \le C_4 N^{s-\beta}\|u(x_1)\|_{H^\beta(\widetilde{I})}, \quad 0 \le s \le \beta, \quad r \ge \beta - s,$$

where $C_2 \sim C_4$ *are positive constants.*

The proof of Lemma 4.9 is given in [25,26]. The others could be found in [27].

§5. Error Estimation for Dirichlet Boundary Condition

In this section, we suppose that $\alpha_1 = \alpha_2, h = O(\frac{1}{N})$ and $\tau = O(\frac{1}{N^2})$. Let $\tilde{f}_1$ and $\tilde{\xi}_0$ be the errors of f_1 and ξ_0. The right side of the first formula of (2.3) has the error $P_c\tilde{f}_2$. Also assume that $\eta^{(N)}(0,x_2,t)=P_c\xi(0,x_2,t)=P_cg_0$, $\eta^{(N)}(1,x_2,t)=P_c\xi(1,x_2,t)=P_cg_1$ and that g_0 and g_1 have the errors $\tilde{g}_0$ and $\tilde{g}_1$. They induce the errors of $\eta^{(N)}$ and $\varphi^{(N)}$, denoted by $\tilde{\eta}^{(N)}$ and $\tilde{\varphi}^{(N)}$. For simplicity, we assume that $\tilde{\varphi}^{(N)}(0,x_2,t)=\tilde{\varphi}^{(N)}(1,x_2,t)=0$. Then

$$\begin{cases} \tilde{\eta}_t^{(N)}+R_rJ_c^{(\alpha)}(R_r(\tilde{\eta}^{(N)}+\delta\tau\tilde{\eta}_t^{(N)}),R_r\varphi^{(N)}+R_r\tilde{\varphi}^{(N)}) \\ \quad +R_rJ_c^{(\alpha)}(R_r(\eta^{(N)}+\delta\tau\eta_t^{(N)}),R_r\tilde{\varphi}^{(N)})-\nu\Delta(\tilde{\eta}^{(N)}+\sigma\tau\tilde{\eta}_t^{(N)})=P_c\tilde{f}_1, \\ -\Delta\tilde{\varphi}^{(N)}=\tilde{\eta}^{(N)}+P_c\tilde{f}_2, \\ \tilde{\eta}^{(N)}(0)=P_c\tilde{\xi}_0. \end{cases} \tag{5.1}$$

Let $\varepsilon>0$ and C denote a positive constant which may be different in different cases. Let m be an undetermined positive constant. By taking the scalar product of the first formula of (5.1) with $2\tilde{\eta}^{(N)}+m\tau\tilde{\eta}_t^{(N)}(t)$, we obtain from (2.13)-(2.15) and Lemmas 4.1 and 4.2 that

$$\begin{aligned} &\|\tilde{\eta}^{(N)}(t)\|_t^2+\tau(m-1-\varepsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2+2\nu|\tilde{\eta}^{(N)}(t)|_1^2+\nu\tau(\sigma+\frac{m}{2})(|\tilde{\eta}^{(N)}(t)|_1^2)_t \\ &+\nu\tau^2(m\sigma-\sigma-\frac{m}{2})|\tilde{\eta}_t^{(N)}(t)|_1^2+\sum_{l=1}^{3}G_l(t)+\sum_{l=1}^{5}D_l(t)+\sum_{l=1}^{4}B_l(t) \\ &\le\|\tilde{\eta}^{(N)}(t)\|^2+(1+\frac{Cm^2\tau}{4\varepsilon})\|\tilde{f}_1\|^2, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} G_1(t)&=(R_r(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t)),J_c^{(\alpha)}(R_r(\eta^{(N)}(t)+\delta\tau\eta_t^{(N)}(t)),R_r\tilde{\varphi}^{(N)}(t))), \\ G_2(t)&=(R_r(2\tilde{\eta}^{(N)}(t)+m\tau\tilde{\eta}_t^{(N)}(t)),J_c^{(\alpha)}(R_r(\tilde{\eta}^{(N)}(t)+\delta\tau\tilde{\eta}_t^{(N)}(t)),R_r\tilde{\varphi}^{(N)}(t))), \\ G_3(t)&=\tau(m-2\delta)(R_r\tilde{\eta}_t^{(N)}(t),J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)), \end{aligned}$$

$$\begin{aligned} D_1(t)&=2\alpha_1A_3(R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)), \\ D_2(t)&=2\alpha_3A_4(R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)), \\ D_3(t)&=2\alpha_1\delta\tau A_3(R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)) \\ &\quad+2\alpha_1\delta\tau A_3(R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)), \\ D_4(t)&=2\alpha_3\delta\tau A_4(R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)) \\ &\quad+2\alpha_3\delta\tau A_3(R_r\tilde{\eta}^{(N)}(t),R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)), \\ D_5(t)&=m\delta\alpha_1\tau^2A_3(R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)) \\ &\quad+m\delta\alpha_3\tau^2A_4(R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\eta}_t^{(N)}(t),R_r\tilde{\varphi}^{(N)}(t)), \end{aligned}$$

$$\begin{aligned} &B_1(t)=-2\nu B(\tilde{\eta}^{(N)}(t),\tilde{\eta}^{(N)}(t)), && B_2(t)=-2\nu\sigma\tau B(\tilde{\eta}^{(N)}(t),\tilde{\eta}_t^{(N)}(t)), \\ &B_3(t)=-m\nu B(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}^{(N)}(t)), && B_4(t)=-m\nu\sigma\tau^2B(\tilde{\eta}_t^{(N)}(t),\tilde{\eta}_t^{(N)}(t)). \end{aligned}$$

By taking the scalar product of the second formula of (5.1) with $R_r^2\tilde{\varphi}^{(N)}(t)$, we have from (2.16) and Lemma 4.5 that

$$|R_r\tilde{\varphi}^{(N)}(t)|_1^2+S(R_r\tilde{\varphi}^{(N)}(t))\le C(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2\|^2). \tag{5.3}$$

We now estimate $|G_l(t)|$. We use the following notations

$$\|\tilde{g}(t)\|^2_{\tilde{I}} = \|g_0(t)\|^2_{\tilde{I}} + \|\tilde{g}_1(t)\|^2_{\tilde{I}}, \qquad \|\tfrac{\partial \tilde{g}}{\partial x_2}(t)\|^2_{\tilde{I}} = \|\tfrac{\partial \tilde{g}_0}{\partial x_2}(t)\|^2_{\tilde{I}} + \|\tfrac{\partial \tilde{g}_1}{\partial x_2}(t)\|^2_{\tilde{I}},$$
$$\|\tilde{g}_t(t)\|^2_{\tilde{I}} = \|g_{0t}(t)\|^2_{\tilde{I}} + \|\tilde{g}_{1t}(t)\|^2_{\tilde{I}}, \qquad |||u|||_{p,q} = \max_{\substack{t\in S_\tau,\\ t\le T}} \|u(t)\|_{W^{p,q}(\Omega)}.$$

We have

$$|G_1(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + C\Big(1+\Big(1+\frac{\tau m^2}{\varepsilon}\Big)|||R_r\eta^{(N)}|||_{1,\infty}\Big)\|\tilde{\eta}^{(N)}(t)\|^2$$
$$+ C\Big(1+\frac{\tau m^2}{\varepsilon}\Big)|||R_r\eta^{(N)}|||^2_{1,\infty}\|\tilde{f}_2\|^2. \tag{5.4}$$

The estimation for $G_2(t)$ is a little difficult. We have

$$\begin{aligned} G_2(t) = {} & 2(R_r\tilde{\eta}^{(N)}(t), J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}(t), R_r\varphi^{(N)}(t))) \\ & + m\delta\tau^2(R_r\tilde{\eta}_t^{(N)}(t), J_c^{(\alpha)}(R_r\tilde{\eta}_t^{(N)}(t), R_r\varphi^{(N)}(t))) \\ & + 2\delta\tau(R_r\tilde{\eta}^{(N)}(t), J_c^{(\alpha)}(R_r\tilde{\eta}_t^{(N)}(t), R_r\varphi^{(N)}(t))) \\ & + m\tau(R_r\tilde{\eta}_t^{(N)}(t), J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}(t), R_r\varphi^{(N)}(t))). \end{aligned}$$

Furthermore

$$|(R_r\tilde{\eta}^{(N)}(t), J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}(t), R_r\varphi^{(N)}(t)))| \le \varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \frac{C}{\varepsilon}|||R_r\varphi^{(N)}|||^2_{2,\infty}\|\tilde{\eta}^{(N)}(t)\|^2.$$

Let $\lambda = N^2 + \frac{1}{h^2}$. By ε-inequality, Lemma 4.4 and Lemma 4.5,

$$\begin{aligned} & |2\delta\tau(R_r\tilde{\eta}^{(N)}(t), J_c^{(\alpha)}(R_r\tilde{\eta}_t^{(N)}(t), R_r\varphi^{(N)}(t)))| \\ & \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C}{\varepsilon}|||R_r\varphi^{(N)}|||^2_{2,\infty}\|\tilde{\eta}^{(N)}(t)\|^2 + C\tau h\|\tilde{g}_t(t)\|^2_{\tilde{I}}. \end{aligned}$$

We can estimate the other two terms similarly and so

$$\begin{aligned} & |G_2(t)| \\ & \le \Big(3\varepsilon\tau + \frac{c\tau^2}{\varepsilon}|||R_r\varphi^{(N)}|||^2_{2,\infty}\Big)\|\tilde{\eta}_t^{(N)}(t)\|^2 + \varepsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \frac{C}{\varepsilon}|||R_r\varphi^{(N)}|||^2_{2,\infty}\|\tilde{\eta}^{(N)}(t)\|^2 \\ & \quad + C\Big(1+\frac{1}{\varepsilon}|||R_r\varphi^{(N)}|||^2_{2,\infty}\Big)\Big(\frac{\tau}{h}\|\tilde{g}(t)\|^2_{\tilde{I}} + \tau h\|\tilde{g}_t(t)\|^2_{\tilde{I}}\Big). \end{aligned} \tag{5.5}$$

By Lemma 4.8,

$$|G_3(t)| \le \varepsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C\tau N(m-2\delta)^2}{\varepsilon h}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2)|\tilde{\eta}^{(N)}(t)|_1^2. \tag{5.6}$$

We now estimate $|D_l(t)|$. Firstly, we have

$$\begin{aligned} D_1(t) = \alpha_1\Big[& \Big(R_r\tilde{\eta}^{(N)}(1,t), P_c\Big(\frac{\partial R_r\tilde{\varphi}^{(N)}}{\partial x_2}(1-h,t)R_r\tilde{\eta}^{(N)}(1-h,t)\Big)\Big)_{\tilde{I}} \\ & + \Big(R_r\tilde{\eta}^{(N)}(1-h,t), P_c\Big(\frac{\partial R_r\tilde{\varphi}^{(N)}}{\partial x_2}(1,t)R_r\tilde{\eta}^{(N)}(1,t)\Big)\Big)_{\tilde{I}} \\ & - \Big(R_r\tilde{\eta}^{(N)}(h,t), P_c\Big(\frac{\partial R_r\tilde{\varphi}^{(N)}}{\partial x_2}(0,t)R_r\tilde{\eta}^{(N)}(0,t)\Big)\Big)_{\tilde{I}} \\ & - \Big(R_r\tilde{\eta}^{(N)}(0,t), P_c\Big(\frac{\partial R_r\tilde{\varphi}^{(N)}}{\partial x_2}(h,t)R_r\tilde{\eta}^{(N)}(h,t)\Big)\Big)_{\tilde{I}}\Big]. \end{aligned}$$

By Lemma 4.8,

$$\Big|\Big(R_r\tilde{\eta}^{(N)}(1,t), P_c\Big(\frac{\partial R_r\tilde{\varphi}^{(N)}}{\partial x_2}(1-h,t)R_r\tilde{\eta}^{(N)}(1-h,t)\Big)\Big)_{\tilde{I}}\Big|$$
$$\le \frac{\epsilon\nu}{2h}\|\tilde{\eta}^{(N)}(1-h,t)\|_{\tilde{I}}^2 + \frac{ChN}{\epsilon}\Big\|\frac{\partial R_r\tilde{\varphi}^{(N)}}{\partial x_2}(1-h,t)\Big\|_{\tilde{I}}^2\|\tilde{g}_1(t)\|_{\tilde{I}}^2.$$

We can deal with the other terms in $D_1(t)$ similarly and thus by Lemma 4.6 and Lemma 4.7,

$$|D_1(t)| \le \epsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{ChN}{\epsilon}\|\tilde{g}(t)\|_{\tilde{I}}^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2). \tag{5.7}$$

By an argument similar to the estimation for $|D_1(t)|$, we get

$$|D_2(t)| \le \epsilon\nu S(\tilde{\eta}^{(N)}(t)) + \frac{ChN}{\epsilon}\Big(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \|\frac{\partial\tilde{g}}{\partial x_2}(t)\|_{\tilde{I}}^2\Big)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2). \tag{5.8}$$

$$|D_3(t)| \le \epsilon\nu S(\tilde{\eta}^{(N)}(t)) + \epsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) + \frac{ChN}{\epsilon}(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \tau^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2), \tag{5.9}$$

$$|D_4(t)| \le \epsilon\nu S(\tilde{\eta}^{(N)}(t)) + \epsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) + \frac{ChN}{\varepsilon}\Big(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \|\frac{\partial\tilde{g}}{\partial x_2}(t)\|_{\tilde{I}}^2 + \tau^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2 + \tau^2\|\frac{\partial\tilde{g}_t}{\partial x_2}(t)\|^2\Big)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2), \tag{5.10}$$

$$|D_5(t)| \le \epsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) + \frac{ChN\tau^2}{\epsilon}\Big(\|\tilde{g}_t(t)\|_{\tilde{I}}^2 + \|\frac{\partial\tilde{g}_t}{\partial x_2}(t)\|_{\tilde{I}}^2\Big)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2). \tag{5.11}$$

We now estimate $B_l(t)$. It can be verified that

$$B_1(t) \ge 2\nu S(\tilde{\eta}^{(N)}(t)) - \frac{C}{\epsilon h}\|\tilde{g}(t)\|_{\tilde{I}}^2, \tag{5.12}$$

$$B_2(t) + B_3(t) \ge \nu\tau\Big(\sigma + \frac{m}{2}\Big)[S(\tilde{\eta}^{(N)}(t))]_t - \nu\tau^2\Big(\sigma + \frac{m}{2}\Big)S(\tilde{\eta}_t^{(N)}(t)) - \epsilon\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) - \epsilon\nu S(\tilde{\eta}^{(N)}(t)) - \frac{C}{\epsilon h}(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \tau h^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2), \tag{5.13}$$

$$B_4(t) \ge m\sigma\nu\tau^2 S(\tilde{\eta}_t^{(N)}(t)) - \frac{C\tau h}{\epsilon}\|\tilde{g}_t(t)\|_{\tilde{I}}^2. \tag{5.14}$$

By substituting (5.4)-(5.14) into (5.2), we obtain

$$\begin{aligned}&\|\tilde{\eta}^{(N)}(t)\|^2 + \tau\Big(m - 1 - 6\epsilon - \frac{C\tau}{\epsilon}\||R_r\varphi^{(N)}\||_{2,\infty}^2\Big)\|\tilde{\eta}_t^{(N)}(t)\|^2 + \nu|\tilde{\eta}^{(N)}(t)|_1^2\\ &+ \nu\tau\Big(\sigma + \frac{m}{2}\Big)(|\tilde{\eta}^{(N)}(t)|_1^2)_t + \nu\tau^2\Big(m\sigma - \sigma - \frac{m}{2}\Big)|\tilde{\eta}_t^{(N)}(t)|_1^2 + \nu(2 - 5\epsilon)S(\tilde{\eta}^{(N)}(t))\\ &+ \nu\tau\Big(\sigma + \frac{m}{2}\Big)[S(\tilde{\eta}^{(N)}(t))]_t + \nu\tau^2\Big(m\sigma - \sigma - \frac{m}{2} - 4\epsilon\Big)S(\tilde{\eta}_t^{(N)}(t))\\ &\le H_0(t)\|\tilde{\eta}^{(N)}(t)\|^2 + H_1(t)|\tilde{\eta}^{(N)}(t)|_1^2 + R(t),\end{aligned} \tag{5.15}$$

where

$$H_0(t) = C\Big(\Big(1+\frac{\tau m^2}{\epsilon}\Big)|||R_r\eta^{(N)}|||_{1,\infty}^2 + \frac{|||R_r\varphi^{(N)}|||_{2,\infty}^2}{\epsilon} + \frac{hN}{\epsilon}\Big(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \|\frac{\partial \tilde{g}}{\partial x_2}(t)\|_{\tilde{I}}^2 + \tau^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2 + \tau^2\|\frac{\partial \tilde{g}_t}{\partial x_2}(t)\|_{\tilde{I}}^2\Big)\Big),$$

$$H_1(t) = -\nu + \epsilon\nu + \frac{C\tau N(m-2\delta)^2}{\epsilon h}(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2(t)\|^2),$$

$$R(t) = C\Big[\Big(1+\frac{\tau m^2}{4\epsilon}\Big)\|\tilde{f}_1(t)\|^2 + |||R_r\eta^{(N)}|||_{1,\infty}^2\|\tilde{f}_2(t)\|^2 + \frac{1}{\epsilon h}(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \tau h^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2) + \Big(1+\frac{|||R_r\varphi^{(N)}|||_{2,\infty}^2}{\epsilon h}\Big)(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \tau h^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2) + \frac{hN}{\epsilon}\Big(\|\tilde{g}(t)\|_{\tilde{I}}^2 + \|\frac{\partial \tilde{g}}{\partial x_2}(t)\|_{\tilde{I}}^2 + \tau^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2 + \tau^2\|\frac{\partial \tilde{g}_t}{\partial x_2}(t)\|_{\tilde{I}}^2\Big)\|\tilde{f}_2(t)\|^2\Big].$$

Let τ and ϵ be suitably small, and choose the value of m as follows. If $\sigma > \frac{1}{2}$, then we take

$$m > m_1 = \max\Big(\frac{2\sigma + 8\epsilon}{2\sigma - 1}, 1 + p_0 + 6\epsilon\Big), \quad p_0 \geq 0.$$

Then (5.15) leads to

$$\|\tilde{\eta}^{(N)}(t)\|_t^2 + p_0\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \nu(|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))) + \nu\tau\Big(\sigma + \frac{m}{2}\Big)[|\tilde{\eta}^{(N)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))]_t \leq H_0(t)\|\tilde{\eta}^{(N)}(t)\|^2 + H_1(t)|\tilde{\eta}^{(N)}(t)|_1^2 + R(t). \tag{5.16}$$

If $\sigma = \frac{1}{2}$, then we take

$$m > m_2 = 1 + p_0 + \frac{1}{2}\nu\tau N^2 + \frac{9\nu\tau}{4h^2} + \frac{2\epsilon\nu\tau}{h^2} + 6\epsilon.$$

By Lemma 4.3 and Lemma 4.4,

$$|\tilde{\eta}_t^{(N)}(t)|_1^2 \leq \Big(N^2 + \frac{4}{h^2}\Big)\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{2}{h}\|\tilde{g}_t(t)\|_{\tilde{I}}^2, \tag{5.17}$$

$$S(\tilde{\eta}_t^{(N)}(t)) \leq \frac{1}{2h^2}\|\tilde{\eta}_t^{(N)}(t)\|^2. \tag{5.18}$$

Thus

$$\tau(m-1-6\epsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2 + \nu\tau^2(m\sigma - \sigma - \frac{m}{2})|\tilde{\eta}_t^{(N)}(t)|_1^2 + \nu\tau^2\Big(m\sigma - \sigma - \frac{m}{2} - 4\epsilon\Big)S(\tilde{\eta}_t^{(N)}(t)) \geq p_0\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 - \frac{C\tau}{h}\|\tilde{g}_t(t)\|_{\tilde{I}}^2. \tag{5.19}$$

Hence (5.16) holds still. If $\sigma < \frac{1}{2}$ and $\tau < \frac{4h^2}{\nu(1-2\sigma)(9+2N^2h^2)}$, then we take

$$m > m_3 = \Big(1 + p_0 + \nu\sigma\tau N^2 + \frac{9\nu\sigma\tau}{2h^2} + \frac{2\epsilon\nu\tau}{h^2} + 6\epsilon\Big)\Big(1 + \nu\tau N^2\Big(\sigma - \frac{1}{2}\Big) + \frac{9\nu\tau}{2h^2}\Big(\sigma - \frac{1}{2}\Big)\Big)^{-1}.$$

By (5.17)-(5.18), the estimation (5.19) still holds and so (5.16) follows also.

Now put

$$\begin{aligned}E_1^{(N)}(t) &= \|\tilde{\eta}^{(N)}(t)\|^2 + \nu\tau(|\tilde{\eta}^{(t)}(t)|_1^2 + S(\tilde{\eta}^{(N)}(t))) \\ &\quad + \tau \sum_{\substack{y\in S_\tau \\ y\le t-\tau}} (p_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2 + \nu|\tilde{\eta}^{(N)}(y)|_1^2 + \nu S(\tilde{\eta}^{(N)}(y))), \\ \rho_1^{(N)}(t) &= \|\tilde{\eta}^{(N)}(0)\|^2 + \tau \sum_{\substack{y\in S_\tau \\ y\le t-\tau}} R(y).\end{aligned}$$

By summing (5.16) up for $t\in S_\tau$, we get

$$E_1^{(N)}(t) \le \rho_1^{(N)}(t) + \tau \sum_{\substack{y\in S_\tau \\ y\le t-\tau}} (H_0(y)E^{(N)}(y) + H_1(y)|\tilde{\eta}^{(N)}(y)|_1^2). \tag{5.20}$$

In particular, if

$$2\delta > \begin{cases} m_1, & \text{for } \sigma > \frac{1}{2}, \\ m_2, & \text{for } \sigma = \frac{1}{2}, \\ m_3, & \text{for } \sigma < \frac{1}{2}, \end{cases} \tag{5.21}$$

then we take $m = 2\delta$ and so $H_1(t) = -\nu + \epsilon\nu < 0$. Finally we apply Lemma 4.16 of [28] to (5.20) to get the result.

Theorem 5.1. *Assume that the following conditions are satisfied:*

(i) $h = O\big(\frac{1}{N}\big)$, $\tau = O\big(\frac{1}{N^2}\big)$ *and* $\alpha_1 = \alpha_2$,

(ii) $\sigma \ge \frac{1}{2}$ *or* $\tau < \frac{4h^2}{\nu(1-2\sigma)(9+2N^2h^2)}$,

(iii) *for all* $t \le T$,

$$\|\tilde{f}_2(t)\|^2 \le b_1, \quad \|\tilde{g}(t)\|_{\tilde{I}}^2 + \|\frac{\partial \tilde{g}}{\partial x_2}(t)\|_{\tilde{I}}^2 \le b_2, \quad \rho_1^{(N)}(t) \le b_3.$$

Then for all $t \le T$,

$$E_1^{(N)}(t) \le b_4 e^{b_5 t} \rho_1^{(N)}(t), \tag{5.22}$$

where b_l *are positive constants depending only on* $|||R_r\eta^{(N)}|||_{1,\infty}$, $|||R_r\varphi^{(N)}|||_{2,\infty}$ *and* ν. *In particular, if* (5.21) *holds, then for all* $\rho_1^{(N)}(t)$ *and* t, (5.22) *holds.*

Remark. If $\alpha_1 = \alpha_2$, then the main nonlinear error terms

$$(Z(t), R_r J_c^{(\alpha)}(R_r Z(t), R_r\tilde{\varphi}^{(N)}(t))), \quad Z = \tilde{\eta}^{(N)} \text{ or } \tilde{\eta}_t^{(N)}$$

depend only on the boundary errors. If in addition, there is no error on the boundary, then both of them vanish.

Now, we consider the convergence. Let

$$\begin{aligned}&\xi^{(N)} = P_N\xi, \quad \psi^{(N)} = P_N\psi, \\ &\tilde{\xi}^{(N)} = \eta^{(N)} - \xi^{(N)}, \quad \tilde{\psi}^{(N)} = \varphi^{(N)} - \psi^{(N)}.\end{aligned}$$

By (1.1) and (2.3),

$$\begin{cases} \quad \tilde{\xi}_t^{(N)} + R_r J_c^{(N)}(R_r(\tilde{\xi}^{(N)} + \delta\tau\tilde{\xi}_t^{(N)}), R_r(\psi^{(N)} + \tilde{\psi}^{(N)})) \\ \quad + R_r J_c^{(\alpha)}(R_r(\xi^{(N)} + \delta\tau\xi_t^{(N)}), R_r\tilde{\psi}^{(N)}) - \nu\Delta(\tilde{\xi}^{(N)} + \sigma\tau\tilde{\xi}_t^{(N)}) \\ = P_c f_1 - P_N f_1 - \sum\limits_{l=1}^{5} M_l, \\ -\Delta\tilde{\psi}^{(N)} = \tilde{\xi}^{(N)} - M_6, \\ \tilde{\xi}^{(N)}(0) = P_c\xi_0 - P_N\xi_0. \end{cases} \tag{5.23}$$

where

$$
\begin{aligned}
M_1 &= \xi_t^{(N)} - \frac{\partial \xi^{(N)}}{\partial t},\\
M_2 &= R_r J_c^{(\alpha)}(R_r \xi^{(N)}, R_r \psi^{(N)}) - P_N\Big(\frac{\partial \psi}{\partial x_2}\frac{\partial \xi}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \xi}{\partial x_2}\Big),\\
M_3 &= \delta\tau R_r J_c^{(\alpha)}(R_r \xi_t^{(N)}, R_r \psi^{(N)}),\\
M_4 &= \nu \frac{\partial^2 \xi}{\partial x_1^2} - \nu \xi_{x_1\bar{x}_1}^{(N)},\\
M_5 &= \nu\sigma\tau\Delta\xi_t^{(N)},\\
M_6 &= \frac{\partial^2 \psi^{(N)}}{\partial x_1^2} - \psi_{x_1\bar{x}_1}^{(N)}.
\end{aligned}
$$

Next, we estimate the terms at the right side of (5.23). Firstly

$$
\begin{aligned}
\tau \sum_{\substack{y\in S_\tau\\ y\le t-\tau}} \|M_1(y)\|^2 &\le C\tau \sum_{\substack{y\in S_\tau\\ y\le t-\tau}} \|\xi_t(y) - \frac{\partial \xi}{\partial t}(y)\|^2\\
&\le C\tau^2 \|\frac{\partial^2 \xi}{\partial t^2}\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

We have $M_2 = \sum\limits_{l=1}^{3} \alpha_l F_l$, where

$$
F_l = R_r J_{l,c}^{(\alpha)}(R_r \xi^{(N)}, R_r \psi^{(N)}) - P_N\Big(\frac{\partial \psi}{\partial x_2}\frac{\partial \xi}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \xi}{\partial x_2}\Big).
$$

Obviously

$$
F_1 = \sum_{l=1}^{6} K_6 - R_r P_c\Big((R_r\psi^{(N)})_{\hat{x}_1} \frac{\partial R_r \xi^{(N)}}{\partial x_2}\Big) + P_N\Big(\frac{\partial \psi}{\partial x_1}\frac{\partial \xi}{\partial x_2}\Big), \tag{5.24}
$$

$$
\begin{aligned}
K_1 &= R_r P_c\Big(\frac{\partial R_r \psi^{(N)}}{\partial x_2}(R_r \xi^{(N)})_{\hat{x}_1}\Big) - R_r P_c\Big(\frac{\partial R_r \psi^{(N)}}{\partial x_2}\frac{\partial R_r \xi^{(N)}}{\partial x_1}\Big),\\
K_2 &= R_r P_c\Big(\frac{\partial R_r \psi^{(N)}}{\partial x_2}\frac{\partial R_r \xi^{(N)}}{\partial x_1}\Big) - R_r P_c\Big(\frac{\partial R_r \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1}\Big),\\
K_3 &= R_r P_c\Big(\frac{\partial R_r \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1}\Big) - R_r P_c\Big(\frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1}\Big),
\end{aligned}
$$

$$
\begin{aligned}
K_4 &= R_r P_c\Big(\frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1}\Big) - P_c\Big(\frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1}\Big),\\
K_5 &= P_c\Big(\frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1}\Big) - \frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1},\\
K_6 &= \frac{\partial \psi^{(N)}}{\partial x_2}\frac{\partial \xi^{(N)}}{\partial x_1} - P_N\Big(\frac{\partial \psi}{\partial x_2}\frac{\partial \xi}{\partial x_1}\Big).
\end{aligned}
$$

Let $\beta > \frac{1}{2}$ and $\mu > 0$. By Lemma 4.9 and embedding theory, we have

$$\begin{aligned}
\|K_1\| &\le Ch^2 |||\psi^{(N)}|||_{C^1(\Omega)} |||\xi^{(N)}(t)|||_{C^3(I,L^2(\widetilde{I}))} \\
&\le Ch^2 |||\psi|||_{H^{2+\mu}(\Omega)} |||\xi|||_{H^{\frac{1}{2}+\mu}(I,L^2(\widetilde{I}))}, \\
\|K_2\| &\le CN^{-\beta} |||\psi^{(N)}|||_{C^1(\Omega)} |||\xi^{(N)}|||_{C^1(I,H^\beta(\widetilde{I}))} \\
&\le CN^{-\beta} |||\psi|||_{H^{2+\mu}(\Omega)} |||\xi|||_{H^{\frac{3}{2}+\mu}(I,H^\beta(\widetilde{I}))}, \\
\|K_3\| &\le CN^{-\beta} |||\psi^{(N)}|||_{C(I,H^{\beta+1}(\widetilde{I}))} |||\xi^{(N)}|||_{C^1(I,L^2(\widetilde{I}))} \\
&\le CN^{-\beta} |||\psi|||_{H^{\frac{1}{2}+\mu}(I,H^{\beta+1}(\widetilde{I}))} |||\xi|||_{H^{\frac{3}{2}+\mu}(I,L^2(\widetilde{I}))}, \\
\|K_4\| &\le CN^{-\beta} |||\psi^{(N)}|||_{C^1(I,H^{\beta+1}(\widetilde{I}))} |||\xi^{(N)}|||_{C^1(I,H^\beta(\widetilde{I}))} \\
&\le CN^{-\beta} |||\psi|||_{H^{\frac{1}{2}+\mu}(I,H^{\beta+1}(\widetilde{I}))} |||\xi|||_{H^{\frac{3}{2}+\mu}(I,H^\beta(\widetilde{I}))}, \\
\|K_5\| &\le CN^{-\beta} |||\psi|||_{H^{\frac{1}{2}+\mu}(I,H^{\beta+1}(\widetilde{I}))} |||\xi|||_{H^{\frac{3}{2}+\mu}(I,H^\beta(\widetilde{I}))}, \\
\|K_6\| &\le CN^{-\beta} |||\xi|||_{H^{\frac{1}{2}+\mu}(I,H^\beta(\widetilde{I}))} |||\psi|||_{H^{\frac{3}{2}+\mu}(I,H^{\beta+1}(\widetilde{I}))}.
\end{aligned}$$

We can estimate the last two terms of (5.24) and $\|F_2\|$, $\|F_3\|$. Thus

$$\begin{aligned}
\|M_2\| \le C(N^{-\beta}+h^2)\Big(&|||\xi|||_{H^{2+\mu}(\Omega)} + |||\xi|||_{H^{\frac{1}{2}+\mu}(I,H^{\beta+1}(\widetilde{I}))} + |||\xi|||_{H^{\frac{3}{2}+\mu}(I,H^\beta(\widetilde{I}))} \\
&+ |||\xi|||_{H^{\frac{7}{2}+\mu}(I,L^2(\widetilde{I}))}\Big)\Big(|||\psi|||_{H^{2+\mu}(\Omega)} + |||\psi|||_{H^{\frac{1}{2}+\mu}(I,H^{\beta+1}(\widetilde{I}))} \\
&+ |||\psi|||_{H^{\frac{3}{2}+\mu}(I,H^\beta(\widetilde{I}))} + |||\psi|||_{H^{\frac{7}{2}+\mu}(I,L^2(\widetilde{I}))}\Big).
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
\|M_3\| &\le C\tau |||\psi^{(N)}|||_{H^{2+\mu}(\Omega)} \Big(|||\frac{\partial\xi}{\partial t}|||_{H^{\frac{3}{2}+\mu}(I,L^2(\widetilde{I}))} + |||\frac{\partial\xi}{\partial t}|||_{H^{\frac{1}{2}+\mu}(I,H^1(\widetilde{I}))}\Big), \\
\|M_4\| &\le Ch^2 |||\xi|||_{C^4(I,L^2(\widetilde{I}))} \le Ch^2 |||\xi|||_{H^{\frac{9}{2}+\mu}(I,L^2(\widetilde{I}))}, \\
\|M_5\| &\le C\tau \Big(|||\frac{\partial\xi}{\partial t}|||_{H^{\frac{5}{2}+\mu}(I,L^2(\widetilde{I}))} + |||\frac{\partial\xi}{\partial t}|||_{H^{\frac{1}{2}+\mu}(I,H^2(\widetilde{I}))}\Big), \\
\|M_6\| &\le Ch^2 |||\psi|||_{C^4(I,L^2(\widetilde{I}))} \le Ch^2 |||\psi|||_{H^{\frac{9}{2}+\mu}(I,L^2(\widetilde{I}))}.
\end{aligned}$$

By an argument as in Theorem 5.1, we come to the following conclusion.

Theorem 5.2. *Let the conditions* (i) *and* (ii) *of Theorem* 5.1 *hold. In addition,* $\beta > \frac{1}{2}$, $\mu > 0$ *and*

$$\begin{aligned}
&\xi, \psi \in C(0,T; H^{2+\mu}(\Omega) \cap H^{\frac{1}{2}+\mu}(I, H^{\beta+1}(\widetilde{I})) \cap H^{\frac{3}{2}+\mu}(I, H^\beta(\widetilde{I})) \cap H^{\frac{9}{2}+\mu}(I, L^2(\widetilde{I}))), \\
&\frac{\partial\xi}{\partial t} \in C(0,T; H^{\frac{1}{2}+\mu}(I, H^2(\widetilde{I})) \cap H^{\frac{5}{2}+\mu}(I, L^2(\widetilde{I}))), \ \frac{\partial^2\xi}{\partial t^2} \in L^2(0,T; L^2(\Omega)), \\
&f_1 \in C(0,T; H^{\frac{1}{2}+\mu}(I, H^\beta(\widetilde{I}))), \quad g_0,\ g_1 \in C(0,T; H^{\beta+1}(\widetilde{I})).
\end{aligned}$$

Then for all $t \le T$, *we have*

$$\|\xi(t) - \eta^{(N)}(t)\|^2 \le b^*(\tau^2 + h^4 + N^{-2\beta}),$$

where b^* *is a positive constant which depends on the norms mentioned above.*

§6. Error Estimations for Other Unsteady Problems

In this section, let $b \geq 0$ and consider the following boundary condition

$$\begin{cases} \quad -\eta_{x_1}^{(N)}(0,x_2,t) + \frac{b}{2}(\eta^{(N)}(0,x_2,t) + \eta^{(N)}(h,x_2,t)) \\ = P_c(-\frac{\partial \xi}{\partial x_1}(0,x_2,t) + b\xi(0,x_2,t)) \\ = P_c g_0(x_2,t), \\ \quad \eta_{\bar{x}_1}^{(N)}(1,x_2,t) + \frac{b}{2}(\eta^{(N)}(1,x_2,t) + \eta^{(N)}(1-h,x_2,t)) \\ = P_c(\frac{\partial \xi}{\partial x_1}(1,x_2,t) + b\xi(1,x_2,t)) \\ = P_c g_1(x_2,t). \end{cases} \tag{6.1}$$

For simplicity, we assume that

$$\alpha_1 = \alpha_2 = \frac{1}{2}, \quad \delta = 0$$

and

$$\tilde{\varphi}^{(N)}(0,x_2,t) = \tilde{\varphi}^{(N)}(1,x_2,t) = 0$$

as before. By an argument as in Section 5, we have

$$\begin{aligned} &\|\tilde{\eta}^{(N)}(t)\|_t^2 + \tau(m-1-\epsilon)\|\tilde{\eta}_t^{(N)}(t)\|^2 + 2\nu|\tilde{\eta}^{(N)}(t)|_1^2 \\ &+ \nu\tau\Big(\sigma + \frac{m}{2}\Big)(|\tilde{\eta}^{(N)}(t)|_1^2)_t + \nu\tau^2\Big(m\sigma - \sigma - \frac{m}{2}\Big)|\tilde{\eta}_t^{(N)}(t)|_1^2 \\ &+ D_1(t) + G_4(t) + G_5(t) + \sum_{l=1}^{4} B_l(t) \\ \leq\ & \|\tilde{\eta}^{(N)}(t)\|^2 + \Big(1 + \frac{m^2\tau}{4\epsilon}\Big)\|\tilde{f}_1\|^2 \end{aligned} \tag{6.2}$$

where $D_1(t),\ B_l(t)$ are the same as in the previous section, and

$$\begin{aligned} G_4(t) &= (2\tilde{\eta}^{(N)}(t) + m\tau\tilde{\eta}_t^{(N)}(t), R_r J_c^{(\alpha)}(R_r\eta^{(N)}(t), R_r\tilde{\varphi}^{(N)}(t)) \\ &\quad + R_r J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}(t), R_r\varphi^{(N)}(t))), \\ G_5(t) &= m\tau(\tilde{\eta}_t^{(N)}(t), R_r J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}(t), R_r\tilde{\varphi}^{(N)}(t))). \end{aligned}$$

We first suppose that $b = 0$. By Lemma 4.4 and Lemma 4.8,

$$\begin{aligned} |G_4(t)| &\leq \epsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \epsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2 \\ &\quad + \frac{C}{\epsilon}(1 + |||R_r\eta^{(N)}|||_{1,\infty}^2 + |||R_r\varphi^{(N)}|||_{2,\infty}^2)(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2 + \tau h\|\tilde{g}\|_{\tilde{I}}^2), \\ |G_5(t)| &\leq \epsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2 + \frac{C\tau N}{\epsilon h}|\tilde{\eta}^{(N)}(t)|_1^2(\|\tilde{\eta}^{(N)}(t)\|^2 + \|\tilde{f}_2\|^2). \end{aligned}$$

By Lemma 4.7 and Lemma 4.8, for any $x_1, x_1', x_1'' \in \bar{I}_h$,

$$\begin{aligned} &|(u(x_1,t), v(x_1',t)w(x_1'',t))_{\tilde{I}}| \\ \leq\ & \|u(x_1,t)\|_{\tilde{I}}^2 + N\|v(x_1',t)\|_{\tilde{I}}^2\|w(x_1'',t)\|^2 \\ \leq\ & \epsilon|u(t)|_1^2 + C\|u(t)\|^2 + N(\epsilon|v(t)|_1^2 + C\|v(t)\|^2)(\epsilon|w(t)|_1^2 + \|w(t)\|^2). \end{aligned}$$

Thus Lemma 4.6 and the second formula of (5.1) lead to

$$|D_1(t)| \leq C(1 + N\|\tilde{\eta}(t)\|^2 + N\|\tilde{f}_2(t)\|^2)(\epsilon\nu|\tilde{\eta}(t)|_1^2 + \|\eta(t)\|^2). \tag{6.5}$$

Next by Lemma 4.7,

$$\sum_{l=1}^{4}|B_l(t)| \le \epsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2 + \epsilon\nu\tau^2|\tilde{\eta}_t^{(N)}(t)|_1^2 + \frac{C}{\epsilon}(\|\tilde{\eta}^{(N)}(t)\|^2 + \tau^2\|\tilde{\eta}_t^{(N)}(t)\|^2 + \|\tilde{g}(t)\|_{\tilde{I}}^2 + \tau^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2).$$

Theorem 6.1. *Assume that*

(i) $b=0,\ h=O(\frac{1}{N}),\ \tau=O\Big(\frac{1}{N^2}\Big),\ \sigma\ge\frac{1}{2}$ *or* $\tau<\frac{2h^2}{\nu(1-2\sigma)(4+N^2h^2)}$,

(ii) $\|\tilde{f}_2(t)\|^2\le\frac{b_6}{N}$ *and* $\rho_2^{(N)}(t)\le\frac{b_7}{N}$ *for all* $t\le T$.

Then for all $t\le T$,

$$E_2^{(N)}(t)\le b_8e^{b_9t}\rho_2^{(N)}(t),$$

where

$$E_2^{(N)}(t)=\|\tilde{\eta}^{(N)}(t)\|^2+\nu\tau|\tilde{\eta}^{(N)}(t)|_1^2+\tau\sum_{\substack{y\in S\tau\\ y\le t-\tau}}(p_0\tau\|\tilde{\eta}_t^{(N)}(y)\|^2+\nu|\tilde{\eta}^{(N)}(y)|_1^2),$$

$$\rho_2^{(N)}(t)=\|\tilde{\eta}^{(N)}(0)\|^2+\sum_{\substack{y\in S_\tau\\ y\le t-\tau}}(\|\tilde{f}_1(y)\|^2+\|\tilde{g}(y)\|_{\tilde{I}}^2+\tau^2\|\tilde{g}_t(y)\|_{\tilde{I}}^2).$$

We next consider the case $b>0$. Let

$$S^*(\tilde{\eta}^{(N)}(t))=\frac{1}{2}(\|\tilde{\eta}^{(N)}(0,t)+\tilde{\eta}^{(N)}(h,t)\|_{\tilde{I}}^2+\|\tilde{\eta}^{(N)}(1-h,t)+\tilde{\eta}^{(N)}(1,t)\|_{\tilde{I}}^2).$$

By (6.1),

$$\|\tilde{\eta}_{x_1}^{(N)}(0,t)\|_{\tilde{I}}^2+\|\tilde{\eta}_{\bar{x}_1}^{(N)}(1,t)\|_{\tilde{I}}^2\le\epsilon S^*(\tilde{\eta}^{(N)}(t))+\frac{C}{\epsilon}\|\tilde{g}(t)\|_{\tilde{I}}^2.$$

As the derivation of (6.3) and (6.4), we have

$$|G_4(t)|+|G_5(t)|\le\epsilon\tau\|\tilde{\eta}_t^{(N)}(t)\|^2+\epsilon\nu|\tilde{\eta}^{(N)}(t)|_1^2+\frac{C}{\epsilon}\Big(|||R_r\eta^{(N)}|||_{1,\infty}^2 + |||R_r\varphi^{(N)}|||_{2,\infty}^2+1\Big)(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2+\tau hS^*(\tilde{\eta}^{(N)}(t)) + \tau h\|\tilde{g}(t)\|_{\tilde{I}}^2)+\frac{C\tau N}{\epsilon h}|\tilde{\eta}^{(N)}(t)|_1^2(\|\tilde{\eta}^{(N)}(t)\|^2+\|\tilde{f}_2(t)\|^2).$$

On the other hand,

$$B_1(t)\ge b\nu(1-\epsilon)S^*(\tilde{\eta}^{(N)}(t))-\frac{C}{\epsilon}\|\tilde{g}(t)\|_{\tilde{I}}^2,$$

$$B_2(t)+B_3(t)\ge\frac{b(m+2\sigma)}{4}S_t^*(\tilde{\eta}^{(N)}(t))-\frac{b\nu\tau^2(m+2\sigma)}{4}(1+\epsilon)S^*(\tilde{\eta}_t^{(N)}(t)) - \frac{C}{\epsilon}(\|\tilde{g}(t)\|_{\tilde{I}}^2+\tau^2\|\tilde{g}_t(t)\|_{\tilde{I}}^2),$$

$$B_4(t)\ge\frac{b\nu\sigma\tau^2m}{2}(1-\epsilon)S^*(\tilde{\eta}_t^{(N)}(t))-\frac{C\tau^2}{\epsilon}\|\tilde{g}_t(t)\|_{\tilde{I}}^2.$$

By an argument as in Theorem 5.1, we reach the following conclusion.

Theorem 6.2. *Assume that*

(i) $b>0,\ h=O\Big(\frac{1}{N}\Big),\ \tau=O\Big(\frac{1}{N^2}\Big),\ \sigma\ge\frac{1}{2}$ or $\tau<\frac{2h^2}{\nu(1-2\sigma)(4+N^2h^2+b)}$,

(ii) $\|\tilde{f}_2(t)\|^2\le\frac{b_{10}}{N}$ *and* $\rho_3^{(N)}(t)\le\frac{b_{11}}{N}$ *for all* $t\le T$.

Then for all $t \le T$,

$$E_3^{(N)}(t) \le b_{12} e^{b_{13} t} \rho_3^{(N)}(t)$$

where

$$\begin{aligned} E_3^{(N)}(t) &= \|\tilde{\eta}^{(N)}(t)\|^2 + \nu\tau(|\tilde{\eta}^{(N)}(t)|_1^2 + S^*(\tilde{\eta}^{(N)}(t))) \\ &\quad + \tau \sum_{\substack{y \in S_\tau \\ y \le t-\tau}} (p_0 \tau \|\tilde{\eta}_t^{(N)}(y)\|^2 + \nu |\tilde{\eta}^{(N)}(y)|_1^2 + \nu S^*(\tilde{\eta}^{(N)}(y))), \\ \rho_3^{(N)}(t) &= \|\tilde{\eta}^{(N)}(0)\|^2 + \sum_{\substack{y \in S_\tau \\ y \le t-\tau}} (\|\tilde{f}_1(y)\|^2 + \|\tilde{g}(y)\|_{\tilde{I}}^2 + \tau^2 \|\tilde{g}_t(y)\|_{\tilde{I}}^2). \end{aligned}$$

We can prove the convergence for $b = 0$ and $b > 0$ as in Theorem 5.2.

§7. The Steady Problem

In this section, we consider the steady problem

$$\begin{cases} \frac{\partial \psi}{\partial x_2}\frac{\partial \xi}{\partial x_1} - \frac{\partial \psi}{\partial x_1}\frac{\partial \xi}{\partial x_2} - \nu\Big(\frac{\partial^2 \xi}{\partial x_1^2} + \frac{\partial^2 \xi}{\partial x_2^2}\Big) = f_1, & \text{in } \Omega, \\ -\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_2^2} = \xi, & \text{in } \Omega. \end{cases} \tag{7.1}$$

For simplicity, suppose that $\xi \equiv \psi \equiv 0$ for $x_1 = 0, 1$. Let

$$H = \{u | u(0, x_2) = u(1, x_2) = 0,\ u(x_1, x_2) \in V_N \qquad \text{for } x_1 \in I_h\}$$

with the scalar product and the norm as follows:

$$\begin{aligned} \langle u, v\rangle &= \frac{1}{2}(u_{x_1}, v_{x_1}) + \frac{1}{2}(u_{\bar{x}_1}, v_{\bar{x}_1}) + \Big(\frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2}\Big) + \frac{1}{2h}(u(h), v(h))_{\tilde{I}} \\ &\quad + \frac{1}{2h}(u(1-h), v(1-h))_{\tilde{I}}, \\ \|u\|_H^2 &= |u|_1^2 + S(u). \end{aligned}$$

The Fourier pseudospectral-finite difference scheme for (7.1) is

$$\begin{cases} R_r J_c^{(\alpha)}(R_r \eta^{(N)}, R_r \varphi^{(N)}) - \nu \Delta \eta^{(N)} = P_c f_1, \\ -\Delta \varphi^{(N)} = \eta^{(N)}. \end{cases} \tag{7.2}$$

For any fixed $w \in H$, $(w, P_c f_1)$ is a linear functional in H and so there exists $F \in H$ such that

$$\langle F, w\rangle = (w, P_c f_1), \qquad |(w, P_c f_1)| \le \|F\|_H \|w\|_H.$$

Theorem 7.1. *Let* $\alpha_1 = \alpha_2$ *and* $\|F\|_M$ *be bounded uniformly for* N *and* h. *Then* (7.2) *has at least one solution which is bounded uniformly for* N *and* h.

Proof. From (7.2) and (2.15) we have

$$\nu\langle \eta^{(N)}, w\rangle + (w, R_r J_c^{(\alpha)}(R_r \eta^{(N)}, R_r \varphi^{(N)})) = (P_c f_1, w).$$

For any fixed $\eta^{(N)}$ and $\varphi^{(N)}$, $(w, R_r J_c^{(\alpha)}(R_r \eta^{(N)}, R_r \varphi^{(N)}))$ is a linear functional in H and thus there exists $A\eta^{(N)} \in H$ such that

$$\langle A\eta^{(N)}, w\rangle = (w, R_r J_c^{(\alpha)}(R_r \eta^{(N)}, R_r \varphi^{(N)})). \tag{7.3}$$

Hence (7.2) is equivalent to the following operator equation

$$\eta^{(N)} + \lambda(A\eta^{(N)} - F) = 0, \quad \lambda = \frac{1}{\nu}.$$

Assume that the sequence $\{\eta_n^{(N)}\}$ satisfies

$$-\Delta\varphi_n^{(N)} = \eta_n^{(N)}$$

and

$$\|\eta_n^{(N)} - \eta_0^{(N)}\|_H \to 0 \ \text{ as } \ n \to \infty.$$

If n is large enough, then $\|\eta_n^{(N)}\|_H$ and $\|\varphi_n^{(N)}\|_H$ are uniformly bounded. Let

$$Z_{m,n} = \langle A\eta_m^{(N)} - A\eta_n^{(N)}, w\rangle.$$

Then by (2.13),

$$|Z_{m,n}| \le C^* \|\eta_m^{(N)} - \eta_n^{(N)}\|_H \|w\|_H,$$

and so

$$\|A\eta_m^{(N)} - A\eta_n^{(N)}\|_H \le C^* \|\eta_m^{(N)} - \eta_n^{(N)}\|_H.$$

Thus A is a continuous operator. On the other hand, from (7.3) and (2.13) it follows that for $\lambda \in (0, \frac{1}{\nu}]$ the possible solution satisfies

$$\|\eta_\lambda^{(N)}\|_H^2 \le \lambda \|\eta_\lambda^{(N)}\|_H \|F\|_H \le \frac{1}{\nu}\|\eta_\lambda^{(N)}\| \|F\|_H$$

and thus $\|\eta_\lambda^{(N)}\|_H$ is bounded. Finally the conclusion follows from Browder's theorem.

Let

$$\|u\|_{l^4}^4 = \|u^2\|^2, \quad |u|_{1,l^4}^4 = \frac{1}{2}\|u_{x_1}\|_{l^4}^4 + \frac{1}{2}\|u_{\bar{x}_1}\|_{l^4}^4 + \|\frac{\partial u}{\partial x^2}\|_{l^4}^4.$$

We can prove as in [21] that $\|u\|_{l4}^4 \le C_5(\|u\|^2 + |u|_1^2)$.

Theorem 7.2. *If* $\alpha_1 = \alpha_2$ *and* $\nu^2 > C_6\|f_1\|$, *then* (7.2) *has only one solution where* $C_6 = C_5^{\frac{1}{2}} C_1^{\frac{7}{4}} (2 + 2C_1)^{\frac{3}{4}}$, *and* C_1 *is the same as in Lemma* 4.5.

Proof. Let $\eta^{(N)}$, $\varphi^{(N)}$ and $\eta_1^{(N)}$, $\varphi_1^{(N)}$ be the solutions of (7.2) and

$$\tilde{\eta}^{(N)} = \eta_1^{(N)} - \eta^{(N)}, \ \tilde{\varphi}^{(N)} = \varphi_1^{(N)} - \varphi^{(N)}.$$

Then

$$\begin{cases} R_r J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}, R_r\tilde{\varphi}^{(N)} + R_r\varphi^{(N)}) \\ +R_r J_c^{(\alpha)}(R_r\eta^{(N)}, R_r\tilde{\varphi}^{(N)}) - \nu\Delta\tilde{\eta}^{(N)} = 0, & \text{in } \Omega, \\ -\Delta\tilde{\varphi}^{(N)} = \tilde{\eta}^{(N)}, & \text{in } \Omega, \\ \tilde{\eta}^{(N)} = \tilde{\varphi}^{(N)} = 0, & \text{for } x_1 = 0, 1. \end{cases} \tag{7.4}$$

Since

$$(\tilde{\eta}^{(N)}, R_r J_c^{(\alpha)}(R_r\eta^{(N)}, R_r\tilde{\varphi}^{(N)})) = -(\eta^{(N)}, R_r J_c^{(\alpha)}(R_r\tilde{\eta}^{(N)}, R_r\tilde{\varphi}^{(N)})),$$

from (7.4) we have

$$\begin{aligned} \nu\|\tilde{\eta}\|_H^2 &\le \|\eta^{(N)}\|_{l^4} |\tilde{\eta}^{(N)}|_1 |\tilde{\varphi}^{(N)}|_{1,l^4} \\ &\le C_5^{\frac{1}{2}} \|\eta^{(N)}\|^{\frac{1}{2}} \|\tilde{\eta}^{(N)}\|_H |\tilde{\varphi}^{(N)}|_1^{\frac{1}{2}} (\|\eta^{(N)}\|^2 \\ &\quad + |\eta^{(N)}|_1^2)^{\frac{1}{4}} (|\tilde{\varphi}^{(N)}|_1^2 + |\tilde{\varphi}^{(N)}|_2^2)^{\frac{1}{4}}. \end{aligned} \tag{7.5}$$

Therefore by the second formula of (7.4), Lemma 4.5 and Lemma 4.6,

$$(\nu - C_5^{\frac{1}{2}} C_1^{\frac{3}{4}} (2+2C_1)^{\frac{1}{4}} \|\eta^{(N)}\|_H) \|\tilde{\eta}^{(N)}\|_H^2 \le 0. \tag{7.6}$$

On the other hand, by (7.2),

$$\nu \|\eta^{(N)}\|_H^2 \le \|\eta^{(N)}\| \|f_1\| \le \sqrt{C_1} \|\eta^{(N)}\|_H \|f_1\|.$$

By the technique as in [5], we have the following results.

Theorem 7.3. *Let $\tilde{f}_1$ be the error of f_1. If the conditions of Theorem 7.2 are fulfilled, then $\|\tilde{\eta}^{(N)}\|_H^2 \le C\|\tilde{f}_1\|^2$.*

Theorem 7.4. *If the conditions of Theorem 7.2 hold, then the iteration*

$$\begin{cases} \eta_{n+1}^{(N)} = \eta_n^{(N)} + \tau[\nu \Delta \eta_{n+1}^{(N)} - R_r J_c^{(\alpha)}(R_r \eta_{n+1}^{(N)}, R_r \varphi^{(N)}) + P_c f_1], \quad \tau > 0, \ n \ge 0, \\ -\Delta \varphi_n^{(N)} = \eta_n^{(N)}. \end{cases}$$

is convergent, and there exists a positive constant $0 < \theta < 1$ such that

$$\|\eta_n^{(N)} - \eta^{(N)}\|_H^2 \le \theta^n \|\eta_0^{(N)} - \eta^{(N)}\|_H^2.$$

References

[1] Gottlieb, D. & Orszag, S. A., Numerical analysis of spectral methods, CBMS-NSF, SIAM, Philadelphia, 1977.
[2] Kreiss, H. O. & Oliger, J., Stability of the Fourier method, *SIAM J. Numer. Anal.,* **16**(1979), 421-433.
[3] Kuo Penyu, Error estimations of the spectral method for solving K.D.V-Burgers equation, Talk in Madrid University, March 1981, Also see *Acta Mathematica Sinica,* **28**(1985), 1-15.
[4] Canuto, C., Hussaini, M.Y., Quarteroni, A. & Zang, T.A., Spectral methods in fluid dynamics, Springer-Verlag, New York, 1988.
[5] Kuo Penyu, The convergence of spectral scheme for solving two-dimensional vorticity equation, *J. Comp. Math.,* **1**(1983), 353-362.
[6] Guo Benyu, Spectral method for solving Navier-Stokes equations, *Scientia Sinica,* **28A**(1985), 1139-1153.
[7] Ma Heping & Guo Benyu, The Fourier pseudospectral method with a restrain operator for the Korteweg-de Vries equation, *J. Comp. Phys.,* **65**(1986), 120-137.
[8] Guo Benyu & Cao Weiming, The Fourier pseudospectral method with a restrain operator for the R. L. W. equation, *J. Comp. Phys.,* **74**(1988), 110-126.
[9] Woodward, P. & Colella, P., The numerical simulation of two-dimensional fluid flows with strong shock, *J. Comp. Phys.,* **54**(1984), 115-173.
[10] Vandeven, H., Family of spectral filters for discontinuous problems, CNRS-756 Centre de Math. Appl., Ecole Polytechnique, 1987.
[11] Ingham, D. B., Free-convection boundary layer on an isothermal horizontal cylinder, *J. Appl. Math. Phys.,* **29**(1978), 871-883.
[12] Ingham, D. B., Unsteady separation, *J. Comp. Phys.,* **53**(1984), 90-99.
[13] Moin, P. & Kim, J., Numerical investigation of turbulent channel flow, *J. Fluid Mech.,* **118**(1982), 341-377.
[14] Macaraeg, M. G., A mixed pseudospectral-finite difference method for the axisymmetric flow in a heated rotating spherical shell, *J. Comp. Phys.,* **62**(1986), 297-320.
[15] Murdock, J. W., A numerical study of nonlinear effects on boundary-layer stability, *AIAA J.,* **15**(1977), 1167-1173.
[16] Guo Benyu, Spectral-difference method for baroclinic primitive equation and its error estimation, *Scientia Sinica,* **30A**(1987), 696-713.
[17] Guo Benyu & Xiong Yueshan, A spectral difference method for two-dimensional viscous flow, *J. Comp. Phys.,* **84**(1989), 259-278.
[18] Guo Benyu, Ma Heping, Cao Weiming & Huang Hui, The Fourier-Chebyshev spectral method for solving two-dimensional unsteady vorticity equations, *J. Comp. Phys.,* **101**(1992), 207-217.
[19] Canuto, C., Maday, Y. & Quarteroni. A., Combinded finite element and spectral approximation of the Navier-Stokes equations, *Numer. Math.,* **44**(1984), 201-217.

[20] Guo Benyu & Cao Weiming, A combined spectral-finite element method for solving two-dimensional unsteady Navier-Stokes equations, *J. Comp. Phys.,* **101**(1992), 375-385.

[21] Guo Benyu & Ma Heping, Pseudospectral-finite element method for solving two-dimensional vorticity equations, *SIAM J. Numer. Anal.,* **28**(1991), 113-132.

[22] Guo Benyu & Zheng Jiadong, Pseudospectral-difference method for baroclinic primitive equation and its error estimation, *Scientia Sinica,* **35A**(1992), 1-13.

[23] Lions, J. L., Quelques Méthods de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.

[24] Guo Benyu, A class of diffenence schemes for two dimensional viscous vorticity equation, *Acta Math. Sinica.,* **17**(1974), 242-258.

[25] Canuto, C. & Quarteroni, A., Approximation results for orthogonal polynomials in Sobolev spaces, *Math. Comp.,* **38**(1982), 67-86.

[26] Ma Heping & Guo Benyu, The Fourier pseudospectral method for two-dimensional vorticity equations, *IMA J. Numer. Anal.,* **7**(1987), 47-60.

[27] Guo Benyu, A spectral-difference method for solving two-dimensional vorticity equation, *J. Comp. Math.,* **6**(1988), 238-257.