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SUPPLEMENT TO "PASSAGE, BLOCKADE, SINK AND SOURCE OF A PLANAR DYNAMICAL SYSTEM"

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Abstract

A Theorem is given on the number of passages passing throgh a multiply-connected region, which corrects a wrong conjecture in a former paper of the author.

Keywords Passage, Multiply-connected region, Critical point, Saddle point.
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In [1] we have given a conjecture on the number of passages of a planar dynamical system D through a multiply-connected region G. Now, we find that the number conjectured is incorrect. Instead, we give a theorem as follows:

Theorem. *For an n-multiply-connected region $G\,(n \geq 2)$ with outer boundary L_1 and inner boundaries $L_2, \cdots, L_n$, the σ_i and ν_i denote the number of inner and outer contact points on L_i (as a boundary of G) with respect to the trajectories of D, respectively. Assume now*

$$\nu_1 = 2m, \quad \sigma_1 = 0, \quad \nu_i = 0$$

for $i = 2, 3, \cdots, n$,

$$\sum_{i=2}^{n} \sigma_i = 2(m+k),$$

where $m \geq 1,\ 0 \leq k \leq n-2$ are integers, so that the sum of indices of critical points in G is $k+2-n$. We assume that there are exactly $n-k-2$ saddle points in G. Then the number of passages is at least

$$p = 2(m+2+k-n)$$

when $2 \leq n < m+2+k$; except the case $m = 1,\ n = 2,\ k = 0$, in which we have $p = 1$. There may exist no passage when

$$n \geq m+2+k.$$

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Proof of the Second Part (the idea comes from Fig. 13 in [1]). When $n \geq m+2+k$, we have $m+k \leq n-2 < n-1$. We may take

$$\sigma_2 = \sigma_3 = \cdots = \sigma_{m+k+1} = 2,$$

and let $l_1,\ l_2$ be two trajectories such that each of them contacts $L_2, L_3, \cdots, L_{m+k+1}$ externally at one point and both of them go into the interior of L_{m+k+2} as shown in Fig. 1. Evidently, there is no passage in G.

Fig. 1

When $m=1,\ n=2$, we must have $k=0$; the theorem reduces to Theorem 3 in [1].

Proof of the First Part. When $2 < n < m+2+k$ we have $m+k > n-2 > 0$. Assume that $\sigma_2 = \cdots = \sigma_{n-1} = 2$ and $L_2, L_3, \cdots, L_{n-1}$ all contact l_1 and l_2; moreover, l_1 and l_2 both go into L_n. Now,

$$\sigma_n = 2(m+k-(n-2));$$

it is easily seen that there will be $2(m+k+2-n)$ passages passing through L_1 and L_n, as is shown in Fig. 2 in case $n=4,\ k=1$ and $m=3$, so $p=4$. In case $n=2$, there is only inner boundary, we need not consider l_1 and l_2.

Fig. 2

Notice that Fig. 2 gives the least number of passages under the condition $n < m+2+k$. If instead of Fig. 2, we have less inner boundaries like L_2, more inner boundaries like L_n, then the number of passages will be greater than $2(m-n+2+k)$. See Fig. 3, where

$$n = 4, \quad m = 3, \quad k = 0, \quad 2(m-n+2+k) = 2, \quad \text{but } p = 3.$$

Fig. 3

Remark 1. Fig. 4 shows that the assumption $\sigma_1 = 0$ in the theorem cannot be dropped.

$n = 3,\ m = 2,\ k = 0,$

$L_1:\ \sigma_1 = \nu_1 = 3;$

$L_2:\ \sigma_2 = \nu_2 = 1;$

$L_3:\ \sigma_3 = \nu_3 = 0.$

$2(m-n+k+2) = 2,$

but $p = 0$.

Fig. 4

If we notice the picture in Fig. 4 and the proof of the theorem, we see at once that the theorem remains true if the conditions on the ν_i and σ_i are changed into:

$$\nu_1 - \sigma_1 = 2m,$$

$$\sum_{i=2}^{n}(\sigma_i - \nu_i) = 2(m+k), \quad m \geq 2.$$

Under these conditions we have in Fig. 4:

$$m = k = 0,\ n = 3 > 2,$$

so there is no passage.

Remark 2. Fig. 5 shows that the condition $k \leq n-2$ cannot be dropped.

Remark 3. Fig. 6 shows that the condition $k \geq 0$ cannot be dropped.

$n=2,\ k=1,\ m=1,\ k+2-n=1$

$n < m+2+k, \quad p=4,$

only two passages

Fig. 5

$m=3,\ n=2,\ k=-2,\ n<m+2+k$

$p=2(3+2-2-2)=2$

only one passage

Fig. 6

References

[1] Ye Yanqian, Passage, blockade, sink and source of a planar dynamical system, *Chin. Ann. of Math.,* **13B:** 3(1992), 257-265.