Chin. Ann. of Math.
15B: 4(1994),493-500.

THE UNIQUENESS OF BIFURCATION TO SEPARATRIX LOOPS IN SUPERCRITICAL CASES

Sun Jianhua*

Abstract

In paper [4] the existence of bifurcation to separatrix loops in supercritical cases on the plane is studied. This note is a continuation of [4]. The author proves the uniqueness of limit cycles in a neighborhood of the separatrix loop, and the results strengthen the relevant conclusions in [1-6].

Keywords Supercritical, Separatrix loop, Bifurcation, Limit cycle, Uniqueness.
1991 MR Subject Classification 34C05.

§1. Introduction

In this paper we consider the two-dimensional system in the general vector form

$$\dot{X} = F(X) + \alpha G(X, \delta), \tag{1.1}$$

or, in coordinates,

$$\begin{aligned} \dot{x} &= F_1(x, y) + \alpha G_1(x, y, \delta), \\ \dot{y} &= F_2(x, y) + \alpha G_2(x, y, \delta), \end{aligned} \tag{1.1$'$}$$

where $|\alpha| << 1$, F and G are sufficiently smooth, and $F(0) = G(0, \delta) = 0$, the parameter $\delta \in \mathbb{R}^m$, $m \in \mathbb{N}^+$. We make the following hypothesis on the unperturbed system.

(A) $\sigma_0 = \text{div}F(0) = 0$, and, for $\alpha = 0$, (1.1) possesses a separatrix loop L_0 passing through the hyperbolic saddle point 0. Set $L_0 = \{X_0(t) | -\infty < t < \infty\}$.

In our preceding paper [4] we studied the existence of the bifurcation to the separatrix loop L_0 for the system (1.1) in the supercritical case. In the present paper we will study the uniqueness of the bifurcation to separatrix loop L_0 in the supercritical case. The analogous issues were considered by [5] for the critical case.

Let

$$\begin{aligned} \sigma &= \int_{-\infty}^{\infty} \text{div}F(X_0 t) dt, \\ M_1(\delta) &= \int_{-\infty}^{\infty} \exp\Big(-\int_0^t \text{div}F(X_0(s))ds\Big) F(X_0(t)) \wedge G(X_0(t), \delta) dt, \\ M_k(\delta) &= \int_{-\infty}^{\infty} \exp\Big(-\int_0^t \text{div}F(X_0(s))ds\Big) F(X_0(t)) \wedge DG(X_0(t), \delta) X_{k-1}(t) dt \quad (k \geq 2) \end{aligned} \tag{1.2}$$

Manuscript received May 6, 1992. Revised November 6, 1993.
*Department of Mathematics, Nanjing University, Nanjing 210008, China

in which

$$X_{k-1}(t) = \begin{cases} X_{k-1}^s(t), & t \in [0, \infty), \\ X_{k-1}^u(t), & t \in (-\infty, 0] \end{cases} \tag{1.3}$$

and $X_{k-1}^s,\ X_{k-1}^u(t)$ satisfy the following equations:

$$\begin{aligned} \dot{X}_k^* &= DF(X_0(t))X_k^* + DG(X_0(t), \delta)X_{k-1}^*, \\ \dot{X}_1^* &= DF(X_0(t))X_1^* + G(X_0(t), \delta) \quad (k \geq 2), \end{aligned} \tag{1.4}$$

where $t \in [0, \infty)$ as $* = s$ and $t \in (-\infty, 0]$ as $* = u$ (see [4]).

We obtain the following main results.

Theorem 1.1. *In addition to the hypothesis* (A)*, suppose that* $\sigma_0 = 0,\ \sigma \neq 0,\ L_0$ *is counter-clockwise and there exists a* $\delta_0 \in \mathbb{R}^m$ *such that* $M_k(\delta_0) = 0,\ k = 1, \cdots, n-1,\ M_n(\delta_0) \neq 0$. *Then for* $0 < |\alpha| << 1,\ \|\delta - \delta_0\| << 1$, *the system* (1.1) *has a unique limit cycle if* $\sigma\alpha^n M_n(\delta) < 0$ *and no limit cycle if* $\sigma\alpha^n M_n(\delta) > 0$ *near* L_0. *And the limit cycle is stable as* $\sigma < 0$ *and unstable as* $\sigma > 0$.

The paper is organized as follows: the preliminaries are given in Section 2; Section 3 is devoted to the proof of our main results and corollary; finally, in Section 4 we show how our results are applied to an example.

§2. Preliminaries

Under our assumptions, the origin is still a hyperbolic saddle point of (1.1) for α near zero. Up to a linear change of coordinates, the system (1.1) can be written in the form

$$\begin{aligned} \dot{x} &= \lambda_1 x + f_1(x, y) + \alpha\bar{\lambda}_1(\delta)x + \alpha g_1(x, y, \delta), \\ \dot{y} &= -\lambda_2 y + f_2(x, y) - \alpha\bar{\lambda}_2(\delta)y + \alpha g_2(x, y, \delta), \end{aligned} \tag{2.1}$$

in which $\lambda_1 = \lambda_2 > 0,\ \bar{\lambda}_i(\delta) > 0$ for $\|\delta\|$ small, f_i and g_i vanish at the origin together with their first derivatives, $i = 1, 2$.

Moreover, according to the results of [7], we have the following

Lemma 2.1. *There exists a* C^3*-transformation in a neighborhood of the origin such that the system* (1.1) *can be changed into the form*

$$\begin{aligned} \dot{x} &= \lambda_1 x + \alpha\bar{\lambda}_1(\delta)x + x^2 y h_1(x, y) + \alpha x^2 y R_1(x, y, \delta) = xP(x, y, \alpha, \delta), \\ \dot{y} &= -\lambda_2 y - \alpha\bar{\lambda}_2(\delta)y + xy^2 h_2(x, y) + \alpha x y^2 R_2(x, y, \delta) = yQ(x, y, \alpha, \delta), \end{aligned} \tag{2.2}$$

where h_i *and* R_i *are continuous functions,* $i = 1, 2$, *near the origin.*

Suppose that $T(X)$ is the transformation changing (1.1) into (2.2) in a neighborhood of the origin. We have

$$T(X) = T_0(X) + T_1(X),$$

where T_0 is an invertible linear transformation and T_1 satisfies

$$T_1(0) = 0, \quad DT_1(0) = 0.$$

Let $\mu : \mathbb{R}^2 \to [0, 1]$ be a C^∞-function defined by

$$\mu(X) = \begin{cases} 1, & \|X\| \leq \frac{1}{2}, \\ 0, & \|X\| \geq 1 \end{cases}$$

and

$$T^*(X) = \begin{cases} T_0(X) + T_1(X\mu(X/\varepsilon)), & \|X\| \le \varepsilon, \\ T_0(X), & \|X\| \ge \varepsilon, \end{cases}$$
$$S(X) = T^*(X) - T_0(X).$$

It follows that $\|DS(X)\|$ can be arbitrarily small when ε is sufficiently small, for all $X \in \mathbb{R}^2$. Thus, T^* is a C^3-diffeomorphism in $\mathbb{R}^2$, provided ε is small enough. Therefore, we have

Lemma 2.2. *There exists a coordinate transformation of* $\mathbb{R}^2$ *which is linear outside a neighborhood of the origin such that the system* (1.1) *can be changed into system* $(*)$ *which has the form* (2.2) *near the origin.*

Consequently, in what follows we suppose that the system (1.1) has the property of the system $(*)$.

It is clear that the local stable manifold of the origin is $W^s_{\rm loc} = \{(x,y)|x = 0,\ |y| < \varepsilon\}$. The unstable manifold of the origin is $W^u_{\rm loc} = \{(x,y)||x| < \varepsilon,\ y = 0\}$ which is included in the stable manifold of the origin ultimately for the unperturbed system $(1.1)_{\alpha=0}$. Let $W^s_{\alpha,\delta}(0)$ and $W^u_{\alpha,\delta}(0)$ be respectively the stable and unstable maniflods of the origin for (1.1) when α near zero. It is easy to see that $\varepsilon > 0$ can be chosen sufficiently small so that $x = \varepsilon$ and $y = \varepsilon$ are cross-sections of the vector field (1.1). We can take $\varepsilon = 1$, by scaling $X = \varepsilon X_1$.

Let $d(\alpha,\delta)$ be the separation of the manifolds $W^s_{\alpha,\delta}(0)$, $W^u_{\alpha,\delta}(0)$ on the cross-section $x = 1$. Then from our paper [4] we have

Lemma 2.3.

$$d(\alpha,\delta) = \frac{1}{\|F(1,0)\|}\sum_{k=1}^{n} \alpha^k M_k(\delta) + o(\alpha^{n+1}), \tag{2.3}$$

where $M_k(\delta)$ $(k = 1, 2, \cdots,)$ *satisfy the formula* (1.2).

We define the Poincaré map P: $l \to l$, where $l = \{(x,y)|x = 1,\ 0 < y << 1\}$ is a cross-section as above. Let $A_1 = ye_2 + A_0,\ A_2 = \beta(y,\alpha,\delta)e_2 + A_0,\ y > 0$, where $e_2 = (0,1);\ A_0$ is the point $(1,0)$ if $d(\alpha,\delta) > 0$, and A_0 is the first intersection point of the stable manifold $W^s_{\alpha,\delta}(0)$ with l if $d(\alpha,\delta) < 0$. A_2 is the first intersection point of the orbit passing through A_1 with l. Note that the stable manifold lies inside (outside) of the unstable manifold if $d(\alpha,\delta) < 0\,(> 0)$.

For the Poincaré map $P:\ (1,y) \to (1,\beta(y,\alpha,\delta))$, we have

Lemma 2.4. *If* $\sigma_0 = 0$, *then for* $|\alpha|$ *sufficiently small and any fixed* $\delta \in \mathbb{R}^m$ *we have*

1) $\beta(0,\alpha,\delta) = d(\alpha,\delta)$, *if* $d(\alpha,\delta) \le 0$;

2) $\beta(0,\alpha,\delta) = Kd(\alpha,\delta) + o(\alpha) > 0$, *if* $d(\alpha,\delta) > 0$;

where $K = \|F(1,0)\|/\|F(0,1)\|$.

Proof. 1) It is trivial.

2) Consider the Poincaré map $P : (1,0) \to (1,\beta(0,\alpha,\delta))$, which can be decomposed to

$$\pi_1 :\ (1,0) \to (d',1) \ \text{ and } \ \pi_0 :\ (d',1) \to (1,\beta(0,\alpha,\delta)),$$

where

$$d' = \sum_{k=1}^{n} \alpha^k M_k(\delta)/\|F(0,1)\| + o(\alpha^{n+1}) = Kd(\alpha,\delta).$$

In order to study the map π_0, we consider the flow defined by the linearization of (1.1) about the origin, which is given by

$$x(t) = d' \exp(\lambda_1 + \alpha\bar{\lambda}_1)t,$$
$$y(t) = \exp(-(\lambda_2 + \alpha\bar{\lambda}_2)t). \tag{2.4}$$

The time of flight, T_0, needed from the point $(d', 1)$ to $(1, \beta(0, \alpha, \delta))$, is given by

$$T_0 = -\frac{1}{\lambda_1 + \alpha\bar{\lambda}_1} \ln d'.$$

Thus, the linear part π_0^L of π_0 is defined by

$$\pi_0^L : (d', 1) \to \Big(1, (d')^{\frac{\lambda_2+\alpha\bar{\lambda}_2}{\lambda_1+\alpha\bar{\lambda}_1}}\Big).$$

So,

$$P^L = \pi_0^L \circ \pi_1 : (1, 0) \to \Big(1, (Kd)^{\frac{\lambda_2+\alpha\bar{\lambda}_2}{\lambda_1+\alpha\bar{\lambda}_1}}\Big).$$

According to the continuity of the solutions with respect to the functions on the right hand side and the parameter α for the system (1.1), we can take the map P^L as the approximate of the Poincaré map P. Therefore, we have

$$\beta(0, \alpha, \delta) = (Kd)^{\frac{\lambda_2+\alpha\bar{\lambda}_2}{\lambda_1+\alpha\bar{\lambda}_1}} + o(\alpha) = Kd + o(\alpha),$$

for α near zero, because $\sigma_0 = \lambda_1 - \lambda_2 = 0$. Also, clearly, $\beta(0, \alpha, \delta) > 0$. This prove the lemma.

§3. Proof of the Main Theorem and Corollary

The existence and nonexistence of the limit cycle can be obtained from our paper [4].

We begin on the proof of the uniqueness of the limit cycle. In order to be specific in the following discussion, we assume that $\sigma < 0$, $\alpha^n M_k(\delta_0) > 0$. Let $L_{\alpha,\delta}$ be any limit cycles generated by the separatrix loop L_0 for the system (1.1) when $0 < |\alpha| << 1$, $\|\delta - \delta_0\| << 1$.

$$L_{\alpha,\delta} \to L_0 \quad \text{as} \quad \alpha \to 0, \ \delta \to \delta_0.$$

The characteristic exponent of the limit cycle $L_{\alpha,\delta}$ is

$$\gamma_{\alpha,\delta} = \oint_{L_{\alpha,\delta}} \text{div} F(X) dt + \alpha \oint_{L_{\alpha,\delta}} \text{div} G(X, \delta) dt \equiv I_1 + I_2. \tag{3.1}$$

We claim that

$$\gamma_{\alpha,\delta} \to \sigma \quad \text{as} \quad \alpha \to 0, \quad \delta \to \delta_0. \tag{3.2}$$

If the above claim holds, we yield

$$\gamma_{\alpha,\delta} = \sigma + o(|\alpha| + \|\delta - \delta_0\|) \quad \text{for} \quad 0 < |\alpha| << 1, \ \|\delta - \delta_0\| << 1,$$

so any limit cycles $L_{\alpha,\delta}$ generated by the separatrix loop L_0 must have the same stability as L_0. Hence, the system (1.1) has a unique limit cycle near L_0, which is stable and unstable as $\sigma < 0$ and $\sigma > 0$, respectively.

To prove the above claim, we first prove

Lemma 3.1.

$$I_1 = \oint_{L_{\alpha,\delta}} \mathrm{div}F(X)dt \to \sigma \quad as \quad \alpha \to 0,\ \delta \to \delta_0. \tag{3.3}$$

Proof. As shown in Fig. 1, let

$$l = \{(x, y)|x = 1,\ y > 0\}$$

as before, $l' = \{(x, y)|x > 0,\ y = 1\}$. $A_0 = (1, 0) \in l \cap W^S_{\alpha,\delta}(0)$, $B_0 = (0, 1) \in l' \cap W^S_{\alpha,\delta}(0)$. The points A, A_1, B_1 and C_1 are taken on the limit cycle $L_{\alpha,\delta}$, $A_1 \in l \cap L_{\alpha,\delta}$, $B_1 \in l' \cap L_{\alpha,\delta}$. Without loss of generality, we can assume that $A \to 0$, $A_1 \to A_0$, $B_1 \to B_0$ and $C_1 \to C_0$ as $\alpha \to 0$, $\delta \to \delta_0$.

$$I_1 = \int_{\widehat{A_1C_1}+\widehat{C_1B_1}} + \int_{\widehat{B_1A}} + \int_{\widehat{AA_1}}.$$

Fig. 1

From the normal form (2.2), we have

$$\mathrm{div}F(X) = xyh(x, y), \quad h \in C \tag{3.4}$$

in a neigborhood of the origin. Suppose that the parameter representation of the trajectory segments $\widehat{B_1A}$ and $\widehat{AA_1}$ are, respectively.

$$\widehat{B_1A}:\ x = \varphi(y), \quad y_A \le y \le 1$$

and

$$\widehat{AA_1}:\ y = \psi(x), \quad x_A \le x \le 1.$$

Clearly, $\varphi \to 0$, $\psi \to 0$ as $\alpha \to 0$, $\delta \to \delta_0$.

It follows that

$$\int_{\widehat{B_1A}} = \int_{\widehat{B_1A}} xyh(x, y)dt = \int_{y_A}^{1} -\frac{\varphi h(\varphi, y)}{Q(\varphi, y, \alpha, \delta)}dy \to 0, \quad \text{as} \quad \alpha \to 0,\ \delta \to \delta_0;$$

$$\int_{\widehat{AA_1}} = \int_{\widehat{AA_1}} xyh(x, y)dt = \int_{x_A}^{1} \frac{\psi h(x, \psi)}{P(x, \psi, \alpha, \delta)}dx \to 0, \quad \text{as} \quad \alpha \to 0,\ \delta \to \delta_0.$$

Therefore,

$$I_1 \to \int_{\widehat{A_0C_0}+\widehat{C_0B_0}} = \oint_{L_0} \mathrm{div}F(X)dt = \sigma.$$

Secondly, we have

Lemma 3.2.

$$I_2 = \alpha \oint_{L_{\alpha,\delta}} \mathrm{div}G(X, \delta)dt \to 0 \quad as \quad \alpha \to 0,\ \delta \to \delta_0. \tag{3.5}$$

Proof. From the normal form (2.2), in a neighborhood of the origin $\mathrm{div}G(X, \delta)$ takes the following form:

$$\mathrm{div}G(X, \delta) = \bar\lambda_1 - \bar\lambda_2 + xyR(x, y, \delta), \quad R \in C. \tag{3.6}$$

It is easy to show that

$$-I_2=\alpha\int_{\widehat{A_1B_1}}\operatorname{div}G(X,\delta)dt+\alpha\int_{\widehat{B_1C_1}+\widehat{C_1A_1}}\operatorname{div}G(X,\delta)dt.$$

Clearly, the second integral can be reduced to a definite integral with respect to the variable y, which is a bounded continuous function of all its arguments. Moreover, for the first integral, we have

$$\begin{aligned}&\int_{\widehat{A_1B_1}}\operatorname{div}G(X,\delta)dt\\&=\int_{\widehat{A_1B_1}}[(\bar\lambda_1-\bar\lambda_2)+xyR]dt\\&=\int_{y_{A_1}}^{1}\Big[-\frac{\bar\lambda_1-\bar\lambda_2}{(\lambda_2+\alpha\bar\lambda_2)y}+\frac{\bar\lambda_1-\bar\lambda_2}{yQ(x,y,\alpha,\delta)}+\frac{\bar\lambda_1-\bar\lambda_2}{(\lambda_2+\alpha\bar\lambda_2)y}\Big]dy+\int_{\widehat{A_1B_1}}xyRdt\\&=\frac{\bar\lambda_1-\bar\lambda_2}{\lambda_2+\alpha\bar\lambda_2}\ln y_{A_1}+\int_{Y_{A_1}}^{1}\frac{(\bar\lambda_1-\bar\lambda_2)(\lambda_2+\alpha\bar\lambda_2)+(\bar\lambda_1-\bar\lambda_2)Q}{(\lambda_2+\alpha\bar\lambda_2)yQ}dy+\int_{\widehat{A_1B_1}}xyRdt\\&=\frac{\bar\lambda_1-\bar\lambda_2}{\lambda_2+\alpha\bar\lambda_2}\ln y_{A_1}+\int_{\widehat{A_1B_1}}\frac{\bar\lambda_1-\bar\lambda_2}{\lambda_2+\alpha\bar\lambda_2}[xyh_2+\alpha xyR_2]dt+\int_{\widehat{A_1B_1}}xyRdt.\end{aligned}$$

Set

$$\overline{R}(x,y,\alpha,\delta)=\frac{\bar\lambda_1-\bar\lambda_2}{\lambda_2+\alpha\bar\lambda_2}(h_2+\alpha R_2)+R.$$

It follows that

$$\int_{\widehat{A_1B_1}}\operatorname{div}G(X,\delta)dt=\frac{\bar\lambda_1-\bar\lambda_2}{\lambda_2+\alpha\bar\lambda_2}\ln y_{A_1}+\int_{\widehat{A_1B_1}}xy\overline{R}dt. \tag{3.7}$$

Similarly to the proof of Lemma 3.1, we can show that

$$\int_{\widehat{A_1B_1}}xy\overline{R}dt\to 0,\quad\text{as}\quad\alpha\to0,\ \delta\to\delta_0.$$

Clearly,

$$\frac{\bar\lambda_1(\delta)-\bar\lambda_2(\delta)}{\lambda_2+\alpha\bar\lambda_2(\delta)}\to\frac{\bar\lambda_1(\delta_0)-\bar\lambda_2(\delta_0)}{\lambda_2},\quad\text{as}\quad\alpha\to0\quad\delta\to\delta_0.$$

By Lemma 2.4, we have $y_{A_1}>\beta(0,\alpha,\delta)>0$. Without loss of generality, suppose that $\alpha>0$. Then $\alpha\ln\beta(0,\alpha,\delta)<\alpha\ln y_{A_1}<0$. By Lemmas 2.3 and 2.4 we get

$$\begin{aligned}\alpha\ln\beta(0,\alpha,\delta)&=\alpha\ln\Big[\frac{1}{\|F(0,1)\|}\sum_{k=1}^{n}\alpha^kM_k(\delta)+o(\alpha^{n+1})\Big]\\&\sim\alpha\ln[\alpha^kM_k(\delta)]\to0,\quad\text{as}\quad\alpha\to0,\ \delta\to\delta_0.\end{aligned}$$

So $\alpha\ln y_{A_1}\to0$, as $\alpha\to0,\quad\delta\to\delta_0$.

Hence we have proved that

$$\begin{aligned}-I_2=&\frac{\bar\lambda_1(\delta)-\bar\lambda_2(\delta)}{\lambda_2+\alpha\bar\lambda_2(\delta)}\alpha\ln y_{A_1}+\alpha\int_{\widehat{A_1B_1}}xy\overline{R}dt+\alpha\int_{\widehat{B_1C_1}+\widehat{C_1A_1}}\operatorname{div}G(X,\delta)dt\\&\to0,\quad\text{as}\quad\alpha\to0,\quad\delta\to\delta_0.\end{aligned}$$

The proof of this lemma is finished.

Similar to Theorem 1.1, we have

Theorem 3.1. *In addition to the hypothesis* (A), *suppose that* $\sigma_0 = 0,\ \sigma \neq 0,\ L_0$ *is clockwise and there exists a* $\delta_0 \in \mathbb{R}^m$ *such that* $M_k(\delta_0) = 0,\ k = 1, \cdots, n-1,\ M_n(\delta_0) \neq 0$. *Then for* $0 < |\alpha| << 1,\ \|\delta - \delta_0\| << 1$, *the system* (1.1) *has a unique limit cycle if* $\sigma\alpha^n M_n(\delta) > 0$ *and no limit cycle if* $\sigma\alpha^n M_n(\delta) < 0$ *near* L_0. *And the limit cycle is stable as* $\sigma < 0$ *and unstable as* $\sigma > 0$.

From the proof of Theorem 1.1, we have

Theorem 3.2. *Suppose that, in addition to the hypothesis* (A), $\sigma_0 = 0,\ \sigma \neq 0$, *and there exists a* $\delta_0 \in \mathbb{R}^m$ *such that* $\sigma_\alpha(\delta_0) = \alpha \mathrm{div} G(0, \delta_0) = 0$. *Then at most one limit cycle is generated by the separatrix loop* L_0 *in the system* (1.1) *for* $0 < |\alpha| << 1,\ \|\delta - \delta_0\| << 1$.

Remark. Our results strengthen the relevant theorems in [1-6].

§4. An Example

In this section, we apply the above theorems to the following system:

$$\dot{x} = 2y - \mu(y^2 - x^2 + x^3)(2x - 3x^2),$$
$$\dot{y} = 2x - 3x^2 + \mu(y^2 - x^2 + x^3)(2y) + \alpha y^2,$$

i.e.,

$$\dot{X} = F_1(X) + \mu F_2(X) + \alpha G(X) = F(X, \mu) + \alpha G(X), \tag{4.1}$$

where $X = (x, y)^T \in \mathbb{R}^2,\ \mu$ and α are real parameters.

For $\alpha = 0$, the system (4.1) has a hyperbolic saddle 0 and a separatrix loop L_0 oriented clockwise, given by

$$X_0(t) = (\mathrm{sech}^2 t, -\mathrm{sech}^2 t \mathrm{th} t). \tag{4.2}$$

In [4], we studied the existence of bifurcation to separatrix loop L_0. We determine the uniqueness of the bifurcation near L_0 here.

Clearly, for the system (4.1), $\sigma_0 = 0,\ \sigma_\alpha = 0$. And we can obtain easily

$$\begin{aligned} \sigma &= \int_{-\infty}^{\infty} \mathrm{div} F(X_0(t), \mu) dt \\ &= \int_{-\infty}^{\infty} \mu[(2x - 3x^2)^2 + (2y)^2]|_{X_0(t)} dt \\ &= a\mu, \end{aligned} \tag{4.3}$$

where a is a positive constant. Thus, from Theorem 3.2, we get

Theorem 4.1. *If* $\mu \neq 0$, *the system* (4.1) *has at most one limit cycle generated by separatrix loop* L_0 *for* α *near zero.*

By [4], we have

$$M_1 = 0 + o(\mu),$$
$$M_2 = \int_{-\infty}^{\infty} 4\mathrm{sech}^3 t \cdot \mathrm{th}^2 t \cdot y(t) dt + o(\mu) \equiv b + o(\mu)$$

in which

$$y(t) = \varphi(t) p_2(t) + \psi(t) q_2(t)$$

and

$$\begin{aligned}\varphi(t) &= \frac{21}{32} + \frac{15\pi}{8} - 3\mathrm{ch}t + \frac{5}{2}\mathrm{sh}t\mathrm{ch}t + \frac{105}{32}\mathrm{sech}t - \frac{5}{4}\mathrm{sech}^2 t + \frac{5}{16}\mathrm{sech}^3 t \\ &\quad + \frac{15}{16}t\mathrm{sech}^3 t\mathrm{th}t - \frac{15}{32}t\mathrm{sech}t\mathrm{th}t - \frac{15}{16}\arcsin e^t, \\ \psi(t) &= \mathrm{sech}t \cdot \mathrm{th}^3 t - \mathrm{sech}^3 t \cdot \mathrm{th}t + \arcsin \mathrm{th}t, \\ p_2(t) &= 2\mathrm{sech}^2 t - 3\mathrm{sech}^4 t, \\ q_2(t) &= -\frac{1}{4}\mathrm{sh}t\mathrm{ch}t - \frac{45}{16}\mathrm{sech}^2 t\mathrm{th}t + \frac{15}{8}t\mathrm{sech}^2 t\mathrm{th}^2 t - \frac{15}{16}t\mathrm{sech}^4 t.\end{aligned}$$

Hence,

$$d = \frac{\alpha^2 b}{\|F_1((X_0(0))\|} + o(\mu)\alpha + o(\mu)\alpha^2 + o(\mu\alpha^3). \tag{4.4}$$

Thus, we obtain the following result, which is more precise than Theorem 4.1.

Theorem 4.2. *For* $0 < |\mu| << |\alpha| << 1$, *the system* (4.1) *has a unique limit cycle near* L_0 *if* $\mu b > 0$ *and no limit cycle if* $\mu b < 0$. *Moreover, the limit cycle is stable* (*unstable*) *as* $\mu < 0\,(> 0)$.

References

[1] Andronov, A. A. et al., Theory of bifurcation of dynamic system on a plane, Wiley, New York, 1973.

[2] Ma Zhien & Wang Ernian, The stability of a loop formed the separatrix of a saddle point and the condition to produce a limit cycle, *Chin. Ann. of Math.,* **4A:**1 (1983), 105-110.

[3] Feng Beiye & Qian Min, On the stability of a saddle point separatrix loop and analytical criteron for its bifurcation limit cycle, *Acta Math. Sinica,* **28** (1985), 53-70.

[4] Sun Jianhua, Bifurcation to separatrix loops in supercritical cases, *Chin. Ann. of Math.,* **12A:** 5 (1991), 114-121.

[5] Luo Dingjun, Han Mao'an & Zhu Deming, The uniqueness of a limit cycle produced by the separatrix loop, *Acta Math. Sinica*, **35** (1992), 407-417.

[6] Chow, S. N. & Hale, J. K., Methods of bifurcation theory, Springer-Verlay, New York, 1982.

[7] Joyal, P., The stability of a loop formed the separatrix of a saddle point and the condition to produce a limit cycle, Universite' Mentreal, *SIAM J. Appl. Math.,* **48** (1988).