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CHAOS FOR MIXING TRANSFORMATIONS**

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Abstract

For a class of mixing transformations of a compact metric space it is proved that each chaotic subset is "small" but the possibility for any finite subset to display chaotic behavior is "large".

Keywords Compact metric space, Chaotic subset, Mixing transformation, Topological mixing map.
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§1. Introduction

In recent years, many authors pay their attention to the existence of chaotic subsets (cf. [1–6] and [8–10]). Some of them mentioned the Lebesgue measure of chaotic subsets (cf. [1], [4–6] and [9]). In this paper for a class of mixing transformations of compact metric space, we investigate the possibility for any finite subset to display chaotic behavior instead of the existence of some chaotic subsets. And the invariant measures of chaotic subsets will be disscussed. We will use the following definition drawn in [9].

Definition 1.1. *Suppose that $f : X \to X$ is a map, where X is a topological space. A subset Y of X is said to be chaotic with respect to a given increasing sequence $\{n_i\}$ of positive integers if for any continuous map $g : Y \to X$ there is a subsequence $\{p_i\}$ of the sequence $\{n_i\}$ such that*

$$\lim_{i\to\infty} f^{p_i}(x) = g(x) \quad \textit{for each} \quad x \in Y.$$

Throughout this paper we use X^m to denote the product set of m X's. If X is a metric space or a probability space, then X^m will denote the product metric space or the product probability space respectively, whose definitions are as usual (cf. [7]). We use f_m to denote the product map of m f's.

The following results will be proved.

Theorem 1.1. *Let $f : X \to X$ be a continuous map, where X is a compact metric space containing infinitely many points, let $\{n_i\}$ be an increasing sequence of positive integers and let Y be a chaotic Borel subset of X with respect to the sequence $\{n_i\}$. Then Y has invariant measure zero, i.e., if μ is an invariant measure on the Borel subsets of X, then $\mu(Y) = 0$.*

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Theorem 1.2. *Let f be a continuous map of the compact metric space X into itself. Then*

(1) *f is weakly topologically mixing iff for any $m > 0$ there is a dense G_δ subset Y of X^m such that m coordinates of each point of Y form a chaotic subset of X with respect to the sequence $1, 2, \cdots$.*

(2) *f is topologically mixing iff for any $m > 0$ and any increasing sequence of positive integers there is a dense G_δ subset Y of X^m such that m coordinates of each point of Y form a chaotic subset of X with respect to this sequence.*

Theorem 1.3. *Let f be a continuous map of the compact metric space X into itself and let f has an invariant probability measure on the Borel subsets of X giving non-zero measure to every non-empty open set. Then*

(1) *if f is weak-mixing, then for any $m > 0$, the m coordinates of almost each point of X^m form a chaotic subset of X with respect to the sequence $1, 2, 3, \cdots$;*

(2) *if f is mixing, then for any $m > 0$ and any increasing sequence of positive integers, the m coordinates of almost each point of X^m form a chaotic subset of X with respect to this sequence.*

For a topologically mixing map of a compact metric space containing infinitely many points, Xiong and Yang[9] proved that each chaotic subset must be F_σ, i.e., it could not be very "large" from the topological viewpoint. Theorem 1.1 supports this conclusion from the viewpoint of invariant measure. Theorems 1.2 and 1.3 say that although each chaotic subset is "small", if we take a finite subset at random then the possibility for this subset to display chaotic behavior must be very "large".

We put the proofs of Theorems 1.1, 1.2 and 1.3 in §2, §3 and §4 respectively.

§2. Proof of Theorem 1.1

Under the hypotheses of Theorem 1.1, we first prove two claims.

Claim 2.1. *No periodic points of f are in Y.*

Proof. Assume that Y contains a periodic point p of f. Let $q \in X$ be any point. Define the continuous map $g : Y \to X$ by $g(y) = q$ for each $y \in Y$. Then there exists a subsequence $\{p_i\}$ of the sequence $\{n_i\}$ such that $f^{p_i}(p) \to q$ as $i \to \infty$, which implies that $q \in O_f(p)$ (the orbit of p under f), and consequently, $X \subset O_f(p)$. This contradicts the infiniteness of f.

Claim 2.2. *For any $n > 0$, $Y \cap f^{-n}(Y) = \emptyset$.*

Proof. Assume that for some $n > 0$, $Y \cap f^{-n}(Y) \neq \emptyset$. Then there exists $p \in Y$ such that $q = f^n(p) \in Y$. Define the continuous map $g : Y \to X$ by $g(y) = p$ for each $y \in Y$. Then there exists a subsequence $\{p_i\}$ of the sequence $\{n_i\}$ such that for $i \to \infty$,

$$f^{p_i}(p) \to p \tag{2.1}$$

and

$$f^{p_i}(q) \to p. \tag{2.2}$$

By (2.1), for $i \to \infty$, $f^n f^{p_i}(p) \to f^n(p) = q$, i.e.,

$$f^{p_i}(q) \to q. \tag{2.3}$$

By combining (2.3) with (2.2) we have $p = q$ and hence $p \in Y$ is a periodic point of f. This contradicts Claim 2.1.

Proof of Theorem 1.1. Suppose that Y is a chaotic Borel subset of X. Then by Claim 2.2, $Y, f^{-1}(Y), f^{-2}(Y), \cdots$ are pairwise disjoint. Therefore if μ is an invariant measure of f, then $\mu(Y) = 0$.

§3. Proof of Theorem 1.2

Definition 3.1. *Let f be a continuous self-map of the topological space X and let $\{n_i\}$ be an increasing sequence of positive integers. f is said to be transitive with respect to the sequence $\{n_i\}$ if for any non-empty open sets U and V of X there is an integer $i > 0$ such that $f^{n_i}(U) \cap V \neq \emptyset$. A map transitive with respect to the sequence $1, 2, 3, \cdots$ is briefly said to be transitive.*

Definition 3.2. *Let f be a continuous self-map of the topological space X. f is said to be weakly topologically mixing if $f_2 = f \times f$ is transitive, i.e., for any non-empty open sets U_1, U_2, V_1 and V_2 there is an integer $n > 0$ such that*

$$f^n(U_1) \cap V_1 \neq \emptyset \quad \text{and} \quad f^n(U_2) \cap V_2 \neq \emptyset.$$

f is said to be topologically mixing if for any non-empty open sets U and V of X there is an integer $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for each $n \geq N$.

It is easy to see that each topologically mixing map is weakly topologically mixing and each weakly topologically mixing map is transitive.

Lemma 3.1. *Let f be a continuous self-map of the compact metric space X and let $\{n_i\}$ be an increasing sequence of positive integers. Then the following are equivalent.*

(1) *f is transitive with respect to the sequence $\{n_i\}$.*

(2) *The set of points x with $\{f^{n_i}(x); i \geq 1\}$ dense in X is a dense G_δ.*

Proof. (1)⇒(2). If $\{U_n\}_1^\infty$ is a base for the topology, then

$$\{x; \{f^{n_i}(x)\}_{i=1}^\infty \text{ is dense}\} = \bigcap_{n=1}^\infty \bigcup_{i=1}^\infty f^{-n_i}(U_n).$$

By (1) we know that $\bigcup\limits_{i=1}^\infty f^{-n_i}(U_n)$ is dense. So the result follows.

(2)⇒(1). Suppose U, V are non-empty open sets. By (2), there exists $x \in U$ such that $\{f^{n_i}(x); i \geq 1\}$ is dense in X, which implies $f^{n_i}(U) \cap V \neq \emptyset$ for some $i > 0$. Hence f is transitive with respect to the sequence $\{n_i\}$.

Lemma 3.2. *Let f be a continuous self-map of the topological space X. Then the following are equivalent.*

(1) *f is weakly topologically mixing.*

(2) *f_2 is transitive.*

(3) *For any $m \geq 2$, f_m is transitive.*

(4) *For any $m > 0$, f_m is weakly topologically mixing.*

Proof. (1)$\Rightarrow$(2) is clear.

(4)$\Rightarrow$(1) is trivial.

Noting that $f_{2m} = f_m \times f_m$, by Definition 3.2 we have (3)$\Rightarrow$(4).

It remains to show (2)$\Rightarrow$(3). We have known that f_2 is transitive. Now assume that for some $k \geq 2$, f_k is transitive. We want to prove that f_{k+1} is transitive. Let $U_1, U_2, \cdots, U_k$, U_{k+1}, $V_1, V_2, \cdots, V_k$, and V_{k+1} be non-empty open sets of X. There is $l > 0$ such that $U_{k+1} \cap f^l(U_k) \neq \emptyset$ and $V_{k+1} \cap f^l(V_k) \neq \emptyset$. Set

$$U = U_k \cap f^{-l}(U_{k+1}), \quad V = V_k \cap f^{-l}(V_{k+1}).$$

Then U, V are non-empty open sets. Since f_k is transitive, there is $m > 0$ such that $V_i \cap f^m(U_i) \neq \emptyset$ for each $i = 1, 2, \cdots, k-1$ and $V \cap f^m(U) \neq \emptyset$. Noting that

$$V_k \cap f^m(U_k) \supset V \cap f^m(U) \neq \emptyset$$

and

$$V_{k+1} \cap f^m(U_{k+1}) \supset f^l(V) \cap f^m(f^l(U)) \supset f^l(V \cap f^m(U)) \neq \emptyset,$$

we see that f_{k+1} is transitive. By induction, the result holds.

Lemma 3.3. *Let f be a continuous self-map of the topological space X, $m > 0$ an integer. Then f is topologically mixing iff so is f_m.*

Proof. The sufficiency is clear, we prove the necessity. Suppose f is topologically mixing and let U, V be any members of topological base of X^m. Then there are non-empty open sets $U_1, U_2, \cdots, U_m$, $V_1, V_2, \cdots, V_m$ of X such that

$$U = U_1 \times U_2 \times \cdots \times U_m, \quad V = V_1 \times V_2 \times \cdots \times V_m.$$

For each $i = 1, 2, \cdots, m$, there is $N_i > 0$ such that for any $n \geq N_i$, $f^n(U_i) \cap V_i \neq \emptyset$. Set

$$N = \max\{N_1, N_2, \cdots, N_m\}.$$

Then for any $n \geq N$,

$$\begin{aligned} &(f_m)^n(U) \cap V \\ &= (f^n(U_1) \cap V_1) \times (f^n(U_2) \cap V_2) \times \cdots \times (f^n(U_m) \cap V_m) \neq \emptyset. \end{aligned}$$

This proves that f_m is topologically mixing.

Lemma 3.4. *Let f be a continuous self-map of the topological space X. Then the following are equivalent.*

(1) *f is topologically mixing.*

(2) *For any increasing sequence of positive integers, f is transitive with respect to this sequence.*

Proof. (1)$\Rightarrow$(2). Suppose $\{n_i\}$ is any increasing sequence of positive integers. Suppose U, V are non-empty open sets. Since f is topologically mixing, there exists $N > 0$ such that for each $n \geq N$, $f^n(U) \cap V \neq \emptyset$. Choose i large enough so that $n_i \geq N$. Then $f^{n_i}(U) \cap V \neq \emptyset$. The result then follows.

(2)$\Rightarrow$(1). Assume that f is not topologically mixing. Then there are two non-empty open sets U and V such that $f^{n_i}(U) \cap V = \emptyset$ for some increasing sequence $\{n_i\}$ of positive integers. This is a contradiction.

Proof of Theorem 1.2. Clearly f_m is a continuous map of the compact metric space X^m into itself. By a simple analysis we know that $x_1, x_2, \cdots, x_m$ form a chaotic subset of X with respect to $\{n_i\}$ iff for $x = (x_1, x_2, \cdots, x_m)$, $\{(f_m)^{n_i}(x); i \geq 1\}$ is dense in X^m. So (1) holds by Lemmas 3.1 and 3.2, (2) by Lemmas 3.1, 3.3 and 3.4.

§4. Proof of Theorem 1.3

Throughout this section we use $(X, \mathcal{B}, \mu)$ to denote a probability space, where X is a non-empty set, $\mathcal{B}$ a σ-algebra, μ a measure.

Definition 4.1. *Let $f: X \to X$ be a measure-preserving transformation of the probability space $(X, \mathcal{B}, \mu)$ and let $\{n_i\}$ be an increasing sequence of positive integers. f is said to be ergodic with respect to the sequence $\{n_i\}$ if for any $A, B \in \mathcal{B}$*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu(A \cap f^{-n_i}(B)) = \mu(A)\mu(B).$$

A transformation ergodic with respect to the sequence $1, 2, 3, \cdots$ is briefly said to be ergodic.

Definition 4.2. *Let $f: X \to X$ be a measure-preserving transformation of the probability space $(X, \mathcal{B}, \mu)$. f is said to be weak-mixing if for any $A, B \in \mathcal{B}$*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} |\mu(A \cap f^{-i}(B)) - \mu(A)\mu(B)| = 0.$$

f is said to be mixing if for any $A, B \in \mathcal{B}$

$$\lim_{n\to\infty} \mu(A \cap f^{-n}(B)) = \mu(A)\mu(B).$$

It is easily seen that a mixing transformation is weak-mixing and a weak-mixing transformation is ergodic.

Lemma 4.1. *Let $f: X \to X$ be a measure-preserving transformation of the probability space $(X, \mathcal{B}, \mu)$. Then for any $m > 0$,*

(1) *if f is weak-mixing then so is f_m and therefore f_m is ergodic.*

(2) *if f is mixing then so is f_m.*

For the proof see [7].

Lemma 4.2. *Let $f: X \to X$ be a measure-preserving transformation of the probability space $(X, \mathcal{B}, \mu)$. Then f is mixing iff for any increasing sequence of positive integers f is ergodic with respect to this sequence.*

Proof. Suppose f is mixing. Then for any increasing sequence $\{n_i\}$ of positive integers and any $A, B \in \mathcal{B}$

$$\lim_{i\to\infty} \mu(A \cap f^{-n_i}(B)) = \mu(A)\mu(B).$$

This implies

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu(A \cap f^{-n_i}(B)) = \mu(A)\mu(B),$$

i.e., f is ergodic with respect to the sequence $\{n_i\}$.

We now show the converse. Suppose f is not mixing. Then there exist $A, B \in \mathcal{B}$, an $\epsilon > 0$ and an increasing sequence $\{n_i\}$ of positive integers such that for all $i > 0$

$$|\mu(A \cap f^{-n_i}(B)) - \mu(A)\mu(B)| \geq \epsilon.$$

So either for some subsequence $\{p_i\}$ of the sequence $\{n_i\}$

$$\mu(A \cap f^{-p_i}(B)) \geq \mu(A)\mu(B) + \epsilon,$$

or for some subsequence $\{q_i\}$ of the sequence $\{n_i\}$

$$\mu(A \cap f^{-q_i}(B)) \leq \mu(A)\mu(B) - \epsilon.$$

The former implies

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu(A \cap f^{-p_i}(B)) \geq \mu(A)\mu(B) + \epsilon.$$

The latter implies

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu(A \cap f^{-q_i}(B)) \leq \mu(A)\mu(B) - \epsilon.$$

Whatever happens, there exists an increasing sequence of positive integers such that f fails to be ergodic with respect to this sequence. The sufficiency then follows.

Lemma 4.3. *Let $f : X \to X$ be a measure-preserving transformation of the probability space $(X, \mathcal{B}, \mu)$ and let f be ergodic with respect to the sequence $\{n_i\}$. Then for every $A \in \mathcal{B}$ with $\mu(A) > 0$ we have*

$$\mu\Big(\bigcup_{i=1}^{\infty} f^{-n_i}(A)\Big) = 1.$$

Proof. Suppose $A \in \mathcal{B}$ with $\mu(A) > 0$. Let $\widetilde{A} = \bigcup\limits_{i=1}^{\infty} f^{-n_i}(A)$ and $B = X - \widetilde{A}$. If $\mu(\widetilde{A}) \neq 1$, then $\mu(B) > 0$. Noting that for each $i > 0$, $B \cap f^{-n_i}(A) = \emptyset$, we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \mu(B \cap f^{-n_i}(A)) = 0 \neq \mu(B)\mu(A)$$

since $\mu(B)\mu(A) > 0$. This contradicts the ergodicity of f.

Proof of Theorem 1.3. Since f_m is a continuous map of the compact metric space X^m into itself, it follows that if $\{n_i\}$ is an increasing sequence of positive integers and $\{U_n\}_1^\infty$ a base for the topology of X^m then

$$\{x; \{(f_m)^{n_i}(x)\}_{i=1}^{\infty} \text{ is dense}\} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} (f_m)^{-n_i}(U_n).$$

As stated in the proof of Theorem 1.2, $x_1, x_2, \cdots, x_m$ form a chaotic subset of X with respect to the sequence $\{n_i\}$ iff for $x = (x_1, x_2, \cdots, x_m)$, $\{(f_m)^{n_i}(x); i \geq 1\}$ is dense in X^m. So (1) holds by Lemma 4.1–(1) and Lemma 4.3, (2) by Lemma 4.1–(2), Lemmas 4.2 and 4.3.

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