Chin. Ann. of Math.
15B: 4(1994),507-516.

CLIFFORD MARTINGALES Φ-EQUIVALENCE BETWEEN $S(f)$ AND f^* ***

LONG RULIN* QIAN TAO**

Abstract

Required by the application in the investigation of the Cauchy integral operators on Lipschitz surfaces, the classical martingales are generalized to ones defined with respect to Clifford algebra valued measures. Meanwhile, very general Φ-equivalences between $S(f)$ and f^*, the same as in the classical case, are established too.

Keywords Clifford algebra, Martingale, Square function, Maximal function, Good λ-inequality.

1991 MR Subject Classification 60G46, 60G42.

§1. Introduction

As well known, martingale theory plays a remarkable role in analysis, especially in harmonic analysis: the former was a source from which many ideas and methods in analysis come out. Recently, R. Coifman, P. Jones and S. Semmes[1] gave a simple proof of L^2-boundedness of the Cauchy integral operator on Lipschitz curves by making use of martingale theory that presents a typical example of this role. In [1], the martingales are not the classical ones, but ones with respect to complex measures. Since the Cauchy integral operators on surfaces are defined by Clifford valued integrals, this leads to the investigations of Clifford martingales. This is the motivation of this paper. For the sake of completeness, the results given in this paper are not restricted to the usage in investigating the Cauchy integral operators, but also in some more aspects: the equivalences between $S(f)$ and f^* should be the same as in the classical case, but not only L^2-equivalence.

Let $(\Omega, \mathcal{F}, \nu)$ be a nonnegative σ-finite measure space, ψ be a bounded Clifford valued measurable function. Consider the Clifford valued measure $d\mu = \psi d\nu$. The martingales we now consider are with respect to the measure $d\mu$ and a family $\{\mathcal{F}_n\}_{-\infty}^{+\infty}$ of sub-σ-fields satisfying

$$\begin{aligned} &\{\mathcal{F}_n\}_{-\infty}^{+\infty} \text{ nondecreasing}, \mathcal{F} = \vee \mathcal{F}_n,\ \cap_n \mathcal{F}_n = \text{trivial one}, \\ &(\Omega, \mathcal{F}_n, \nu) \text{ complete},\ \sigma\text{ -finite},\ \forall n. \end{aligned} \tag{1.1}$$

Manuscript received April 6, 1992. Revised May 17, 1993.
*Institute of Mathematics, Academia Sinica, Beijing 100080, China.
**School of Information Science and Technology, The Flinders University of South Australia.
***The first author was supported by the National Natural Science Foundation of China and the second by Australian National Research Fellowship

And the estimates we will do will be in terms of the integrals with respect to ν. Before we proceed, some basic facts of Clifford algebra should be recalled.

Let $e_1, \cdots, e_n$ be the basis vectors of $\mathbb{R}^n$. For each subset A of $\{1, 2, \cdots, n\}$ (ordered increasingly) we correspond a vector e_A to it, with identification $e_{\{i\}} = e_i, i = 1, 2, \cdots, n$. Denote $e_\phi = e_0 = 1$. Then the Clifford algebra $A_{(n)}$ shall be the algebra consisting of the real linear spaces generated by $\{e_A\}$ with the mulitiplication subject to the relations

$$e_i e_j = -e_j e_i, e_i^2 = -1, i, j = 1, \cdots, n.$$

$$e_{A_1} e_{A_2} = (-1)^{n(A_1 \cap A_2)} (-1)^{p(A_1, A_2)} e_{A_1 \triangle A_2}, \tag{1.2}$$

$$\lambda\mu = \sum_{A,B} \lambda_A \mu_B e_A e_B, \quad \text{for} \quad \lambda = \sum_A \lambda_A e_A, \mu = \sum_B \mu_B e_B, \tag{1.3}$$

where $n(A)$ denotes the element number of A, and

$$p(A_1, A_2) = \sum_{j \in A_2} n(\{i \in A_1 : i > j\}),$$

and Δ is the symmetric difference of sets. Obviously $A_{(n)}$ is an associative but not commutative algebra, called Clifford algebra. A conjugate operation is defined in $A_{(n)}$:

$$\overline{e_A} = (-1)^{\frac{n(A)(n(A)+1)}{2}} e_A, \quad A \subset \{1, \cdots, n\}, \tag{1.4}$$

$$\overline{\lambda} = \sum_A \lambda_A \overline{e}_A, \quad \text{for} \quad \lambda = \sum_A \lambda_A e_A. \tag{1.5}$$

For this operation, we have

$$\overline{e_0} = e_0, \quad \overline{e}_i = -e_i, i = 1, \cdots, n,$$

$$\overline{e_A e_B} = \overline{e_A}\ \overline{e}_B, \quad e_A \overline{e_A} = \overline{e_A} e_A = e_0.$$

For the simplicity we will use the following norm in $A_{(n)}$:

$$|\lambda| = \Big(\sum_A \lambda_A^2\Big)^{\frac{1}{2}}, \quad \lambda = \sum_A \lambda_A e_A. \tag{1.6}$$

For it, we have

$$|\lambda\mu| \le k|\lambda||\mu|, \quad \forall \lambda, \mu \in A_{(n)}. \tag{1.7}$$

Here k is a constant depending only on n. When at least one of λ, μ, say λ, is of form $\lambda = \sum_{i=0}^{n} \lambda_i e_i$, (it is just the element of $\mathbb{R}^{n+1}$, usually it is called Clifford number) we have

$$k^{-1}|\lambda||\mu| \le |\lambda\mu|, \quad \text{one of } \lambda, \mu \quad \text{is in} \quad \mathbb{R}^{n+1}. \tag{1.8}$$

In what follows we often use the fact that for $a = \prod_{i=1}^{4} a_i, a_i \in \mathbb{R}^{n+1}$ we have $|a| \approx \prod_{i=1}^{4} |a_i|$.

§2. Clifford Conditional Expectations, Clifford Martingales

We begin with the definition of conditional expectations. $(\Omega, \mathcal{F}, \nu)$ is a σ-finite measure space, $d\mu = \psi d\nu$ is an $\mathbb{R}^{n+1}$ valued measure the domain of which is not $\mathcal{F}$ when $|\Omega|_\nu = \infty$, but a subring of $\mathcal{F}$. This does not bring us any trouble for the definition of conditional expectations. Let $\mathcal{J}$ be a sub-σ-field of $\mathcal{F}$ such that $(\Omega, \mathcal{J}, \nu)$ is σ-finite and complete. Denote the conditional expectations with respect to ν and μ by $\widetilde{E}$ and E respectively. The definition of $\widetilde{E}$ is standard and $\widetilde{E}$ has all the good properties as in the classical case. Assume that ψ is bounded. Then

$$\widetilde{E}(\psi|\mathcal{J}) = \sum_{i=0}^{n} \widetilde{E}(\psi_i|\mathcal{J})e_i, \quad \text{with} \quad \psi = \sum_{i=0}^{n} \psi_i e_i.$$

Furthermore, assume $\widetilde{E}(\psi|\mathcal{J}) \neq 0$, a.e. Then for Clifford valued (or say $A_{(n)}$ valued) functions, define

$$_lE(f|\mathcal{J}) = \widetilde{E}(\psi|\mathcal{J})^{-1}\widetilde{E}(\psi f|\mathcal{J}), \quad \forall A_{(n)} \text{ valued } f \in L^1_{\text{loc}}(\nu), \tag{2.1}$$

$$_rE(f|\mathcal{J}) = \widetilde{E}(f\psi|\mathcal{J})\widetilde{E}(\psi|\mathcal{J})^{-1}, \quad \forall A_{(n)} \text{ valued } f \in L^1_{\text{loc}}(\nu). \tag*{(2.1)$'$}$$

Obviously, E has the following properties (a)—(f).

(a) $_lE$ is right Clifford linear, left and right real linear, and

$$_lE(fg|\mathcal{J}) = {_lE}(f|\mathcal{J})g, \text{ provided } g \text{ is } \mathcal{J}\text{-measurable Clifford valued.}$$

And similarly for $_rE$.

(b) $_lE(1|\mathcal{J}) = 1 = {_rE}(1|\mathcal{J})$.

(c) Both $_lE(f|\mathcal{J})$ and $_rE(f|\mathcal{J})$ are $\mathcal{J}$-measurable, and

$$\int_A {_lE}(f|\mathcal{J})d_l\mu = \int_A f d_l\mu, \quad \forall A \in \mathcal{J}, \forall f \in L^1(A,\nu), \tag{2.2}$$

$$\int_A {_rE}(f|\mathcal{J})d_r\mu = \int_A f d_r\mu, \quad \forall A \in \mathcal{J}, \forall f \in L^1(A,\nu), \tag*{(2.2)$'$}$$

where

$$\int_A f d_l\mu = \int_A \psi f d\nu, \quad \int_A f d_r\mu = \int_A f\psi d\nu. \tag{2.3}$$

(d) When $\mathcal{J}_1 \subset \mathcal{J}_2$, we have (denoting $_lE$ or $_rE$ by E)

$$E(E(f|\mathcal{J}_2)|\mathcal{J}_1) = E(f|\mathcal{J}_1). \tag{2.4}$$

As a result of (2.4), we have ($E = {_lE}$ or $_rE$)

$$E(E(f|\mathcal{J}_2) - E(f|\mathcal{J}_1)|\mathcal{J}_1) = 0. \tag{2.5}$$

Now assume that we have a nondecreasing family $\{\mathcal{F}_n\}_{-\infty}^{\infty}$. In the classical case, as an important result of (2.4), the martingale difference operator $\widetilde{\Delta_n}, \widetilde{\Delta}_n = \widetilde{E}_n - \widetilde{E}_{n-1}, \widetilde{E}_n = \widetilde{E}(\cdot|\mathcal{F}_n)$ are orthorgonal:

$$\widetilde{E}(\widetilde{\Delta}_n f \widetilde{\Delta}_m g|\mathcal{F}_k) = 0, \ n \neq m, \ n, m \geq k, \forall f, g \in L^2. \tag{2.6}$$

In Clifford algebra case, because of the noncommutability, only the following substitution holds. Let $\langle , \rangle$ denote the following "inner-product"

$$\langle f, g \rangle = \int_\Omega f\psi g d\nu, \ \ \forall A_{(n)} \text{ valued nice } f, g. \tag{2.7}$$

Then we have

(e) Let $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ be nondecreasing and $(\Omega, \mathcal{F}_n, \nu)$ be complete and σ-finite, and $\widetilde{E}(\psi|\mathcal{F}_n) \neq 0$, a.e. $\forall n$. With ${}_r\Delta_n, {}_l\Delta_m$ naturally defined, we have

$$\widetilde{E}({}_r\Delta_n f \psi {}_l\Delta_m g|\mathcal{F}_k) = 0, \quad n \neq m, n, m \geq k, \tag{2.8}$$

especially

$$\langle {}_r\Delta_n f, {}_l\Delta_m g\rangle = 0, \forall n \neq m. \tag*{(2.8)$'$}$$

The typical case of ψ is the case where ψ is complex valued and $d\mu$ is absolutely continuous with respect to $d\nu$. In this case, $|\psi| = 1$, a.e. So in this paper, we assume the condition $C_0^{-1} \leq |\psi| \leq C_0$, a.e. Thus we have

(f) Let $1 \leq p \leq \infty$ and $\mathcal{J}$ be any sub-σ-field we consider. Then $E(= {}_lE(\cdot|\mathcal{J})$ or ${}_rE(\cdot|\mathcal{J}))$ is L^p-bounded, if and only if

$$C^{-1}C_0^{-1} \leq |\widetilde{E}(\psi|\mathcal{J})| \leq CC_0, \quad \text{a.e.}$$

Now we turn to the inverstigation of Clifford martingales. Let $(\Omega, \mathcal{F}, \nu)$ be a σ-finite (but not finite) space endowed with a nondecreasing family $\{\mathcal{F}_n\}_{-\infty}^{\infty}$ satisfying (1.1). From the property (f), it is natural to assume

$$C_0^{-1} \leq |\widetilde{E}(\psi|\mathcal{F}_n)| \leq C_0, \quad \text{a.e.} \quad \forall n. \tag{2.9}$$

The definitions of martingales and the related operators are similar to ones in the classical case. Let $f = (f_n)_{-\infty}^{\infty}$ be an $A_{(n)}$ valued process. It is said to be an l- or r-martingale, if for $E = {}_lE$ or ${}_rE$ respectively

$$f_n = E(f_{n+1}|\mathcal{F}_n), \text{ a.e. } \forall n. \tag{2.10}$$

For martingale $f = (f_n)_{-\infty}^{\infty}$ (l-, or r-), the maximal and square functions are defined respectively by

$$f_n^* = \sup_{k \leq n} |f_k|, \quad f^* = f_\infty^*, \tag{2.11}$$

$$S_n(f) = (|f_{-\infty}|^2 + \sum_{-\infty}^{n} |\Delta_k f|^2)^{\frac{1}{2}}, \quad S(f) = S_\infty(f), \tag{2.12}$$

where $f_{-\infty} = \lim_{n \to -\infty} f_n$ pointwise. $f = (f_n)_{-\infty}^{\infty}$ is called L^p-bounded, $1 \leq p \leq \infty$, if

$$\|f\|_p = \sup \|f_n\|_p < \infty. \tag{2.13}$$

All arguments in what follows are the same for $l-$ and $r-$ martingales, so we omit the subscript. We want to show that $*$ is of type (p, p) for $1 < p \leq \infty$, and weak type (1,1). And for $1 < p \leq \infty$, each L^p-bounded martingale $f = (f_n)_{-\infty}^{\infty}$ is generated by some function $f \in L^p(\nu)$, i.e.,

$$f_n = E(f|\mathcal{F}_n), \forall n. \tag{2.14}$$

And for $1 \leq p \leq \infty$, all L^p-bounded martingales have pointwise limits $\lim_{n \to \infty} f_n$ and $\lim_{n \to -\infty} f_n$. We state these as propositions.

Proposition 2.1. *Let $1 < p \leq \infty$. Then $*$ is of type (p,p) and weak type* (1, 1). *And for $1 < p \leq \infty$, each L^p-bounded martingale $f = (f_n)_{-\infty}^{\infty}$ is generated by some $f \in L^p(\nu)$, with $\|f\|_p \approx \sup_n \|f_n\|_p$.*

Proof. Let $f=(f_n)_{-\infty}^{\infty}$ be a martingale, say a left one. Then

$$f_n = E(f_{n+1}|\mathcal{F}_n) = \widetilde{E}(\psi|\mathcal{F}_n)^{-1}\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n), \forall n,$$
$$f_n = E(f_{n+2}|\mathcal{F}_n) = \widetilde{E}(\psi|\mathcal{F}_n)^{-1}\widetilde{E}(\widetilde{E}(\psi f_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n), \forall n,$$

and hence

$$\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n) = \widetilde{E}(\widetilde{E}(\psi f_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n), \forall n,$$

which means that $(\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n))_{-\infty}^{\infty}$ is a martingale (with respect to the measure ν) and it is L^p-bounded, since

$$\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n) = \widetilde{E}(\psi|\mathcal{F}_n) f_n.$$

Now the type (p,p) and weak type (1,1) of $*$ follows from the corresponding assertions in the classical case. Now for $1<p\le\infty$, for any $M>0$, decompose $\Omega=\cup\Omega_k, \Omega_k\in\mathcal{F}_{-M}, |\Omega_k|_\nu<\infty$. Since $\forall k, (\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n)\chi_{\Omega_k})_{n\ge -M}$ are classical (although Clifford algebra valued) L^p-bounded martingales, we can get some $\psi f\in L^p(\Omega_k,\nu)$, such that on Ω_k

$$\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n) = \widetilde{E}(\psi f|\mathcal{F}_n), \quad n\ge -M.$$

Thus

$$\begin{aligned} f_n &= \widetilde{E}(\psi|\mathcal{F}_n)^{-1}\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n) \\ &= \widetilde{E}(\psi|\mathcal{F}_n)^{-1}\widetilde{E}(\psi f|\mathcal{F}_n) \\ &= E(f|\mathcal{F}_n), n\ge -M. \end{aligned}$$

Let $M\to\infty$, then (2.14) follows. In addition, we have obviously $\|f\|_p \approx \sup_n \|f_n\|_p$ (and hence the two meanings of $\|f\|_p$ are essentially the same). The proof of the proposition is finished.

Proposition 2.2. *Let* $1\le p\le\infty, f=(f_n)_{-\infty}^{\infty}$ *be* L^p*-bounded martingale. Then the following hold a.e.*

$$\lim_{n\to\infty} f_n = f, \ \textit{for } 1<p\le\infty, \ \textit{and} \ \lim_{n\to\infty} f_n \ \textit{exists, for } p=1, \tag{2.15}$$

$$\lim_{n\to-\infty} f_n = 0, \ \textit{for } 1\le p<\infty. \tag{2.15$'$}$$

Proof. Let $\Omega=\cup\Omega_k, \Omega_k\in\mathcal{F}_0, |\Omega_k|<\infty$, for all k. Then both of $(\widetilde{E}(\psi|\mathcal{F}_n)\chi_{\Omega_k})_{n\ge 0}$ and $(\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n)\chi_{\Omega_k})_{n\ge 0}$ are L^p-bounded martingales with respect to

$$(\Omega_k, \mathcal{F}\cap\Omega_k, \nu|_{\Omega_k}, \{\mathcal{F}_n\cap\Omega_k\}_{n\ge 0}),$$

and have their limits respectively

$$\lim_{n\to\infty}\widetilde{E}(\psi|\mathcal{F}_n) = \psi, \quad \text{a.e. on each } \Omega_k,$$
$$\lim_{n\to\infty}\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n) = \psi g, \quad \text{a.e. on each } \Omega_k,$$

where $g=f$, when $1<p\le\infty$. Thus

$$\begin{aligned} \lim_{n\to\infty} f_n &= \lim_{n\to\infty}\widetilde{E}(\psi|\mathcal{F}_n)^{-1}\widetilde{E}(\psi f_{n+1}|\mathcal{F}_n) \\ &= \begin{cases} g, & \text{for } \ p=1, \\ f, & \text{for } \ 1<p\le\infty. \end{cases} \end{aligned}$$

Now prove (2.15)'. Denote $\theta(\omega) = \overline{\lim}_{n\to-\infty}|f_n|$. Then $\theta(\omega) \le f^*(\omega)$, and $\theta(\omega)$ is $\cap\mathcal{F}_n$ measurable, and hence $\theta(\omega) = a \ge 0$, a.e. By the weak type (p,p) of $*$, for $1 \le p < \infty$, we have

$$|\{\theta(\omega) > \lambda\}|_\nu \le |\{f^* > \lambda\}|_\nu \le \Big(\frac{C}{\lambda}\|f\|_p\Big)^p, \forall\lambda > 0.$$

So, $a = 0$. This proves the assertion (2.15)'. The proposition is proved.

Remark. In the classical case, for $1 < p < \infty$, the assertion $\lim_{n\to-\infty} f_n = 0$, a.e, was proved in Edward-Gaudry[3].

§3. L^2-Equivalence Between f and $S(f)$ for Clifford Martingales

Now we have $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{-\infty}^{\infty})$ as our underlying space, ψ an $I\!R^{n+1}$ valued measurable function satisfying

$$C_0^{-1} \le |\widetilde{E}(\psi|\mathcal{F}_n)| \le C_0, \quad \text{a.e.} \quad \forall n, \tag{3.1}$$

and $f = (f_n)_{-\infty}^{\infty}$ an $A_{(n)}$ valued martingale with respect to $d\mu = \psi d\nu$ (we call such a martingale as Clifford martingale in what follows). In this section, we want to establish

$$\|f\|_{L^2(\nu)} \approx \|S(f)\|_{L^2(\nu)}, \quad \forall f = (f_n)_{-\infty}^{\infty}.$$

At first, we do the argument on the underlying space $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{n\ge0})$. We want to show that all the related inequalities and the assertions only depend on the C_0 in (3.1), but not on $\{\mathcal{F}_n\}_{n\ge0}$, neither the martingales under consideration. Once it is shown to be the case, then for $\{\mathcal{F}_n\}_{n\ge-M}$, we have the same inequalities and assertions with the involved coefficients independent of M. Then a limint argument goes to the case $\{\mathcal{F}_n\}_{-\infty}^{\infty}$.

The following is the conditioned L^2-equivalence between $S(f)$ and f.

Theorem 3.1. *Let $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{n\ge0})$ and an $I\!R^{n+1}$ valued ψ be as above, and $f = (f_n)_{n\ge0}$ be a Clifford l- or r-martingale. Then we have*

$$\begin{aligned} &C\widetilde{E}(S(f)^2|\mathcal{F}_0)\\ \le\ &\widetilde{E}(|f|^2|\mathcal{F}_0)\\ \le\ &C\widetilde{E}(S(f)^2|\mathcal{F}_0), \end{aligned} \tag{3.2}$$

with C depending only on C_0 (and the dimension n of $A_{(n)}$, but we ignore this dependence in what follows).

Proof. Consider the l-martingale case. Let $f = (f_n)_{n\ge0}$ be L^2-bounded. We have

$$\begin{aligned} \Delta_n f = &(\widetilde{E}(\psi|\mathcal{F}_n)^{-1} - \widetilde{E}(\psi|\mathcal{F}_{n-1})^{-1})\widetilde{E}(\psi f|\mathcal{F}_{n-1})\\ &+ \widetilde{E}(\psi|\mathcal{F}_n)^{-1}(\widetilde{E}(\psi f|\mathcal{F}_n) - \widetilde{E}(\psi f|\mathcal{F}_{n-1})), \end{aligned}$$

$$|\Delta_n f|^2 \le C|\widetilde{\Delta}_n(\psi f)|^2 + C|\widetilde{E}(\psi f|\mathcal{F}_{n-1})|^2|\widetilde{\Delta}_n(\psi)|^2, \quad \forall n \ge 0. \tag{3.3}$$

Here the facts

$$\begin{aligned} a^{-1} - b^{-1} &= a^{-1}(b-a)b^{-1}\\ &= b^{-1}(b-a)a^{-1},\\ |a_1a_2a_3a_4| &\le k^3|a_1||a_2||a_3||a_4| \end{aligned}$$

have been used. So,

$$
\begin{aligned}
&\widetilde{E}\Big(\sum_{n=0}^{\infty}|\Delta_n f|^2|\mathcal{F}_0\Big)\\
&\le C\widetilde{E}\Big(\sum_{n=0}^{\infty}|\widetilde{\Delta}_n(\psi f)|^2|\mathcal{F}_0\Big)+C\widetilde{E}\Big(\sum_{n=1}^{\infty}|\widetilde{E}(\psi f|\mathcal{F}_{n-1})|^2|\widetilde{\Delta}_n(\psi)|^2|\mathcal{F}_0\Big)\\
&\le C\widetilde{E}(|f|^2|\mathcal{F}_0)+CJ,\\
J&=\widetilde{E}\Big(\sum_{n=1}^{\infty}\widetilde{E}^*_{n-1}(\psi f)^2\Big(\sum_{k=n}^{\infty}|\widetilde{\Delta}_k\psi|^2-\sum_{k=n+1}^{\infty}|\widetilde{\Delta}_k\psi|^2\Big)|\mathcal{F}_0\Big)\\
&=\widetilde{E}\Big(\sum_{n=1}^{\infty}\widetilde{E}\Big(\sum_{k=n}^{\infty}|\widetilde{\Delta}_k\psi|^2|\mathcal{F}_n\Big)(\widetilde{E}^*_{n-1}(\psi f)^2-\widetilde{E}^*_{n-2}(\psi f)^2)|\mathcal{F}_0\Big)\\
&\le C\|\psi\|_\infty^2\widetilde{E}\Big(\sum_{n=1}^{\infty}(\widetilde{E}^*_{n-1}(\psi f)^2-\widetilde{E}^*_{n-2}(\psi f)^2)|\mathcal{F}_0\Big)\\
&\le C\|\psi\|_\infty^2\widetilde{E}(|f|^2|\mathcal{F}_0).
\end{aligned}
$$

Thus, we have proved that for l-, or r-martingale f,

$$\widetilde{E}(S(f)^2|\mathcal{F}_0)\le C\widetilde{E}(|f|^2|\mathcal{F}_0), \tag{3.4}$$

with desired C. Now prove the reciprocal of (3.4). We have

$$
\begin{aligned}
\widetilde{E}(|f|^2|\mathcal{F}_0)^{\frac12}&\le C\widetilde{E}(|\psi f|^2|\mathcal{F}_0)^{\frac12}\\
&=C\sup_{g:\widetilde{E}(|g|^2|\mathcal{F}_0)\le 1}|\widetilde{E}(g\psi f|\mathcal{F}_0)|\\
&=C\sup_g|\widetilde{E}(\sum_0^{\infty}{}_r\Delta_n g\psi_l\Delta_n f|\mathcal{F}_0)|\\
&\le C\sup_g\widetilde{E}({}_rS(g)^2|\mathcal{F}_0)^{\frac12}\widetilde{E}({}_lS(f)^2|\mathcal{F}_0)^{\frac12}\\
&\le C\widetilde{E}({}_lS(f)^2|\mathcal{F}_0)^{\frac12},
\end{aligned}
$$

where the orthogonality (2.8) and the inequality (3.4) have been used. The theorem is proved.

Remark. [1] obtained the result in the case $A_{(2)}=\mathbb{C}$.

§4. Φ-Equivalence Between $S(f)$ and f^*

Let Φ be a function from $\mathbb{R}^+$ to $\mathbb{R}^+$, which is nondecreasing, continuous, and of moderate growth in the sense

$$\Phi(2u)\le C_1\Phi(u),\quad \forall u\ge 0, \tag{4.1}$$

and $\Phi(0)=0$. In what follows, we will call such Φ general ones. At first, we want to establish a general Φ-inequality between $S(f)$ and f^* for those martingales f which are predictably dominated in the sense

$$|\Delta_n f|\le D_{n-1},\quad \forall n, \tag{4.2}$$

where $D = (D_n)$ is a nonnegative nondecreasing and adapted process. Still, we need only to consider the case $\{\mathcal{F}_n\}_{n\geq 0}$. (In this case for any process $\lambda = (\lambda_n)_{n\geq 0}$, we define $\lambda_{-1} = 0$, so any f which fits (4.2) must satisfy $f_0 = 0$. This is not an essential restriction, of course).

Theorem 4.1. *Let $f = (f_n)_{n\geq 0}$ be an l- or r-martingale which satisfies (4.2), Φ be general one. Then*

$$\int_\Omega \Phi(S(f))d\nu \leq C \int_\Omega \Phi(f^* + D_\infty)d\nu, \tag{4.3}$$

$$\int_\Omega \Phi(f^*)d\nu \leq C \int_\Omega \Phi(S(f)+D_\infty)d\nu, \tag{4.3$'$}$$

with the implied constants depending only on C_0, C_1.

Proof. We will use the stopping time argument and the good λ-inequality method as in [4]. Let α be an arbitrary real number that is bigger than 1 and $\beta > 0$ to be determined later and $\lambda > 0$ be any level. Notice that

$$|f_n| \leq |f_{n-1}| + |\Delta_n f| \leq f^*_{n-1} + D_{n-1} = \rho_{n-1}, \quad \forall n \geq 0.$$

Define a stopping time by

$$\tau = \inf\{n : \rho_n > \beta\lambda\},$$

and the associated stopped martingale

$$f^{(\tau)} = (f^{(\tau)}_n)_{n\geq 0} = (f_{n\wedge\tau})_{n\geq 0}.$$

Then we have

$$\{\tau < \infty\} = \{\rho_\infty > \beta\lambda\}, \quad f^{(\tau)*} = \sup_n |f_{n\wedge\tau}| \leq f^*_\tau \leq \rho_{\tau-1} \leq \beta\lambda.$$

Now consider the adapted process $(S_n(f^{(\tau)}))_{n\geq 0}$, and define another stopping time

$$T = \inf\{n : S_n(f^{(\tau)}) > \lambda\}.$$

Then we have

$$\{T < \infty\} = \{S(f^{(\tau)}) > \lambda\}, \quad S_{T-1}(f^{(\tau)}) \leq \lambda.$$

Thus, we have

$$\{S(f) > \alpha\lambda\} \subset \{\tau < \infty\} \cup \{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2 - 1)\lambda^2\}, \tag{4.4}$$

and

$$\begin{aligned} &\widetilde{E}(\chi_{\{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2-1)\lambda^2\}} | \mathcal{F}_T) \\ \leq\ & \frac{1}{(\alpha^2-1)\lambda^2} \widetilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 | \mathcal{F}_T). \end{aligned} \tag{4.5}$$

Now consider a new underlying space $(\Omega, \mathcal{F}, \nu, \{\mathcal{J}_n\}_{n\geq 0})$ with $\mathcal{J}_n = \mathcal{F}_{T+n}$, and a new Clifford martingale

$$g = (g_n)_{n\geq 0} \quad \text{with} \quad g_n = f^{(\tau)}_{T+n} - f^{(\tau)}_{T-1}.$$

Then we have

$$\Delta_n g = f^{(\tau)}_{T+n} - f^{(\tau)}_{T-1} - (f^{(\tau)}_{T+n-1} - f^{(\tau)}_{T-1}) = \Delta_{T+n} f^{(\tau)}, \quad \forall n \geq 0,$$

and

$$S(g)^2 = \sum_{n=0}^{\infty} |\Delta_n g|^2 = \sum_{n=0}^{\infty} |\Delta_{T+n} f^{(\tau)}|^2$$
$$= \sum_{k=T}^{\infty} |\Delta_k f^{(\tau)}|^2 = S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2.$$

By making use of (3.4), we get

$$\widetilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 | \mathcal{F}_T)$$
$$= \widetilde{E}(S(g)^2 | \mathcal{J}_0) \le C\widetilde{E}(|g|^2 | \mathcal{J}_0)$$
$$= C\widetilde{E}(|f^{(\tau)} - f^{(\tau)}_{T-1}|^2 | \mathcal{F}_T) \le C\beta^2\lambda^2.$$

Now, since $\{S(f^{(\tau)}) > \alpha\lambda\} \subset \{T < \infty\}$, we have

$$|\{S(f^{(\tau)}) > \alpha\lambda\}|_\nu \le \int_{\{T<\infty\}} \widetilde{E}(\chi_{\{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2-1)\lambda^2\}} | \mathcal{F}_T) d\nu$$
$$\le \frac{C\beta^2}{\alpha^2-1} |\{S(f^{(\tau)}) > \lambda\}|_\nu$$
$$\le \frac{C\beta^2}{\alpha^2-1} |\{S(f) > \lambda\}|_\nu,$$

and hence

$$|\{S(f) > \alpha\lambda\}|_\nu \le |\{\rho_\infty > \beta\lambda\}|_\nu + C\frac{\beta^2}{\alpha^2-1} |\{S(f) > \lambda\}|_\nu,$$

which is the desired good λ-inequality of the couple $(S(f), f^* + D_\infty)$. The one for $(f^*, S(f) + D_\infty)$ is similar. From them, we get (4.3) and (4.3)'. The theorem is proved.

Now we want to get rid of D_∞ in (4.3) and (4.3)' in the following two cases: one is the case when Φ is convex, the other is the case when $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{-\infty}^{\infty})$ is regular in some sense. For the simplicity, we consider only the simplest regularity, i.e., the dyadic type one: each $\mathcal{F}_n$ is atomic with its atom $I^{(n)} = I_1^{(n+1)} \cup I_2^{(n+1)}$ satisfying $||I_1^{(n+1)}|_\mu| \approx ||I_2^{(n+1)}|_\mu| \approx ||I^{(n)}|_\mu|$, $\forall n$, although a little more generality as in Long[4] will also do. We have

Theorem 4.2. *Let $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_n\}_{-\infty}^{\infty})$ and $d\mu = \psi d\nu$ be as above, and $\Phi(u)$ be moderately convex. Then we have*

$$\int_\Omega \Phi(S(f)) d\nu \approx \int_\Omega \Phi(f^*) d\nu, \quad \forall \text{ Clifford martingales} \quad f. \tag{4.6}$$

For the dyadic type case, we have, for general Φ,

$$\int_\Omega \Phi(S(f)) d\nu \approx \int_\Omega \Phi(f^*) d\nu, \quad \forall \text{ Clifford martingales} \quad f. \tag{4.7}$$

Here all the equivalence constants depend only on C_0, C_1.

Proof. Only consider the dyadic type case. We claim that in such case (4.2) is always true for every mantingale $f = (f_n)_{-\infty}^{\infty}$ (with some suitably defined $D = (D_n)_{-\infty}^{\infty}$). In fact, for any $f = (f_n)_{-\infty}^{\infty}$,

$$D_{n-1}|_{I^{(n-1)}} = \sup_{k \le n} \max(|\Delta_k f||_{I_1^{(k)}}, |\Delta_k f|_{I_2^{(k)}}) \tag{4.8}$$

is a nonnegative, nondecreasing, and adapted process s.t.

$$|\Delta_n f| \le D_{n-1},$$

and

$$D_\infty \le C\min(f^*, S(f)).$$

Only the last inequality should be verified. This is because of

$$\begin{aligned}\int_{I^{(k-1)}} \Delta_k f d\mu = 0 &\Longrightarrow \int_{I_1^{(k)}} \Delta_k f d\mu = -\int_{I_2^{(k)}} \Delta_k f d\mu \\ &\Longrightarrow \Delta_k f|_{I_1^{(k)}} |I_1^{(k)}|_\mu = -\Delta_k f|_{I_2^{(k)}} |I_2^{(k)}|_\mu \\ &\Longrightarrow \frac{|\Delta_k f||_{I_1^{(k)}}}{|\Delta_k f||_{I_2^{(k)}}} = \frac{||I_2^{(k)}|_\mu|}{||I_1^k|_\mu|},\end{aligned}$$

and hence, on $I^{(k-1)}$

$$\max(|\Delta_k f||_{I_1^{(k)}}, |\Delta_k f||_{I_2^{(k)}}) \le C|\Delta_k f|,$$

so,

$$D_\infty \le C\sup_k |\Delta_k f| \le C\min(S(f), f^*).$$

Thus, the theorem has been proved.

Remark. In the case of $A_{(2)}$, the result for $\Phi(u) = u^p, 1 < p < \infty$, has been obtained in Cowling-Gaudry-Qian[2].

References

[1] Coifman, R., Jones, P. & Semmes, S., Two elementary proofs of the L^2-boundedness of Cauchy integrals on Lipschitz curves, *Journal of A. M. S.,* **2:**3 (1989), 553–564.

[2] Cowling, M., Gaudry, G. & Qian, T., A note on martingale with respect to complex measures, *Proceeding of the Centre for Mathematical Analysis, Australian National University,* **24** (1989), 10–27.

[3] Edwards, R. E. & Gaudry, G. I., Littlewood-Paley and Multiplier Theory, Springer-Verlag, 1977.

[4] Long, R., Martingales régulières et Φ-inégalitès ovec poids entre $f^*, S(f)$ et $\sigma(f)$, *C. R. Acad. Sc. Paris,* **291**(1980), 31–34.