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A COUNTER-EXAMPLE CONCERNING THE ANALYTIC RADON-NIKODYM PROPERTY**

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Abstract

It is shown that there exists a J-convex subset C of a complex Hilbert space X, such that the J-convex hull of the set of all Jensen boundary points of C is different from C.

Keywords Analytic Radon-Nikodym property, J-convex subset, Jensen measure, Jensen boundary point, Banach spaces.
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In the last ten years, several remarkable results have been established in the geometrical theory of complex Banach spaces. In [1], A.V.Bukhvalov and A.A.Danilevich have introduced the analytic Radon-Nikodym property (see the definition below) in complex Banach spaces as the analytic analogue of the well known Radon-Nikodym property concerning the geometrical structure of real Banach spaces. Let X be a complex Banach space; X is said to have the analytic Radon-Nikodym property (see [1]) if, for every uniformly bounded analytic function from the open unit disk with values in X, $f\colon \mathbf{D} \to X$, f has radial limits a.e. on the torus $\mathbf{T}$ in X, this means that for almost all $\theta \in \mathbf{T}$, $\lim\limits_{r\uparrow 1} f(re^{\theta})$ exists. It is known that every Banach space with the Radon-Nikodym property has the analytic Radon-Nikodym property, and the Lebesgue-Bochner integrable functions space L^1 has the analytic Radon-Nikodym property (see [2]), as well as the predual of Von Neumann algebra (see [3]) and the predual of James tree space J_*T (see [4]). Let us first recall some basic notions on the geometrical structure of complex Banach spaces (see [2] for more detailed discussions).

Let X be a complex Banach space and let f be a real function on X. f is plurisubharmonic if f is upper semi-continuous and if for every $x, y \in X$,

$$f(x) \le \int_0^{2\pi} f(x + ye^{i\theta})\frac{d\theta}{2\pi}.$$

Let μ be a Borel probability measure on X and $x_0 \in X$. μ is a Jensen measure on X with barycenter x_0, if for every plurisubharmonic function ϕ on X we have

$$\phi(x_0) \le \int_X \phi(x)d\mu(x).$$

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It is easy to see that for every X-valued polynomial

$$P(z)=\sum_{i=0}^{N}a_iz^i,\quad a_i\in X,\quad z\in\mathbf{C},$$

the image measure of normalized Lebesgue measure on the torus $\mathbf{T}=\{e^{i\theta}:\ \theta\in[0,\ 2\pi]\}$ by P is a Jensen measure on X with barycenter $P(0)=a_0$. In particular, the Dirac measure δ_x is a Jensen measure on X with barycenter x for every $x\in X$. Another obvious fact about Jensen measures is the following: if f is a continuous linear functional on X and μ is a Jensen measure on X with barycenter x_0, then the image measure of μ by f is a Jensen measure on $\mathbf{C}$ with barycenter $f(x_0)$. We shall use frequently this fact in this paper.

Let X be a complex Banach space, C a closed bounded subset of X and $x_0\in C$. x_0 is a Jensen boundary point of C, if the Dirac measure δ_{x_0} is the only Jensen measure on X supported on C with barycenter x_0. It is not hard to verify that for every closed bounded subset C, every strongly PSH-exposed point of C is a Jensen boundary point of C, so the set of all Jensen boundary points of a nonempty closed bounded subset of a complex Banach space with the analytic Radon-Nikodym property is not empty. A closed bounded subset C of X is Jensen convex (J-convex, in short), if the barycenter of any Jensen measure on X supported on C belongs to C. If D is a bounded subset of X, the J-convex hull of D in X is defined as the smallest J-convex subset of X containing D.

As every upper semi-continuous convex function is plurisubharmonic, every closed bounded convex subset is J-convex; it is known that every PSH-convex subset is J-convex (see [4]). In [5], we have shown that every closed bounded denumerable subset of a complex Banach space is J-convex and every point of such subset is a Jensen boundary point. We also use this fact frequently in this paper.

It is known that a Banach space X has the Radon-Nikodym property if and only if for every closed bounded convex subset C of X, C is the closed convex hull of its strongly linear exposed points (see [6]). The analoguous result in the analytic setting has been obtained in [4]: A complex Banach space X has the analytic Radon-Nikodym property if and only if for every PSH-convex subset C of X, C is the PSH-convex hull of its strongly PSH-exposed points. It is natural to ask whether this remains true in the J-convex case, i.e., whether the analytic Radon-Nikodym property is equivalent to the following property: every J-convex sebset is the J-convex hull of its strongly PSH-exposed points (or more generally, its Jensen boundary points). The aim of this paper is to give a negative answer to this question. We shall construct a J-convex subset of $l^2(I)$ (I is an index set), so that the subset $Jr(C)$ consisting of all Jensen boundary points of C is not empty and the J-convex hull of $Jr(C)$ is different from C. We shall use an argument used in [7], where in [7] we have constructed a J-convex subset C of $l^1(I)$ for some index set I, so that $0\in C$ is not the barycenter of any Jensen measure on $l^1(I)$ supported on the set of all Jensen boundary points of C.

Let ω be an abstract element and for each $n\in\mathbf{N}$ let $\mathbf{T}_n$ be a copy of $\mathbf{T}$. For different values of $n,\ m\in\mathbf{N}$, the elements in $\mathbf{T}_n$ and the elements in $\mathbf{T}_m$ will be considered different.

Let $I = \{\omega\} \cup \left(\bigcup_{n=1}^{\infty} \mathbf{T}_n\right)$ and let $X = l^2(I)$ be the complex Hilbert space

$$l^2(I) = \left\{f: I \to \mathbf{C}: \sum_{i \in I} |f(i)|^2 < +\infty\right\}$$

with the norm

$$\|f\| = \left(\sum_{i \in I} |f(i)|^2\right)^{1/2};$$

$l^2(I)$ thus defined has the Radon-Nikodym property (see [6]). Hence $l^2(I)$ has also the analytic Radon-Nikodym property. For every element $f \in l^2(I)$, the support of f is defined as the subset $\{i \in I: f(i) \neq 0\}$ of I. Let

$$C_\omega = \left\{f \in l^2(I): |f(\omega)| \le 1 \text{ and for every } \theta \in \bigcup_{n=1}^{\infty} \mathbf{T}_n, \ f(\theta) = 0\right\},$$

and let

$$C_n = \Big\{f \in l^2(I): \text{ the support of } f \text{ is contained in } \{\omega, \alpha_1, \alpha_2, \cdots, \alpha_n\},$$
$$\alpha_i \in \mathbf{T}_i \text{ for every } i = 1, 2, \cdots, n, \text{ and}$$
$$f(\omega) = e^{i\alpha_1}, \ f(\alpha_1) = \frac{1}{2} e^{i\alpha_2}, \cdots, \ f(\alpha_{n-1}) = \frac{1}{2^{n-1}} e^{i\alpha_n}, \ |f(\alpha_n)| \le \frac{1}{2^n}\Big\}$$

for $n \in \mathbf{N}$. Let

$$\widetilde{C} = C_\omega \cup \left(\bigcup_{n=1}^{\infty} C_n\right)$$

and let

$$C' = \Big\{f \in l^2(I): \text{ the support of } f \text{ is contained in}$$
$$\{\omega, \alpha_1, \alpha_2, \cdots\}, \ \alpha_n \in \mathbf{T}_n, \ f(\omega) = e^{i\alpha_1}$$
$$\text{and for each } n \in \mathbf{N}, \ f(\alpha_n) = \frac{1}{2^n} e^{i\alpha_{n+1}}\Big\}.$$

We shall work with the subset of $l^2(I)$,

$$C = C' \cup \widetilde{C}.$$

From the definition, it is not hard to see that the subset C of $l^2(I)$ has the following elementary properties:

1. For each $n \in \mathbf{N}$ and for each $f \in C$, there exists at most one element $\alpha \in \mathbf{T}_n$ so that $f(\alpha) \neq 0$.

2. If $\alpha \in \mathbf{T}_n$, $\alpha_1 \in \mathbf{T}_{n+1}$, $f \in C$ and $f(\alpha) = \frac{1}{2^n} e^{i\alpha_1}$ (so there exist no other elements $\beta \in \mathbf{T}_n$, $f(\beta) \neq 0$), then the only possible element β in $\mathbf{T}_{n+1}$ such that $f(\beta) \neq 0$ is α_1.

3. If $\alpha \in \mathbf{T}_{n+1}, f \in C$ and $f(\alpha) \neq 0$, then there exists a unique element $\beta \in \mathbf{T}_n$ such that $f(\beta) \neq 0$; in this case, we have $f(\beta) = \frac{1}{2^n} e^{i\alpha}$.

4. If for each $\alpha \in \mathbf{T}_n, |f(\alpha)| < \frac{1}{2^n}$, then for each $\theta \in \bigcup_{k>n} \mathbf{T}_k$ we have $f(\theta) = 0$.

The main result in this paper is the following

Theorem. *C is a J-convex subset of $l^2(I)$, $Jr(C)$ is different from C and the J-convex hull of $Jr(C)$ is contained in C'.*

We shall divide the proof of the theorem above into three steps.

Lemma 1. *C is closed and bounded in $l^2(I)$.*

Proof. The boundeness of C is trivial since each element in C has a norm less or equal to 2. Let $f_n \in C$ be a converging sequence, $f = \lim_{n\to\infty} f_n$, and suppose that $f \notin C$.

If there exists a subsequence f_{n_k} of f_n so that $f_{n_k} \in C_\omega$ for every $k \in \mathbf{N}$, the limit f of f_{n_k} belongs also to C_ω since C_ω is closed in $l^2(I)$. Without loss of generality, we can suppose that the sequence f_n belongs to $C\backslash C_\omega$, and there exists, for each $n \in \mathbf{N}, \alpha_n \in \mathbf{T}_1$ so that

$$f_n(\omega) = e^{i\alpha_n},\ f_n(\alpha_n) \neq 0$$

and for every $\theta \in \mathbf{T}_1, \theta \neq \alpha_n,\ f_n(\theta) = 0$. As $|f_n(\omega)| = 1$ for $n \in \mathbf{N}$, and $f = \lim_{n\to\infty} f_n$, one can find $\beta_1 \in \mathbf{T}_1$ such that $f(\omega) = e^{i\beta_1}$.

If there exists a subsequence f_{m_k} such that for each $k \in \mathbf{N}, \alpha_{m_k} \neq \beta_1$, and for $k \neq h$, $\alpha_{m_k} \neq \alpha_{m_h}$, then for each $\theta \in \mathbf{T}_1$, $\theta \neq \beta_1$, as α_{m_k} converges to β_1, we must have $\alpha_{m_k} \neq \theta$ when k is big enough. So $f_{m_k}(\theta) = 0$ when k is big enough. This implies that $f(\theta) = 0$. On the other hand, $\alpha_{m_k} \neq \beta_1$. Hence $f_{m_k}(\beta_1) = 0$ for every $k \in \mathbf{N}$. We can deduce that $f(\beta_1) = 0$. Hence for every $\theta \in \mathbf{T}_1, f(\theta) = 0$, there exists $N \in \mathbf{N}$ such that for every $k \geq N$ we have for each $\theta \in \mathbf{T}_1$,

$$|f_{m_k}(\theta)| = |f(\theta) - f_{m_k}(\theta)| \leq \|f - f_{m_k}\| < 1/4.$$

This means that $f_{m_k} \in C_1$ when $k \geq N$ by the fourth property disccused just before the theorem. We have for each $\theta \in \bigcup_{k\geq 2} \mathbf{T}_k$ and $k \geq N, f(\theta) = 0$, i.e., the only $\beta \in I$ such that $f(\beta) \neq 0$ is ω. This implies that $f \in C_\omega \subset C$.

Without loss of generality, we can suppose that the sequence α_n is a constant sequence, so $\alpha_n = \beta_1$ for every $n \in \mathbf{N}$. For each $\theta \in \mathbf{T}_1,\ \theta \neq \beta_1,\ n \in \mathbf{N}$, we have $f_n(\theta) = 0,\ f_n(\omega) = e^{i\beta_1}$, and hence $f(\theta) = 0,\ f(\omega) = e^{i\beta_1}$.

Suppose that for some $n \in \mathbf{N}$ there exists $\beta_1 \in \mathbf{T}_1,\ \beta_2 \in \mathbf{T}_2,\ \cdots,\ \beta_n \in \mathbf{T}_n$ so that for every $k \in \mathbf{N}$,

$$f_k(\omega) = f(\omega) = e^{i\beta_1},$$

$$f_k(\beta_1) = f(\beta_1) = \frac{1}{2}e^{i\beta_2},$$

$$\cdots\cdots,$$

$$f_k(\beta_{n-1}) = f(\beta_{n-1}) = \frac{1}{2^{n-1}}e^{i\beta_n},$$

$$|f_k(\beta_n)| \leq \frac{1}{2^n}, \qquad |f(\beta_n)| \leq \frac{1}{2^n},$$

and for every $\theta \in \bigcup_{k=1}^{n} \mathbf{T}_k,\ \theta \neq \omega,\ \beta_1,\ \beta_2,\ \cdots,\ \beta_n$, $f_k(\theta) = f(\theta) = 0$ for every $k \in \mathbf{N}$. Using the same argument as in the beginning of the proof and the hypothesis that $f \notin C$, we can show that there exists a unique $\beta_{n+1} \in \mathbf{T}_{n+1}$ so that for every $k \in \mathbf{N}$,

$$f_k(\beta_n) = f(\beta_n) = \frac{1}{2^n}e^{i\beta_{n+1}},$$

$$|f(\beta_{n+1})| \leq \frac{1}{2^{n+1}}, \quad |f_k(\beta_{n+1})| \leq \frac{1}{2^{n+1}},$$

and for every $\theta \in \mathbf{T}_{n+1},\ \theta \neq \beta_{n+1}, f_k(\theta) = f(\theta) = 0$ for every $k \in \mathbf{N}$. This enables us to construct inductively a sequence $\beta_i \in \mathbf{T}_i$ so that for every $k \in \mathbf{N}$,

$$f_k(\omega) = f(\omega) = e^{i\beta_1},$$
$$f_k(\beta_1) = f(\beta_1) = \frac{1}{2}e^{i\beta_2},$$
$$\cdots\cdots,$$
$$f_k(\beta_n) = f(\beta_n) = \frac{1}{2^n}e^{i\beta_{n+1}},$$
$$\cdots\cdots,$$

and for every $\theta \in I$, $\theta \neq \omega, \beta_1, \beta_2, \cdots$, $f_k(\theta) = f(\theta) = 0$ for each $k \in \mathbf{N}$. This means that f is an element of C', which leads to a contradiction with the hypothesis that $f \notin C$. Hence $f \in C$, C is closed and bounded.

Lemma 2. *C is J-convex in $l^2(I)$.*

Proof. By Lemma 1, C is closed and bounded. It remains to show that every Jensen measure on $l^2(I)$ supported on C has a barycenter in C. Let μ be a Jensen measure on $l^2(I)$ supported on C with barycenter $f \in l^2(I)$. Suppose that $f \notin C$.

First note that for each g in C we have $|g(\omega)| \leq 1$, so $|f(\omega)| \leq 1$. If for every $\theta \in \bigcup_{n=1}^{\infty} \mathbf{T}_n$, $f(\theta) = 0$, then $f \in C_\omega \subset C$; this is impossible since we have supposed that $f \notin C$. There exists then $\theta \in \bigcup_{n=1}^{\infty} \mathbf{T}_n$ so that $f(\theta) \neq 0$. If $\theta \in \mathbf{T}_n$ for some $n \geq 2$, consider the projection

$$P_\theta\colon\ l^2(I) \to \mathbf{C},$$
$$g \to g(\theta),$$

where P_θ is a continuous linear functional on $l^2(I)$, the image measure μ_θ of μ by P_θ on $\mathbf{C}$ is a Jensen measure with barycenter $f(\theta) \neq 0$. We have $\mu_\theta(\{0\}) = 0$ (see [5]), i.e., μ is supported by $\{g \in C\colon\ g(\theta) \neq 0\}$. By the special structure of C, μ is supported by

$$\Big\{g \in C\colon\ \text{there exists } \alpha \in \mathbf{T}_{n-1} \text{ so that } g(\alpha) = \frac{1}{2^{n-1}}e^{i\theta}\Big\}.$$

Let

$$F_\alpha = \Big\{g \in C\colon\ g(\alpha) = \frac{1}{2^{n-1}}e^{i\theta}\Big\}$$

for each $\alpha \in \mathbf{T}_{n-1}$. We have

$$\mu\Big(\bigcup_{\alpha\in\mathbf{T}_{n-1}} F_\alpha\Big) = 1.$$

For every $\alpha, \beta \in \mathbf{T}_{n-1}$, $\alpha \neq \beta$,

$$\operatorname{dist}(F_\alpha,\ F_\beta) = \inf\{\|x-y\|\colon\ x \in F_\alpha,\ y \in F_\beta\} \geq \frac{1}{2^n}.$$

Since μ is a Borel probability measure on $l^2(I)$, μ is supported on some separable subset of $l^2(I)$, and there exist $\alpha_1, \alpha_2, \cdots \in \mathbf{T}_{n-1}$ so that $\mu\Big(\bigcup_{k=1}^{\infty} F_{\alpha_k}\Big) = 1$. This implies that μ is supported by

$$\{g \in C\colon\ g(\sigma) = 0 \text{ for every } \sigma \in \mathbf{T}_{n-1},\ \sigma \neq \alpha_1,\ \alpha_2,\ \cdots\}.$$

We can deduce that $f(\theta) = 0$ for every $\theta \in \mathbf{T}_{n-1}$, $\theta \neq \alpha_1, \alpha_2, \cdots$. On the other hand, for each $i \in \mathbf{N}$ the image measure of μ by

$$P_{\alpha_i}\colon\ l^2(I) \to \mathbf{C},$$
$$g \to g(\alpha_i)$$

is a Jensen measure on $\mathbf{C}$ supported on $\left\{0, \frac{1}{2^{n-1}}e^{i\theta}\right\}$, μ_{α_i} is then a Dirac measure (see [5]). If for each $i \in \mathbf{N}$, $\mu_{\alpha_i} = \delta_0$, then μ is supported by $\{g \in C\colon\ g(\alpha_i) = 0\}$ for each $i \in \mathbf{N}$. This implies that $\mu(F_{\alpha_i}) = 0$ and so $\mu\left(\bigcup\limits_{i=1}^{\infty} F_{\alpha_i}\right) = 0$, which leads to a contradiction. There exists then $i_0 \in \mathbf{N}$ so that

$$\mu_{\alpha_{i_0}} = \delta_{\frac{1}{2^{n-1}}e^{i\theta}}.$$

In this case, μ is supported by

$$\{g \in C\colon\ g(\alpha_{i_0}) = \frac{1}{2^{n-1}}e^{i\theta}\},$$

so the barycenter of μ verifies also the same condition: $f(\alpha_{i_0}) = \frac{1}{2^{n-1}}e^{i\theta}$.

Starting from the hypothesis that there exists $\theta \in \mathbf{T}_n$ (for some $n \geq 2$) so that $f(\theta) \neq 0$, we have shown that there exists $\alpha_{i_0} \in \mathbf{T}_{n-1}$ so that $f(\alpha_{i_0}) \neq 0$. By induction, there exists $\theta \in \mathbf{T}_1$ so that $f(\theta) \neq 0$. Fix then such a $\theta \in \mathbf{T}_1$.

Consider the image measure μ_θ of μ by the projection P_θ

$$P_\theta\colon\ l^2(I) \to \mathbf{C},$$
$$g \to g(\theta).$$

μ_θ is a Jensen measure on $\mathbf{C}$ with barycenter $f(\theta) \neq 0$. We must have $\mu_\theta(\{0\}) = 0$ (see [5]). μ is supported by

$$E = \{g \in C\colon\ g(\theta) \neq 0\}.$$

For every element g in E, $g(\theta) \neq 0$, by the special structure of C, we have then $g(\omega) = e^{i\theta}$. This implies that μ is supported by

$$\{g \in C\colon\ g(\omega) = e^{i\theta},\ g(\theta) \neq 0\}$$

and so $f(\omega) = e^{i\theta}$, $f(\beta) = 0$ for each $\beta \in \mathbf{T}_1$, $\beta \neq \theta$.

Suppose that for some $n \in \mathbf{N}$ there exists $\alpha_i \in \mathbf{T}_i$, $i = 1, 2, \cdots, n$, such that

$$f(\omega) = e^{i\alpha_1},\quad f(\alpha_1) = \frac{1}{2}e^{i\alpha_2}, \cdots,\quad f(\alpha_{n-1}) = \frac{1}{2^{n-1}}e^{i\alpha_n}$$
$$\text{and for every } \sigma \in \bigcup_{k=1}^{n} \mathbf{T}_k, \sigma \neq \alpha_1, \alpha_2, \cdots, \alpha_n,\ f(\sigma) = 0,$$

and μ is supported by

$$A = \Big\{g \in C\colon\ g(\omega) = e^{\alpha_1},\quad g(\alpha_1) = \frac{1}{2}e^{i\alpha_2}, \cdots,\quad g(\alpha_{n-1}) = \frac{1}{2^{n-1}}e^{i\alpha_n},\quad g(\alpha_n) \neq 0$$
$$\text{and for every } \sigma \in \bigcup_{k=1}^{n} \mathbf{T}_k, \sigma \neq \alpha_1,\ \alpha_2, \cdots, \alpha_n,\ g(\sigma) = 0\Big\}.$$

Using the same argument as in the beginning of the proof and the hypothesis that $f \notin C$, we can show that there exists $\alpha_{n+1} \in \mathbf{T}_{n+1}$ so that $f(\alpha_n) = \frac{1}{2^n}e^{i\alpha_{n+1}}, f(\sigma) = 0$ for every

$\sigma \in \mathbf{T}_{n+1}, \sigma \neq \alpha_{n+1}$, and that μ is supported by

$$A \cap \Big\{ g \in C: \ g(\alpha_n) = \frac{1}{2^n} e^{i\alpha_{n+1}}, \ g(\alpha_{n+1}) \neq 0 \text{ and } g(\sigma) = 0 \text{ for each } \sigma \in \mathbf{T}_{n+1}, \ \sigma \neq \alpha_{n+1} \Big\}.$$

This enables us to construct inductively $\alpha_i \in \mathbf{T}_i$, so that

$$f(\omega) = e^{i\alpha_1}, \ f(\alpha_1) = \frac{1}{2} e^{i\alpha_2}, \cdots, \ f(\alpha_n) = \frac{1}{2^n} e^{i\alpha_{n+1}}, \ \cdots$$

and for every $\sigma \neq \omega, \ \alpha_1, \ \alpha_2, \ \cdots, \ f(\sigma) = 0$, i.e., $f \in C'$, which leads to a contradiction with the hypothesis that $f \notin C$. Hence $f \in C$, and C is J-convex.

Lemma 3. *The set of all Jensen boundary points $Jr(C)$ of C is contained in C' and C' is J-convex.*

Proof. It is clear that every point $g \in \widetilde{C}$ is not a Jensen boundary point by the special structure of C, $Jr(C)$ is contained in C'. We are going to show that C' is J-convex.

First note that C' is closed. Indeed, if $f_n \in C'$ is a converging sequence and $f = \lim_{n\to\infty} f_n$, then $f \in C$ since C is closed by Lemma 1. Suppose that $f \notin C'$, i.e., $f \in \widetilde{C}$. There exists then $N \in \mathbf{N}$ so that $f \in C_\omega \cup C_N$. We have for each $n > N$, and for each $\theta \in \mathbf{T}_n$, $f(\theta) = 0$. In particular, for each $\theta \in \mathbf{T}_{N+1}$, $f(\theta) = 0$, so $|f_k(\theta)| < \frac{1}{2^{N+2}}$ when k is big enough. This is a contradiction since $f_k \in C'$ and each $g \in C'$ verifies the following condition: there exists $\theta \in \mathbf{T}_{N+1}$ so that $|g(\theta)| = \frac{1}{2^{N+1}}$. Hence $f \in C'$ and C' is closed.

Let μ be a Jensen measure on $l^2(I)$ supported on C' with barycenter f. We have $f \in C$ since C is J-convex by Lemma 2. Suppose that $f \notin C'$.

For each $\alpha \in \mathbf{T}_1$ let

$$D_\alpha = \{ f \in C': \ f(\omega) = e^{i\alpha} \}.$$

Each D_α is closed and for $\alpha, \ \beta \in \mathbf{T}_1, \ \alpha \neq \beta$,

$$\operatorname{dist}(D_\alpha, \ D_\beta) = \inf\{ \|x - y\|: \ x \in D_\alpha, \ y \in D_\beta \} \geq 1.$$

μ is a Borel measure on $l^2(I)$. Then μ is supported by some separable subset of $l^2(I)$. This implies that there exist $\alpha_1, \ \alpha_2, \ \cdots \in \mathbf{T}_1$ so that

$$\mu\Big(\bigcup_{i=1}^{\infty} D_{\alpha_i} \Big) = 1.$$

Consider the image measure μ_ω of μ by the projection P_ω

$$P_\omega: \ l^2(I) \to \mathbf{C}, \quad g \to g(\omega),$$

μ_ω is a Jensen measure on $\mathbf{C}$ with barycenter $f(\omega)$ and μ_ω is supported by $\{e^{i\alpha_1}, e^{i\alpha_2}, \cdots\}$. There exists then $\beta_1 \in \{\alpha_1, \ \alpha_2, \ \cdots\}$, so that $\mu_\omega = \delta_{e^{i\beta_1}}$ (see [5]). This means that μ is supported by D_{β_1} and $f(\omega) = e^{i\beta_1}$.

Suppose that for some $n \in \mathbf{N}$ there exists $\beta_i \in \mathbf{T}_i, \ i = 1, \ 2, \ \cdots, \ n$, so that

$$f(\omega) = e^{i\beta_1}, \quad f(\beta_1) = \frac{1}{2} e^{i\beta_2}, \cdots, \ f(\beta_{n-1}) = \frac{1}{2^{n-1}} e^{i\beta_n}$$

and for $\sigma \in \bigcup_{i=1}^{n} \mathbf{T}_i$, $\sigma \neq \beta_1, \beta_2, \cdots, \beta_n$, $f(\sigma) = 0$, and μ is supported by

$$\Big\{g \in C'\colon\ g(\omega) = e^{i\beta_1},\ g(\beta_1) = \frac{1}{2}e^{i\beta_2}, \cdots,\ g(\beta_{n-1}) = \frac{1}{2^{n-1}}e^{i\beta_n}\Big\}.$$

Using the same argument as in the beginning of the proof and the hypothesis that $f \notin C'$, we can show that there exists $\beta_{n+1} \in \mathbf{T}_{n+1}$ so that

$$f(\beta_n) = \frac{1}{2^n}e^{i\beta_{n+1}}$$

and for every $\theta \in \mathbf{T}_n$, $\theta \neq \beta_n$, $f(\theta) = 0$, and μ is supported by

$$\Big\{g \in C'\colon\ g(\omega) = e^{i\beta_1},\ g(\beta_1) = \frac{1}{2}e^{i\beta_2},$$
$$\cdots,\ g(\beta_{n-1}) = \frac{1}{2^{n-1}}e^{i\beta_n},\ g(\beta_n) = \frac{1}{2^n}e^{i\beta_{n+1}}\Big\}.$$

This enables us to construct inductively $\beta_i \in \mathbf{T}_i$, $i \in \mathbf{N}$, so that

$$f(\omega) = e^{i\beta_1},\ f(\beta_1) = \frac{1}{2}e^{i\beta_2}, \cdots,\ f(\beta_n) = \frac{1}{2^n}e^{i\beta_{n+1}}, \cdots,$$

i.e., $f \in C'$. This contradicts our hypothesis that $f \notin C'$. Hence $f \in C'$, C' is J-convex.

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