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TRIPLE INTERACTIONS OF CONORMAL WAVES FOR HIGHER ORDER SEMILINEAR HYPERBOLIC EQUATIONS

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Abstract

The interaction of three conormal waves for semi-linear strictly hyperbolic equations of third order is considered. Let Σ_i, $i = 1, 2, 3$, be smooth characteristic surfaces for $P = D_t(D_t^2 - \Delta)$ intersecting transversally at the origin. Suppose that the solution u to $Pu = f(t, x, y, D^\alpha u), |\alpha| \leq 2$ is conormal to $\Sigma_i, i = 1, 2, 3$, for $t < 0$. The author uses Bony's second microlocalization techniques and commutator arguments to conclude that the new singularities a short time after the triple interaction lie on the surface of the light cone Γ over the origin plus the surfaces obtained by flow-outs of the lines of intersection $\Gamma \cap \Sigma_i$ and $\Sigma_i \cap \Sigma_j$, $i, j = 1, 2, 3$.

Keywords Hyperbolic equation, Conormal wave, Interaction, Characteristic surface.
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§1. Introduction

Let u be a solution belonging to H^s, $s > \frac{9}{2}$, in an open set Ω of $I\!R^3$ containing the origin, of

$$Pu = f(t, x, y, u, \partial^\alpha u)_{|\alpha|=1,2}, \tag{1.1}$$

where $P = \partial_t \square = \partial_t(\partial_{tt} - \partial_{xx} - \partial_{yy})$, f is assumed to be a C^∞ function of its arguments, the open set Ω satisfies that any null bicharacteristic curve of P issuing from a point of $\Omega_+ = \Omega \cap \{t > 0\}$ meets $\Omega_- = \Omega \cap \{t < 0\}$ before it goes out Ω. The aim of this paper is to study the triple interaction of the conormal waves for the higher order hyperbolic equation (1. 1). From [1, 2], we know that, in the future, u will have conormal singularity along Σ_1 when u has conormal singularity along the characteristic surface Σ_1 in the past. If u, in the past, has conormal singularities along two characteristic surfaces Σ_1 and Σ_2, then, in the future, the singularities of u are localized on all characteristic surfaces starting from $\Sigma_1 \cap \Sigma_2$. When u is conormal, in the past, with respect to three characteristic surfaces Σ_1, Σ_2, Σ_3 intersecting transversally at a point, there is no result for higher order equations. We know that, for two conormal singularities, the result taht u will be regular outside $\Sigma_1 \cup \Sigma_2$ in the future and will have conormal singularities near Σ_1 and Σ_2 is valid for equations of order 2 in any dimension, but when u is conormal in the past with respect to three smooth characteristic surfaces Σ_1, Σ_2, Σ_3 intersecting transversally at the origin for wave equations, the nonlinear phenomenon can occur in the future even for this restricted singularities, i. e., u has singularities not only on $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ in the future, but also on Γ, the surface of the light cone over the origin. For the detailed proof about this problem one can refer to [3–9]. For higher order equations, as

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mentioned above, the interaction of two singularities will create additional singularities on $\Sigma_1 \cap \Sigma_2$. Without doubt, this brings a lot of difficulties to the study of the triple interaction for higher order equations. On the other hand, the complicated natural geometry for higher order equations themselves must be another main difficulty. In this paper we handle the triple interactions for the model case (1.1) of the third order equations. Even in this case the singularity structure is still very complicated. In fact, we shall prove that the new singularities appear not only on a forward light cone emanating from the point of intersection but also on many other new characteristic surfaces issuing from the intersecting lines of any interaction two by two. To describe such a singularity structure we need to introduce vector fields simultaneously tangent to all these characteristics. However, these vector fields are too degenerate at the intersection point. So, to express the commutator by a linear combination of $\partial_t\Box$ and generators of singular tangential vector fields in some sense, we need to use second microlocalization developed by J. M. Bony[4,5], and establish the commutation relation in the sense of second microlocalization. Since the second microlocal operators include the pseudodifferential operators and singular vector fields, they are most effective in dealing with the functions with flowery singularity structure. To solve the problem of this paper, we also need to use the paradifferential calculus. Because the equation is fully semilinear, there is an operator coefficient $B = B' \circ T_a \circ B''$ on the lower order terms, where $B' \in Op(\Sigma^{0,0})$, $B'' \in Op(\Sigma^{2,0})$, and $a \in C^\rho$, $\rho > 0$.

The contents of this paper are arranged as follows: In Section 2 we will briefly review some basic conception and properties of 2-microdifferential calculus and state the main results of the paper. In Section 3 we will establish the commutation relation of singular vector fields with the operator P, and in Section 4 we will give the proof of the main results inductively by using the commutation theorem and the propagation theorem of regularity.

§2. Statement of Main Results

In the sequal of this paper we will often use the concepts and propositions established in [5] without repeating detailed explanation.

The 2-microlocal Sobolev space $H^{s,s'}$ is defined as $H^{s,k} = \{u \in H^s; x^\alpha u \in H^{s+|\alpha|}, |\alpha| \leq k\}$ for non-negative integer k. It is defined by duality and interpolation for general real s'. The conormal distribution $H^{s,s'}(\Sigma, k)$ is defined as $\{u \in H^{s,s'}; z^\alpha u \in H^{s,s'}, |\alpha| \leq k\}$, where z represents singular vector field tangent to Σ. We denote by $\Sigma^{m,m'}$ the space of functions $a(x,\xi) \in C^\infty(\mathbb{R}^n\backslash\{O\} \times \mathbb{R}^n\backslash\{O\})$ such that

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha\beta}|\xi|^{m-|\alpha|+|\beta|}(|x||\xi|)^{m'-|\beta|},$$

where α and β are multiindeces. We say that $A \in Op(\Sigma^{m,m'})$ (the class of 2-microlocal operators) if the commutators

$$[\partial_{i_1}, [\partial_{i_2}, [\cdots[\partial_{i_p}, [x_{j_1}, [\cdots[x_{j_{q-1}}, [x_{j_q}, A]\cdots]$$

map $H^{s,s'}$ into $H^{s-m-q+p,s'-m'+q}$ for any positive integers p, q. A normal proper $Op(S^m_{1,0})$ pseudodifferential operator is an $Op(\Sigma^{m,0})$ operator, and a multiplier of homogeneous function of degree m is an $Op(\Sigma^{-m,m})$ operator. In [5] the diffeomorphism between the class of $Op(\Sigma^{m,m'})/Op(\Sigma^{m,-\infty})$ onto the class $\Sigma^{m,m'}/\Sigma^{m,-\infty}$ is established. Under this diffeomophism the operator $Op(S^m_{1,0})$ corresponds to its ordinary symbol and the multiplier given by a homogeneous function $b(x)$ corresponds to $b(x)\sigma(|x\xi|)$, where $\sigma(t)$ is a C^∞ function which is equal to one if $t > 1$, and vanishes if $t < 1/2$.

We shall say that u belongs to $H^{s,s'}$ 2-microlocally at $(O;\xi_0;\delta x_0)$ if $A \circ \psi(D) \cdot (\phi(x)u(x)) \in H^{s,s'}$, where $\phi \in C_0^\infty$, $\phi(0) \neq 0$, $\psi(D)$ is a pseudodifferential operator of order 0, and $\psi(\xi_0) \neq 0$, $A \in Op(\Sigma^{0,0})$ with symbol $a(x)$ being homogeneous of degree zero and $a(x) \neq 0$ in a small conic neighborhood of δx_0.

We shall call singular vector field an element $z(x,D)$ of $Op(\Sigma^{0,1})$ whose symbol (mod $\Sigma^{0,-\infty}$) is as follows: $z(x,\xi) = \Sigma a_j(x)(i\xi_j)$, $|D_x^\alpha a_j(x)| \leq C_{j,\alpha}|x|^{1-|\alpha|}$.

To describe our results precisely, let us introduce some notations. Assume that Σ_1, Σ_2, Σ_3 are three smooth characteristic surfaces for higher order equation (1.1), intersecting transversally at the origin. We denote

Γ : the full light cone.

$\Gamma^\pm$: the forward (backward) light cone.

$$L_k^\pm(t) = \Sigma_i \cap \Sigma_j \cap \{t \gtrless 0\},$$
$$L_k'^\pm(t) = \Sigma_k \cap \Gamma \cap \{t \gtrless 0\}, \quad i \neq j \neq k, \quad i,j,k = 1,2,3.$$

$\overline{L}_k^\pm(t)$(resp.$\overline{L}_k'^\pm(t)$): the intersecting lines of $\Gamma^\pm(t)$ and the plane which passes through the lines $L_k^\pm(t)$(resp. $L_k'^\pm(t)$) and the half t-taxis $T^\pm(t)$, $k = 1,2,3$.

$\Pi_k^\pm$(resp.$\Sigma_k'^\pm$): the convex hull of $L_k^\pm(t)$(resp.$L_k'^\pm(t)$) and $\overline{L}_k^\pm(t)$(resp.$\overline{L}_k'^\pm(t)$), $k = 1,2,3$.

$$\Pi_k = \Pi_k^+ \cup \Pi_k^- (\text{resp.} \Sigma_k' = \Sigma_k'^+ \cup \Sigma_k'^-), \quad k = 1,2,3. \quad T(t) = T^+ \cup T^-.$$

Definition 2.1. *Let Σ be either a smooth submanifold or the union of surfaces intersecting only two by two and transversally. We shall say that u belongs to the space of conormal distributions $H^s(\Sigma,k)$ if $X^I u \in H^s$ for $|I| \leq k$, where X^I is a product of $|I|$ smooth vector fields tangent to Σ.*

From the assumption of our problem in this paper, we only investigate the case that u has no additional singularities for intersections two by two before the triple intersection began. For example, when $f = f_1(t,x,y) + \chi f_2(u)$, where f_1 is a C^∞ function of its arguments, $i = 1,2$, $\chi \in C^\infty(\mathbb{R})$, supp$\chi \subseteq \{t \geq 0\}$, it belongs to this case. The main result of this paper can be stated as follows.

Theorem 2.2. *Assume that u is a solution of* (1.1) *belonging to H^s and that $u \in H^\sigma(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, k)$ in Ω_- for $\sigma > 9/2$ and $k \in \mathbb{N}$, i.e.,*

$$u \in H^{\sigma+k} \ in \ \Omega_- \setminus \bigcup_{i=1}^{3} \Sigma_i,$$

$$u \in H^\sigma(\Sigma_i, k) \ near \ \Sigma_i \setminus \bigcup_{j\neq i} \Sigma_j,$$

$$u \in H^\sigma(\Sigma_i \cup \Sigma_j, k) \ near \ \Sigma_i \cap \Sigma_j.$$

Then, one has, near O in Ω_+ and for each $\sigma' < \sigma$:

a) $u \in H^{\sigma'+k}$ *outside* $\bigcup_{i=1}^{3} \Sigma_i \cup \bigcup_{j=1}^{3} \Pi_j^+ \cup \bigcup_{i=1}^{3} \Sigma_i'^+ \cup \Gamma^+$,

b) $u \in H^{\sigma'}(\Sigma_j, k)$ *near* $\Sigma_j \setminus \left(\bigcup_{i\neq j}^{3} \Sigma_i \cup \bigcup_{i=1}^{3} \Pi_i^+ \cup \bigcup_{i=1}^{3} \Sigma_i'^+ \cup \Gamma^+ \right)$,

c) $u \in H^{\sigma'}(\Pi_i^+, k)$ *near* $\Pi_i^+ \setminus \left(\bigcup_{j\neq i}^{3} \Pi_j^+ \cup \bigcup_{j=1}^{3} \Sigma_j \cup \bigcup_{j=1}^{3} \Sigma_j'^+ \cup \Gamma^+ \right)$,

d) $u \in H^{\sigma'}(\Sigma_j'^+, k)$ *near* $\Sigma_j'^+ \backslash \Big(\bigcup_{i \neq j}^{3} \Sigma_i'^+ \cup \bigcup_{i=1}^{3} \Pi_i^+ \cup \bigcup_{i=1}^{3} \Sigma_i \cup \Gamma^+ \Big)$,

e) $u \in H^{\sigma'}(\Sigma_j, k)$ *near* $\Gamma^+ \backslash \Big(\bigcup_{i=1}^{3} \Sigma_i \cup \bigcup_{i=1}^{3} \Pi_i^+ \cup \bigcup_{i=1}^{3} \Sigma_i'^+ \Big)$.

Remark 2.1. From the results of M. Beals[10] and J. M. Bony[2,4], actually we can improve c), d), and e) up to

c') $u \in H^{\sigma'+g}(\Pi_i^+, k-g)$ near $\Pi_i^+ \backslash \Big(\bigcup_{j \neq i}^{3} \Pi_j^+ \cup \bigcup_{j=1}^{3} \Sigma_j \cup \bigcup_{j=1}^{3} \Sigma_j'^+ \cup \Gamma^+ \Big)$,

d') $u \in H^{\sigma'+g}(\Sigma_j'^+, k-g)$ near $\Sigma_j'^+ \backslash \Big(\bigcup_{i \neq j}^{3} \Sigma_i'^+ \cup \bigcup_{i=1}^{3} \Pi_i^+ \cup \bigcup_{i=1}^{3} \Sigma_i \cup \Gamma^+ \Big)$,

e') $u \in H^{\sigma'+g}(\Sigma_j, k-g)$ near $\Gamma^+ \backslash \Big(\bigcup_{i=1}^{3} \Sigma_i \cup \bigcup_{i=1}^{3} \Pi_i^+ \cup \bigcup_{i=1}^{3} \Sigma_i'^+ \Big)$,

where $g = \min(k, [\sigma - 7/2])$.

Remark 2.2. There is no singularity on the prolongation line of

$$\overline{L_k^+(t_4)\overline{L}_k^+(t_4)} \text{ (resp. } \overline{L'^+_k(t_4)\overline{L}'^+_k(t_4)})$$

for any $t_4 > 0$. In fact, for any point $P = (t_4, x_0, y_0)$ on the prolongation line, $P \notin \Gamma^+ \cup \Sigma_3$, we have

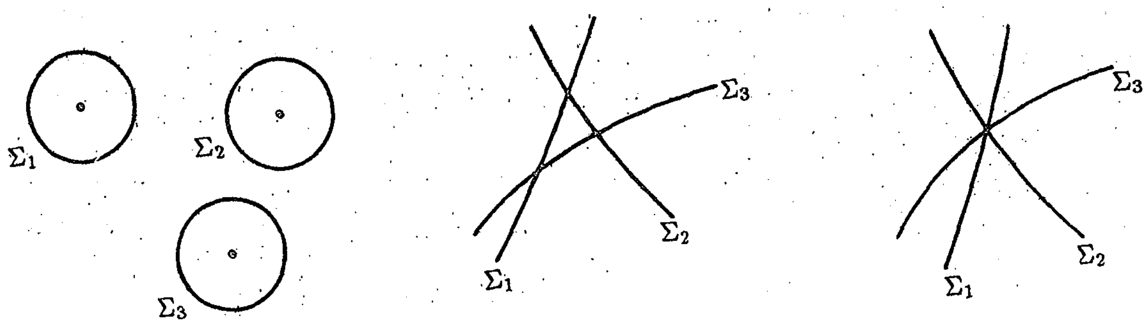
$$\text{dist}(P, \Gamma^+ \cap \{t = t_4\}) = 2\delta > 0.$$

Let $B_{t_4+\delta}(0, x_0, y_0)$ be a ball with center at $(0, x_0, y_0)$ and radius $t_4 + \delta$, and

$$\omega = B_{t_4+\delta}(0, x_0, y_0) \cap \{t = 0\}.$$

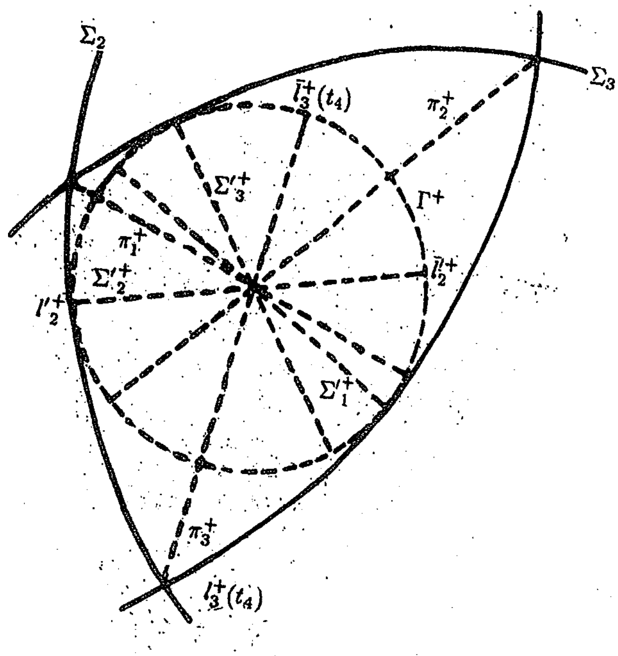
Obviously $\omega \cap \{O\} = \emptyset$. (t_4, x_0, y_0) is in the determinacy domain of ω. In this domain, there is no interaction phenomena. By the result of [11], there is no singularity at point P (resp. although there is an interaction two by two in the determinacy domain of ω, noting that the point P is not on the third characteristic Π_k which is additional singularity on $\Sigma_i \cap \Sigma_j$, $i \neq j \neq k$, we see the conclution holds also).

The result of Theorem 2.2 can be pictured through some sections of t =const. as follows:



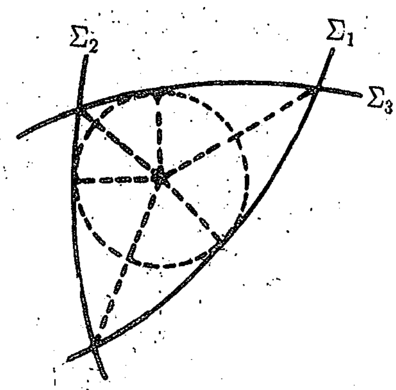
$t_1 < 0$ $\qquad$ $t_1 < t_2 < 0$ $\qquad$ $t_3 = 0$

Between t_1 and t_2, interactions two by two create no new singularity by our assumption. Between t_3 and t_4, interaction creates new singularities, but the new singularities are less strong than the incident ones.



$t_4 > 0$

Remark 2.3. If the nonlinear term $f = f(t, x, y, \partial_t u, \partial\partial_t u)$ in (1,1), the singularities of u can be controlled more exactly, which can be pictured, for $t_4 > 0$, as follows:



$t_4 > 0$

In fact, in this case equation (1.1) is equivalent to

$$\begin{cases} \Box v = f(t, x, y, v, \partial v), \\ \partial_t v = v. \end{cases}$$

Then using the result of [8] and Hormander's singularity propagation theorem, one may immediately obtain the result.

§3. Commutation Relation

The key to the proof of Theorem 2.2 is to prove the commutation theorem below. We denote by $\mathbb{Z}$ the Lie-algebra of $Op(\Sigma^{0,1})$ generated (over $Op(\Sigma^{0,0})$) by 1 and a finite number of singular vector fields $z_1, \cdots, z_m$ which are tangent, outside 0, to Σ_1, Σ_2, Σ_3, Π_1, Π_2, Π_3, Σ_1', Σ_2', Σ_3', and Γ. The conormal distribution space $H^{s,s'}(\mathbf{Z},k)$ is defined as $\{u|z^I u \in H^{s,s'}$ for $|I| \le k, z \in \mathbf{Z}\}$, where $I = (i_1, \cdots, i_l)$, $|I| = l$, $k \in \mathbf{N}$, and $z^I = z_{i_1} \circ \cdots \circ z_{i_l}$. When $s + \inf(s',0) > 3/2$, $H^{s,s'}(\mathbf{Z},k)$ is a C^∞ algebra.

Theorem 3.1. *There exists a system $\{M_i\}_{1\le i\le N}$ of generators of* $\mathbf{Z}$ *such that*

a) $[P, M_i] = \Sigma_j A_{ij} M_j + B_i P + A_{io}$; $A_{ij}, A_{i0} \in Op(\Sigma^{3,-1})$, $B_i \in Op(\Sigma^{0,0})$;

b) $A_{ij}, A_{i0} \in Op(\Sigma^{3,-2}) + Op(\Sigma^{2,0})$ *2-microlocally near the set of* $\widetilde{\Gamma}$ *of points* $(O;\tau,\xi,\eta;\delta t, \delta x, \delta y)$ *satisfying:* $\tau(\tau^2-\xi^2-\eta^2)=0$, $\delta t:\delta x:\delta y = 3\tau^2-\xi^2-\eta^2 : -2\tau\xi : -2\tau\eta$.

Proof. It is not difficult to know that the conclusions of the theorem hold microlocally near $(O;\mathrm{CChar}\ P)$. To prove that the theorem is valid microlocally near $(O;\mathrm{Char}P)$, we construct a 2-microlocal partition of identity $1 = \sum \chi_{\alpha\beta}$ near $(O;\mathrm{Char}\ P)$, as refined as we want, by operators in $Op(\Sigma^{0,0})$ as follows.

For each $(\tau,\xi,\eta) \in \mathrm{Char}\ P$, there exists a conic neighbourhood γ of (τ,ξ,η) such that γ contains the points which satisfy one of the following conditions only: i). $\tau^2 = \xi^2+\eta^2$, $\tau \ne 0$, ii). $\tau = 0$. For this γ, there exists an open cone $\mathbf{g}$ in $\mathbb{R}^3$ satisfying:

a) $(O;\tau',\xi',\eta';\delta t,\delta x,\delta y) \in \widetilde{\Gamma}$ and $(\tau',\xi',\eta') \in \gamma$ imply $(\delta t,\delta x,\delta y) \in \mathbf{g}$;

b) either $\mathbf{g}$ contains one of the intersection lines $\{\Sigma_i \cap \Sigma_i' \cap \Gamma, \Sigma_i' \cap \Gamma, \Pi_i \cap \Gamma, \bigcap_{i=1}^{3} \Pi_i \cap \bigcap_{i=1}^{3} \Sigma_i'\}$ only and $\bar{\mathbf{g}}$ only intersects the surfaces which pass through this intersection line, or $\bar{\mathbf{g}}$ intersects only one of the surfaces $\{\Sigma_i, \Sigma_i', \Pi_j, \Gamma, i,j=1,2,3\}$ (see figure).

Take finite conic neighbourhoods $\{\gamma_\alpha\}_{\alpha\in\Lambda}$ which cover $\mathrm{Char}P$, there exist corresponding open cones $\{\mathbf{g}_\alpha\}_{\alpha\in\Lambda}$ which satisfy a), b). Assume that the set $\{\mathbf{g}_{\alpha\beta}\}_{\beta\in\Theta}$ satisfying $\mathbf{g}_{\alpha\alpha} = \mathbf{g}_\alpha$ is a conic covering of $\mathbb{R}^3$. Now we can first take a partition of unity $1 = \sum h_\alpha(\tau,\xi,\eta)$ near $\mathrm{Char}P$ with the h_α supported in a small cone γ_α, where h_α is homogeneous of degree 0 and smooth on the complement of the origin. Next, let $\{\phi_{\alpha\beta}\}_{\alpha\in\Lambda,\beta\in\Theta}$ be a conic partition of unity $1 \equiv \sum_\beta \phi_{\alpha\beta}$ subordinated to the covering $\{\mathbf{g}_{\alpha\beta}\}_{\beta\in\Theta}$ for each α, where each $\phi_{\alpha\beta}$ is homogeneous of degree zero and smooth away from the origin. Let $\chi_{\alpha\beta} = \phi_{\alpha\beta} h_\alpha(\tau,\xi,\eta)$. We have

$$\{P, \chi_{\alpha\beta}\} \in \Sigma^{3,-2} + \Sigma^{2,0} \text{ 2-microlocally near the set } \widetilde{\Gamma}.$$

Now we say that it is sufficient to examine the existence of 2-microlocal generators satisfying a) and b) near the support of each $\chi_{\alpha\beta}$. In fact, if we can find a finite family of operators $\{M_{\alpha\beta i}\}$ such that a) and b) are valid 2-microlocally near this support, then $\chi_{\alpha\beta} \circ M_{\alpha\beta i}$ will satisfy a) and b) in a fully neighbourhood of O.

If $\mathrm{supp}\phi_{\alpha\beta}$ only intersects Σ_i (resp. Π_i, Σ_i'), then the 2-microlocal generators, near the support of corresponding $\chi_{\alpha\beta}$, are the usual (C^∞) vector fields which are tangent to the parts of the Σ_i (resp. Π_i, Σ_i').

If $\mathrm{supp}\phi_{\alpha\beta}$ contains $\Sigma_i \cap \Sigma_j$, without loss of generality, let $i=1$, $j=2$. Then, from Theorem 12 in [2], there exists a Lie sub-algebra of pseudodifferential operator of order 1 generated by $\{M_i\}_{1\le i\le N}$, where M_i is a pseudodifferential operator of order 1 and the principal symbol of M_i is equal to zero on

$$N^*\Sigma_1 \cup N^*\Sigma_2 \cup N^*\Pi_3 \text{ and } N^*(\Sigma_1 \cap \Sigma_2),$$

such that

$$[P, M_i] = \sum A_{ij} M_j + B_i P, \quad A_{ij} \in OpS^2, \quad B_i \in OpS^0.$$

Let $A \in Op(\Sigma^{-1,1})$ be elliptic and $\tilde{\chi}_{\alpha\beta} \in Op(\Sigma^{0,0})$ with the symbol being equal to 1 near $\mathrm{supp}\chi_{\alpha\beta}$. Then, $\tilde{\chi}_{\alpha\beta} A M_i$ is a 2-microlocal generator near $\mathrm{supp}\chi_{\alpha\beta}$.

If $\mathrm{supp}\phi_{\alpha\beta}$ contains $\Gamma \cap \Sigma_j \cap \Sigma_j'$, then, in this case, by a coordinate transformation which is similar to the one in [8], we can assume that Σ_j is $x + t = 0$ (in a conic neighbourhood of $\Gamma \cap \Sigma_j \cap \Sigma_j'$), and Σ_j' is $y = 0$. **Z** is generated 2-microlocally by $M_1 = t\partial_t + x\partial_x + y\partial_y$, $M_2 = x\partial_t + t\partial_x$, $M_3 = y((x+t)\partial_y + y(\partial_t - \partial_x))/r$ near the support of corresponding $\chi_{\alpha\beta}$, where $r = (t^2 + x^2 + y^2)^{1/2}$. Thus, we have

$$[\Box, M_i] = \sum \bar{A}_{ij} M_j + \bar{B}_i \Box + \bar{A}_{i0}, \quad \bar{A}_{ij}, \quad \bar{A}_{i0} \in Op(\Sigma^{2,-1}), \quad \bar{B}_i \in Op(\Sigma^{0,0}),$$

and $\bar{A}_{ij}, \bar{A}_{i0} \in Op(\Sigma^{2,-2}) + Op(\Sigma^{1,0})$ 2-microlocally near the set of points $(O; \tau, \xi, \eta; \delta t, \delta x, \delta y)$ satisfying $\tau^2 - \xi^2 - \eta^2 = 0$, $\delta t : \delta x : \delta y = -\tau : \xi : \eta$.

$$\begin{aligned} [P, M_1] &= [\partial_t, M_1]\Box + \partial_t[\Box, M_1] \\ &= \partial_t \Box + \partial_t(\bar{A}_{1j} M_j + \bar{B}_1 \Box + \bar{A}_{10}) \\ &= \Sigma A_{1j} M_j + B_1 P + A_{10}, \end{aligned}$$

where $A_{1j} = \partial_t \bar{A}_{1j}$, $B_1 = 1 + \bar{B}_1$, $A_{10} = \partial_t \bar{A}_{10} + [\partial_t, \bar{B}_1]\Box$.

$$\begin{aligned} [P, M_2] &= \bar{\psi} t \partial_x \Box + \partial_t[\Box, M_2] \\ &= (\bar{\psi} M_2 - \bar{\psi} x \partial_t)\Box + \partial_t(\bar{A}_{2j} M_j + \bar{B}_2 \Box + \bar{A}_{20} \\ &= \Sigma A_{2j} M_j + B_2 P + A_{20}, \end{aligned}$$

where $\bar{\psi}$ is homogeneous of degree -1 and smooth except at the origin, and $t\bar{\psi} = 1$ near $\mathrm{supp}\phi_{\alpha\beta}$.

$$\begin{aligned} &A_{20} = \partial_t \bar{A}_{20} + \bar{\psi}[M_2, \Box] + [\partial_t, \bar{B}_2]\Box, \quad A_{21} = \partial_t \bar{A}_{21}, \\ &A_{22} = \bar{\psi}\Box + \partial_t \bar{A}_{22}, \quad A_{23} = \partial_t \bar{A}_{23}, \quad B_2 = \bar{B}_2 - \bar{\psi} x. \end{aligned}$$

$$\begin{aligned} [P, M_3] = &\, r[P, 1/r]M_3 + (1/r)(\Box - 2(\partial_t^2 - \partial_t \partial_x))M_1 \\ &+ (1/r)(t + 2x + x^2 \bar{\psi})P - (1/r)(2(\partial_t \partial_x - \partial_t^2) - x\bar{\psi}\Box)M_2 + Op(\Sigma^{3,-1}). \end{aligned}$$

If $\Gamma \cap \Pi_i$ (resp. $\Gamma \cap \Sigma_i'$) $\subset \mathrm{supp}\phi_{\alpha\beta}$, without loss of generality, let Π_i (resp. Σ_i'): $x = 0$, Γ: $f = t^2 - x^2 - y^2 = 0$. Then the 2-microlocal generators of **Z**, near $\mathrm{supp}\chi_{\alpha\beta}$, are $N_1 = t\partial_t + x\partial_x + y\partial_y$, $N_2 = y\partial_t + t\partial_y$, $N_3 = x(t\partial_x + x\partial_t)/r$, $N_4 = x(y\partial_x - x\partial_y)/r$. The others can be obtained more easily, so we omit it.

$$[P, x(t\partial_x + x\partial_t)] = 2\partial_t \partial_x (t\partial_x + x\partial_t) + x\partial_x \Box.$$

Notice that

$$\begin{aligned} x\partial_x \Box &= (y^2 \psi - t)P + \Box N_1 - y\psi\Box N_2 + Op(\Sigma^{2,0}), \\ 2\partial_t \partial_x (t\partial_x + x\partial_t) &= -2\partial_t \partial_y N_2 + 2\partial_t^2 N_1 - 2tP + Diff(2), \end{aligned}$$

where ψ is a smooth function away from the origin and homogeneous of degree -1, and $t\psi = 1$ near $\mathrm{supp}\phi_{\alpha\beta}$. We can obtain

$$\begin{aligned} [P, N_3] = &\, r[P, 1/r]N_3 + ((y^2 \psi - 3t)/r)P + ((2\partial_t^2 + \Box)/r)N_1 \\ &- (1/r)(y\psi\Box + 2\partial_t \partial_y)N_2 + Op(\Sigma^{3,-1}). \end{aligned}$$

Using the same method for N_4, we have

$$[P,N_4]=r[P,1/r]N_4-(y/r)P-(\partial_t\partial_y/r)N_1+(\partial_t^2/r)N_2+Op(\Sigma^{3,-1}).$$

If supp$\phi_{\alpha\beta}$ only intersects Γ, then the $\mathbf{Z}$ is generated 2-microlocally by $t\partial_y+y\partial_t$, $t\partial_x+x\partial_t$, and $t\partial_t+x\partial_x+y\partial_y$ near the support of corresponding $\chi_{\alpha\beta}$.

If supp$\phi_{\alpha\beta}$ contains $\Pi_1\cap\Pi_2\cap\Pi_3$, then there exists a Lie sub-algebra $\mathcal{M}$ of pseudodifferrntial operator of order 1 generated by $\{M_i\}_{1\le i\le N}$, where the principal symbol of M_i is equal to zero on $\bigcup_{i=1}^{3}N^*\Pi_i\cup\bigcup_{i=1}^{3}N^*\Sigma_i'$, and $N^*(\bigcap_{i=1}^{3}\Pi_i\cap\bigcap_{i=1}^{3}\Sigma_i')$. In this case, we can obtain the 2-microlocal generators as before, and the commutation relations hold also.

§4. Conormal Regularity

Having the above preparation, in this section we can prove the main results of this paper. Obviously, Theorem 2.2 is a corollary of the following second microlocal regularity theorem.

Theorem 4.1. *Assume that u is a solution of* (1.1) *belonging to $H^s(\Omega)$, and that $u\in H^s(\mathbf{Z},k)$ locally in Ω_-, with $s>9/2$ and $k\in\mathbf{N}$. Then, $u\in H^{s'+1/2,-1/2}(\mathbf{Z},l)$, $l\le k$, near O, for each $s'<s$.*

For $l=0$, the conclusion of the theorem holds. In fact, we can show that u belongs to H^s near O[14]. But we can obtain the stronger result $u\in H^{s'+1/2,-1/2}$. This is trivial microlocally at non-characteristic points, and follows from the propagation of 2-microlocal singularities (see Remark of Theorem 5.2 in [5]) at characteristic points.

For $l=1$, $u\in H^{s'+1/2,-1/2}\Rightarrow u\in H^{s'+1/2,-1/2}(\mathbf{Z},1)$. In fact, for each $M_i\in\mathbf{Z}$, $M_i=\sum C_{ij}\cdot z_j$.

$$M_if(t,x,y,u,\partial^\alpha u)_{|\alpha|=1,2}=\sum C_{ij}z_jf(t,x,y,u,\partial^\alpha u)+(z_jf)(t,x,y,u,\partial^\alpha u).$$

From the Leibniz formula we have

$$\begin{aligned}M_if(t,x,y,u,\partial^\alpha u)&=\sum C_{ij}\partial f/\partial u(t,x,y,u,\partial^\alpha u)z_ju+(z_jf)(t,x,y,u,\partial^\alpha u)\\&\quad+\sum_{j,|\alpha|=1,2}C_{ij}\partial f/\partial u_\alpha(t,x,y,u,\partial^\alpha u)z_j\partial^\alpha u\\&\quad+\sum C_{ij}\widetilde{M}_j\Big[\frac{\partial f}{\partial u}\Big]+\sum_{j,\alpha}C_{ij}\widetilde{M}_{j,\alpha}\Big[\frac{\partial f}{\partial u_\alpha}\Big](H^{s'-3/2,\infty}),\end{aligned}$$

where $\widetilde{M}_j$ and $\widetilde{M}_{j,\alpha}$ belong to $Op(\Sigma^{0,0})$. Bcause of $u\in H^{s'+1/2,-1/2}$, z_jf, z_ju, $f_u\partial^\alpha u$, $u\in H^{s'-3/2,-1/2}(\mathbf{Z},l)$.

$$M_if(t,x,y,u,\partial^\alpha u)_{|\alpha|=1,2}=\sum_{j,j',|\alpha|=1,2}C_{ij}(a_\alpha\widetilde{C}_{j,j'}M_ju)(H^{s'-3/2,-1/2}),$$

where $C_{ij}\in Op(\Sigma^{0,0})$, $\widetilde{C}_{ij'}\in Op(\Sigma^{2,0})$, $a_\alpha\in H^{s'-3/2,-1/2}(\mathbf{Z},l)$.

$$PM_ju=M_jPu+[P,M_j]u=M_jf+\sum A_jz_ju.$$

Using the commutation relation we immediately obtain

$$PU_1+R_1U_1+B_1U_1\in H^{s'-5/2,1/2}(\mathbf{Z},l)$$

for $s'<s$, where $U_1=Mu$. Generally, letting U_j be the vector valued function whose components are M^Iu, for $0\le|I|\le j$, we have

$$PU_1+R_1U_1+B_1U_1\in H^{s'-5/2,1/2}(\mathbf{Z},l+1-j),$$

where R_j is a matrix-valued $Op(\Sigma^{3,-1})$ operator, and

$$R_j \in Op(\Sigma^{3,-2}) + Op(\Sigma^{2,0})$$

2-micolocally near $\widetilde{\Gamma}$. $B_j = C_1 \circ a_\alpha \circ C_2$ with $C_1 \in Op(\Sigma^{0,0})$, $C_2 \in Op(\Sigma^{2,0})$. By the property of paradifferential calculus, we have $av - T_a v \in H^{s'-5/2,1/2}$ if $a \in H^{s'-2}$ and $v \in H^{s'-3/2,-3/2}$. Therefore, to prove Theorem 4.1, we only need to prove the following theorem.

Theorem 4.2. *Let $s > 9/2$, and $v \in H^{s'+1/2,-3/2}(\Omega)$, for $s' < s$, be the solution of $Pv + R_{3,-1}v + B_2 v \in H^{s'-5/2,1/2}$ where $R_{3,-1}$ is the operator of $Op(\Sigma^{3,-1})$ with its symbol being zero, 2-microlocally near $\widetilde{\Gamma}$, and $B_2 = C_1 \circ T_a \circ C_2$, $a \in C^{s'-7/2}$. Then, if $v \in H^s$ in the past, we have, for $s' < s$, $v \in H^{s'+1/2,-1/2}(\Omega)$.*

Using the method of [8], we can prove the propagation theorem of regularity as follows.

Theorem 4.3. *Let $v \in H^{s'-5/2+\sigma,-1/2}(\Omega)$ with $\sigma \in [0,1)$, such that $WF(v) \subset \mathbf{K}$ and $P_1 v + Bv = f \in H^{s'-5/2,1/2}_{loc}(\Omega)$, where $\mathbf{K}$ is a small conic neighbourhood of some bicharacteristics issuing from the origin satisfying either $\mathbf{K} \cap \{(O;\tau,\xi,\eta)\} = \emptyset$ with $\tau^2 - \xi^2 - \eta^2 = 0$ or $\mathbf{K} \cap \{(O;0,\xi,\eta)\} = \emptyset$, $B = C_1 \circ T_a \circ C_2'$. Then, we have $v \in H^{s'-5/2+\sigma/2}(\Omega)$.*

From this theorem, it is not difficult to show that the proof of Theorem 4.2 follows the following theorem.

Theorem 4.4. *Let $v \in H^{s'-5/2,-1/2}(\Omega)$, for all $s' < s$ $(s > 9/2)$, satisfy*

$$P_1 v + Bv = f \in H^{s'-5/2,1/2},$$

$WF(v) \subset \mathbf{K}$, with $B = C_1 \circ T_a \circ C_2'$, C_1 and $C_2' \in Op(\Sigma^{0,0})$, $a \in C^{s'-2}$, where if $WF(v) \cap \{\tau = 0\} = \emptyset$, $P_1 = \partial_t - K_1$, $\sigma(K_1)$ does not depend on τ, and $\sigma(K_1)(0,\xi,\eta) = \sqrt{\xi^2+\eta^2}$, and if $WF(v) \cap \{\tau = 0\} \neq \emptyset$, $P_1 = \partial_t$. If $v \in H^{s'-2}$ in the past, then, for all $s' < s$, $u \in H^{s-3/2,-1/2}_{loc}(\Omega)$.

Proof of Theorem 4.4. If $WF(v) \cap \{\tau = 0\} = \emptyset$, by the assumption of

$$v \in H^{s'-5/2-\epsilon/2+\epsilon/2,-1/2} \text{ and } v \in H^{s'-5/2+1/2}_{loc}(\Omega_-),$$

we have $v \in H^{s'-5/2-\epsilon/2+\epsilon/4}(\Omega)$ from Theorem 4.3. $P_1 v \in L^1([t_0,T]; H^{s'-5/2-\epsilon/2+\epsilon/4})$ and $\gamma_{t_0} v \in H^{s'-5/2+1/2}$ from Proposition 1.6 of [12]. Using Theorem 23.1.2 of [13], one has

$$v \in L^\infty([t_0,T]; H^{s'-5/2-\epsilon/2+\epsilon/4}).$$

$WF(v) \subset \mathbf{K}$ assures that $v \in H^{s'-5/2-\Sigma_1^2 \epsilon/2^i+\epsilon/4+1/2,-1/2}$. Let $\sigma = \epsilon/4 + 1/2$. We can obtain $v \in H^{s'-5/2-\Sigma_1^3 \epsilon/2^i+\sigma/2+1/2,-1/2}$ by the same fashion. Generally, we have

$$v \in H^{s'-5/2-\Sigma_1^{n+1} \epsilon/2^i+\epsilon/2^{n+1}+\Sigma_1^n 1/2^i,-1/2},$$

and letting $n \to \infty$ one has $v \in H^{s'-5/2-\epsilon+1,-1/2}$. In view of the arbitrarity of ϵ, we obtain the required result. If $Wf(v) \cap \{\tau = 0\} \neq \emptyset$, we have $v \in H^{s'-5/2,-1/2}$, for $s' < s$, satisfying $\partial_t v \in H^{s'-5/2,1/2}$, and $v \in H^{s'-2}$ in the past. By virtue of the propagation Theorem 5.2 of 2-microlocal singularity in [5], one can obtain the result of this theorem as well.

Remark 4.1. If the nonlinear term f of (1.1) is $f(t,x,y,u,\partial u)$, then the restriction of s in Theorem 2.2 can be relaxed to $s > 5/2$. In fact, in this case we can first prove by induction, for $l \leq k$, the following equation with R_l matrix-valued: $PU_l + \mathcal{R}_l U_l = F_l$ with $U_l \in Op(\Sigma^{3,-1})$ and $P_l \in Op(\Sigma^{3,-2}) + Op(\Sigma^{2,0})$ 2-microlocally near $\widetilde{\Gamma}$, $F_l \in H^{s'-5/2,1/2}$ for $s' < s$, and then by noting that $u \in H^{s+1/2,-1/2}$ implies $U_l \in H^{s+1/2,-1/2-l}$ we see that at noncharacteristic points $P + \mathcal{R}_l$ has a microlocal inverse belonging to $Op(\Sigma^{-3,0})$, and thus $U_l \in H^{s'+1/2,1/2}$ microlocally. At characteristic points, all the assumptions of Theorem 5.2 in [5] are satisfied, and one has $U_l \in H^{s'+1/2,1/2}$ microlocally. Thus we have proved

Theorem 4.5. *Assume that u is a solution of* (1.1) *with the nonlinear term* $f(t,x,y,u,\partial u)$ *belonging to* $H^{5/2+\epsilon}(\Omega)$, *and that* $u \in H^{s'+1/2,1/2}(\mathbf{Z},k)$ *locally in* Ω_-, *with* $s > 5/2$ *and* $k \in \mathbf{N}$. *Then,* $u \in H^{s'+1/2,1/2}(\mathbf{Z},k)$ *near O for each* $s' < s$.

Using the conclusion of this theorem, we immediately obtain Theorem 2.2 for $s > 5/2$.

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