*Chin. Ann. of Math.*
**16B**: 1(1995),33-42.

# RANDOM ITERATION OF HOLOMORPHIC SELF-MAPS OVER BOUNDED DOMAINS IN $C^N$ ***

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**Abstract**

This paper studies the asymptotic properties of the random iterations of both the form $G_n = f_1 \circ f_2 \circ \cdots \circ f_n$ and the form $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$, where $\{f_n\} \subset H(\Omega, \Omega)$ and $\Omega \subset C^N$ is a bounded domain. It is found that, under some conditions, $G_n$ or $F_n$ tends to a point in $\overline{\Omega}$ as $n \to \infty$. Some examples are also given to show that the conditions that we have given can not be dropped in general. Moreover, a complete description is given for $F_n$ or $G_n$ to tend to a point in $\Omega$ under the condition $f_n \to f$.

**Keywords** Random iteration, Holomorphic map, Kobayashi metric.
**1991 MR Subject Classification** 32h02, 32H50.

## §1. Introduction

The iteration theory of analytic functions in one complex variable has a long history of more than 70 years and now has entered a period of great development. This achievement stimulated the creation of iteration theory of several complex variables. Many interesting results on iteration theory of several complex variables have been published (see, for example, [11, 5, 1, 10], etc.). On the other hand, due to the development of fractal geometry and the advent of computer graphics, the study of random iteration of analytic functions has got its own position (see [2, 3, 4], etc.). In this paper, we shall discuss the asymptotic properties of random iteration of holomorphic self-maps over bounded domains in $C^N$ .

Let $\Omega \subset C^N$ be a bounded domain, $\mathcal{F} \subset H(\Omega, \Omega)$. Given $\{f_n\} \subset \mathcal{F}$ , we consider the sequence

$$G_n = f_1 \circ f_2 \circ \cdots \circ f_n, \quad n = 1, 2, \cdots .$$

The question is when $G_n$ converges to a point in $\Omega$ . We shall present several theorems describing conditions on the $f_n{}'$s to ensure the convergence of $G_n$ in §2 and §3. Similar problems for one complex variable have been studied in [6, 7, 9], and [4]. And in paper [6], Gill also listed some reasons why such form of iterations is of interest. Some of our results in this paper are somewhat stronger than those corresponding results they got for $N = 1$ , and the proofs are more simple.

This paper is organized as follows. In §2 , we discuss the random iteration of contractions in an arbitrary bounded domain. In §3 , we deal with some family of holomorphic mappings

Manuscript received November 19, 1992.
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***Project supported by the National Natural Science Foundation of China.

in bounded convex domains. In both cases, we have examples to show that the conditions in our theorems can not be cancelled in general. In §4 , we give some more results for random iterations, including the asymptotic properties of random iterations of the form $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$ .We give a complete description for $F_n$ or $G_n$ to tend to a point in $\Omega$ under the condition $f_n \to f$.

## §2. Random Iteration of Contractions Over Bounded Domains

**Theorem 2.1.** *Let $\Omega \subset C^N$ be a bounded domain, $E \subset\subset \Omega$, and*

$$\mathcal{F} = \{f \in H(\Omega, \Omega) | f(\Omega) \subset E\}.$$

*Then for any sequence $\{f_n\} \subset \mathcal{F}, \exists a \in \Omega$, such that $G_n = f_1 \circ f_2 \circ \cdots \circ f_n$ converges to the point $a$ uniformly on $\Omega$ .*

**Note.** The maps in $\mathcal{F}$ are usually called contractions.

**Proof.** Denote $\varepsilon = \mathrm{dist}(E, \partial\Omega)$ , and let $\delta = \frac{\varepsilon}{2M}$, where $M$ is the diameter of $\Omega$ . For fixed $z \in \Omega$ , define

$$g_n(w) \equiv f_n(w) + \delta(f_n(w) - f_n(z)), \quad \forall w \in \Omega.$$

Notice

$$|\delta(f_n(w) - f_n(z))| \leq \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2}.$$

So $g_n \in H(\Omega, \Omega)$.

Now let $F_K : \Omega \times C^N \longrightarrow R$ be the infinitesmal form of the Kobayashi metric of $\Omega$ which is defined as

$$F_K(z, \xi) \equiv \inf\Big\{\frac{|\xi|}{|Df(0)e_1|} : f \in H(B, \Omega), f(0) = z, Df(0)e_1 \text{ is a constant multiple of } \xi\Big\}.$$

Since holomorphic maps are distance decreasing in the Kobayashi metric, we have

$$F_K(g_n(z), Dg_n(z)\xi) \leq F_K(z, \xi), \quad \forall \xi \in C^N. \tag{2.1}$$

Notice

$$g_n(z) = f_n(z), \quad Dg_n(z) = (1 + \delta)Df_n(z).$$

(2.1) can be rewritten as

$$F_K(f_n(z), Df_n(z)\xi) \leq \frac{1}{1+\delta} F_K(z, \xi), \quad \xi \in C^N, z \in \Omega. \tag{2.2}$$

Recall also that the Kobayashi distance $K_\Omega$ of $\Omega$ is

$$K_\Omega(z, w) = \inf_\gamma \int_0^1 F_K(\gamma(t), \gamma'(t))dt, \quad \forall z, w \in \Omega, \tag{2.3}$$

where the infimum is taken over all $C^1$ curves $\gamma$ with $\gamma(0) = z, \gamma(1) = w$.

Now for any such curve $\gamma$, $f_n \circ \gamma$ is a corresponding curve with endpoints $f_n(z)$ and $f_n(w)$, so (2.2) and (2.3) produce

$$F_K(f_n(\gamma(t)), Df_n(\gamma(t))\gamma'(t)) \leq \frac{1}{1+\delta} F_K(\gamma(t), \gamma'(t)),$$

or, in other words,

$$F_K(f_n(\gamma(t)), [f_n \circ \gamma(t)]') \leq \frac{1}{1+\delta} F_K(\gamma(t), \gamma'(t)). \tag{2.4}$$

Integrate (2.4) in both sides against $t \in [0,1]$ , by (2.3) we have

$$\begin{aligned}K_\Omega(f_n(z), f_n(w)) &\le \inf_\gamma \int_0^1 F_K(f_n \circ \gamma(t), [f_n \circ \gamma(t)]')dt \\ &\le \inf_\gamma \int_0^1 \frac{1}{1+\delta} F_K(\gamma(t), \gamma'(t))dt \\ &= \frac{1}{1+\delta} K_\Omega(z,w), \quad \forall z, w \in \Omega.\end{aligned}$$

So for any $n, p$, the following holds:

$$\begin{aligned}K_\Omega(G_n(z), G_n(w)) &\le (\frac{1}{1+\delta})^{n-1} K_\Omega(f_n(z), f_n(w)) \\ &\le A(\frac{1}{1+\delta})^{n-1}, \quad \forall z, w \in \Omega.\end{aligned} \tag{2.5}$$

$$\begin{aligned}K_\Omega(G_n(z), G_{n+p}(z)) &= K_\Omega(G_n(z), G_n(f_{n+1} \circ \cdots \circ f_{n+p}(z))) \\ &\le (\frac{1}{1+\delta})^{n-1} K_\Omega(f_n(z), f_n \circ \cdots \circ f_{n+p}(z)) \\ &\le A(\frac{1}{1+\delta})^{n-1}, \quad \forall z \in \Omega,\end{aligned} \tag{2.6}$$

where $A = \sup_{z;w \in E} K_\Omega(z,w) < \infty$.

Now since $\Omega$ is bounded, there exists an $R > 0$ such that for any $z \in E$ we have

$$\Omega \subset B(z,R) = \{w \in C^N | \, |w-z| < R\}.$$

It is well-known that

$$K_\Omega(z,w) \ge K_{B(z,R)}(z,w) = \omega(0, \frac{|w-z|}{R}) \ge \frac{|w-z|}{R}, \quad \forall z, w \in \Omega, \tag{2.7}$$

where $\omega$ is the Poincaré distance on the unit disk $\Delta \subset C$ which is defined as

$$\omega(u,v) = \frac{1}{\sqrt{2}} \log \frac{|1-\overline{u}v| + |u-v|}{|1-\overline{u}v| - |u-v|}, \quad u, v \in \Delta.$$

So from (2.5), (2.6), (2.7), we know

$$|G_n(z) - G_n(w)| \le AR(\frac{1}{1+\delta})^{n-1}, \quad \forall z, w \in \Omega. \tag{2.8}$$

$$|G_n(z) - G_{n+p}(z)| \le AR(\frac{1}{1+\delta})^{n-1}, \quad \forall z \in \Omega. \tag{2.9}$$

By (2.9), $\{G_n(z)\}$ is a uniform Cauchy sequence in the Euclidean norm, so converges to some point $z_0 \in \overline{E}$ and (2.8) implies that $G_n(w) \longrightarrow z_0$, $w \in \Omega$, uniformly on $\Omega$.

**Remark 2.1.** It is worth while noticing that Theorem 2.1 is true for bounded domains in any complex Banach space without changing the proofs.

**Remark 2.2.** If we iterate these $f_n$'s in the opposite direction, we can not expect the truth of Theorem 2.1. In fact we have

**Example 2.1.** Let $\Omega = B$ be the unit ball of $C^N$,

$$f_{2n}(z) = \frac{1}{5}z, f_{2n-1}(z) = \frac{1}{5}(z - \frac{3}{5}e_1).$$

Obviously, $F_n(z) = f_n \circ f_{n-1} \circ \cdots \circ f_1(z)$ can not converge at any point of $\Omega$ .

**Remark 2.3.** From the hypothesis of Theorem 2.1 we know that each $f_n \in \mathcal{F}$ has a unique attracting fixed point lying in $E$. It is natural to ask:

If we let

$$\mathcal{F} = \{f \in H(\Omega, \Omega) | f \text{ has a unique attracting fixed point in } E \subset\subset \Omega\},$$

is Theorem 2.1 true for $\mathcal{F}$? The following example denies it.

**Example 2.2.** Let $f_n(z) = (1 - \frac{1}{(n+1)^2})z, \Omega = B = \{z \in C^N | \ |z| < 1\}$. Then $f_n \in \mathcal{F}$, but

$$G_n = f_1 \circ \cdots \circ f_n = \prod_{k=2}^{n+1} (1 - \frac{1}{k^2})z$$

converges to a linear map $f(z) = \frac{1}{2}z$, not a constant.

**Remark 2.4.** Similar result for one complex variable was obtained in [9] by Lisa Lorentzen, an expert in complex analysis well-known for continued fractions under her former name Lisa Jacobsen. But in [9] , Lorentzen only proved that the convergence is uniform on some subdomain of $\Omega$ . Surely, our methods are completely different from that in [9]. In fact her proof used the famous Riemann mapping theorem which is invalid in the high dimensional case, and consequently the method can only be applied to circular star-shaped domains in $C^N$.

## §3. Random Iterations Over Bounded Convex Domains

In what follows, we denote by $\|A\|$ for an $N \times N$ matrix $A$ its operator norm, that is, $\|A\| = \max\limits_{|z| \le 1} |Az|$. In order to avoid notational complexity, we always regard a vector in $C^N$ as the column vector when a matrix acts on it. We hope that this will not bring any confusion.

**Theorem 3.1.** *Let $\Omega \subset\subset C^N$ be a convex domain. $\mathcal{F} \subset H(\Omega, \Omega)$ satisfies:*

(I) *Any $f \in \mathcal{F}$ can be extended continuously to $\overline{\Omega}$ and $\|Df(z)\| \le 1, \forall z \in \Omega$.*

(II) *There exists a $z_0 \in \Omega$, such that*

$$\sup_{f \in \mathcal{F}} \|Df(z_0)\| < 1. \tag{3.1}$$

*Then for any sequence $f_n \in \mathcal{F}, G_n = f_1 \circ \cdots \circ f_n$, we have $G_n(z) \longrightarrow w_0 \in \overline{\Omega}$ .*

**Proof.** The theorem is a consequence of the following assertions.

**Assertion 3.1.**

$$\sup_{f \in \mathcal{F}} \|Df(z)\| < 1, \quad \forall z \in \Omega. \tag{3.2}$$

In fact, If $\exists a \in \Omega$ such that $\sup\limits_{f \in \mathcal{F}} \|Df(a)\| = 1$, then

$$\exists f_n \in \mathcal{F}, \quad f_n \longrightarrow f \in H(\Omega, \Omega), \quad \|Df(a)\| = 1.$$

So

$$\exists P \in C^N, \quad |P| = 1, \quad |Df(a)P| = |P| = 1.$$

Consider the holomorphic map $g(z) = Df(z)P \equiv (g_1(z), \ldots, g_N(z)), \ z \in \Omega$. Clearly $g \in H(\Omega, \overline{B})$. By $|g(a)| = 1$ we have $g(a) \stackrel{\text{def}}{=} b \in \partial B$. Now let

$$h(z) = \langle g(z), b \rangle = \sum_{j=1}^{N} g_j(z) \overline{b}_j.$$

Then $h$ is a holomorphic function on $\Omega$ , and

$$|h(z)| \leq |g(z)||b| = |g(z)| \leq 1, |h(a)| = |b|^2 = 1.$$

So by the maximum modulus principle, $h(z) \equiv h(a) = 1$ , and this in turn shows $g(z) \equiv b$. So $|Df(z)P| \equiv |b| = 1$, and $\|Df(z)\| \geq 1$. This means $\|Df(z)\| = 1$, which contradicts (II), so (3.2) is true.

**Assertion 3.2.** For any $f \in \mathcal{F}$, we have

$$|f(z) - f(w)| < |z - w|, \quad \forall z, w \in \overline{\Omega}. \tag{3.3}$$

If $z, w \in \Omega$, the closed straight line segment $L$, with endpoints $z$ and $w$, is a compact subset of $\Omega$. So by Assertion 3.1, $\exists\ \varepsilon > 0$, such that $\|Df(z)\| < 1 - \varepsilon$, $\forall z \in L$. For any $a \in L$, by the definition of $Df$ we have

$$\lim_{u \to a} \frac{|f(u) - f(a) - Df(a)(u - a)|}{|u - a|} = 0,$$

so $\exists\ \delta_a > 0$ such that when $u \in B(a, \delta_a)$,

$$\begin{aligned} |f(u) - f(a)| &\leq (1 + \varepsilon)|Df(a)(u - a)| \\ &\leq (1 + \varepsilon)\|Df(a)\||(u - a)| \\ &< (1 - \varepsilon^2)|u - a| \end{aligned}$$

holds. Now $L$ is covered by all these $B(a, \delta_a)'$s, $a \in L$ . So the compactness allows us to choose finitely many points $a_1 = z, a_2, ..., a_{n-1}, a_n = w$, ordered from $z$ to $w$ along $L$ such that

$$|f(a_j) - f(a_{j+1})| < (1 - \varepsilon^2)|a_j - a_{j+1}|, \quad j = 1, 2, ..., n - 1.$$

This implies

$$\begin{aligned} |f(z) - f(w)| &= |\sum_{j=1}^{n-1} f(a_j) - f(a_{j+1})| \\ &\leq \sum_{j=1}^{n-1} (1 - \varepsilon^2)|a_j - a_{j+1})| \\ &= (1 - \varepsilon^2)|z - w| \\ &< |z - w|. \end{aligned}$$

Next, if $z \in \partial\Omega$ or $w \in \partial\Omega$ or both $z, w \in \partial\Omega$, then we choose two points $a, b \in \Omega \cap L$ along the segment $L$ in the direction $z$ to $w$. By the above discussion, $\exists \varepsilon > 0$, such that

$$|f(a) - f(b)| < (1 - \varepsilon^2)|a - b|. \tag{3.4}$$

Now by the continuity of $f$ on $\overline{\Omega}$, $\exists a_1 \in [z, a], b_1 \in [b, w]$, such that

$$\begin{aligned} |f(z) - f(a_1)| &< \frac{\varepsilon^2}{2}|a - b|, \\ |f(w) - f(b_1)| &< \frac{\varepsilon^2}{2}|a - b|. \end{aligned} \tag{3.5}$$

Repeating the above discussion implies

$$\begin{aligned} |f(a_1) - f(a)| &< |a - a_1|, \\ |f(b) - f(b_1)| &< |b - b_1|. \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we have

$$\begin{aligned}|f(z)-f(w)| &\le |f(z)-f(a_1)|+|f(a_1)-f(a)| \\ &\quad +|f(a)-f(b)|+|f(b)-f(b_1)|+|f(b_1)-f(w)| \\ &< \frac{\varepsilon^2}{2}|a-b|+|a_1-a|+(1-\varepsilon^2)|a-b| \\ &\quad +|b-b_1|+\frac{\varepsilon^2}{2}|a-b| \\ &= |a_1-a|+|a-b|+|b-b_1| \\ &= |a_1-b_1|<|z-w|.\end{aligned}$$

(**Note.** Assertion 3.2 is true for all connected domains in $C^N$).

**Assertion 3.3.** Let $d=\operatorname{diam}(\overline{\Omega})$, consider the modulus of equicontinuity $\omega$ of $\mathcal{F}$

$$\omega(r)=\sup\{|f(z)-f(w)|\ |f\in\mathcal{F}, z,w\in\overline{\Omega}, |z-w|\le r\},\quad r\in[0,d].$$

We have

(a) $\omega$ is increasing on $[0,d]$;

(b) $\omega(r_1+r_2)\le\omega(r_1)+\omega(r_2),\quad r_1,r_2\ge 0,\quad r_1+r_2\le d$;

(c) $\omega(r)\le r,\quad 0\le r\le d$;

(d) $\omega$ is continuous on $[0,d]$;

(e) $\omega(r)<r,\quad 0<r\le d$.

In fact, (a) follows immediately from the definition of $\omega$, (c) is a consequence of (3.3).

To prove (b), for any $z,w\in\overline{\Omega}$ with $|z-w|\le r_1+r_2$, by the convexity of $\overline{\Omega}$, there must exist a $u\in\overline{\Omega}$ such that $|z-u|\le r_1$ and $|u-w|\le r_2$. So (12) gives

$$\begin{aligned}|f(z)-f(w)| &\le |f(z)-f(u)|+|f(u)-f(w)| \\ &\le \omega(r_1)+\omega(r_2),\quad \forall f\in\mathcal{F},\end{aligned}$$

this in turn proves (b).

Next, by (a), (b), (c) we know that if $r_2\ge r_1$, then

$$|\omega(r_2)-\omega(r_1)|=\omega(r_2)-\omega(r_1)\le\omega(r_2-r_1)\le r_2-r_1=|r_2-r_1|.$$

So $\omega$ is continuous on $[0,d]$ , and (d) is proved.

Finally, since $\Omega$ is convex, and so taut, $H(\Omega,\Omega)$ and $\mathcal{F}$ are normal. Furthermore by the hypotheses (I), (II) and the equicontinuity of $\mathcal{F}$ we see that $\mathcal{F}$ is closed. To prove (e), we use the proof by contradiction. Suppose now there exists an $r\in(0,d]$, such that $\omega(r)=r$. Then $\exists z_n,w_n\in\overline{\Omega}, f_n\in\mathcal{F}$ with $|z_n-w_n|\le r$ and $\lim\limits_{n\to\infty}|f_n(z_n)-f_n(w_n)|=r$ . Taking a subsequence if necessary, we may assume that both $z_n,w_n$ and $f_n$ converge. Let $z_n\to z, w_n\to w, z,w\in\overline{\Omega}, f_n\to f\in\mathcal{F}$. Then $|f(z)-f(w)|=r\ge|z-w|$. This contradicts (3.3), and the proof of Assertion 3.3 is completed.

Now we go to the proof of Theorem 3.1. First of all, for any compact $E\subset\overline{\Omega}$, any $f\in\mathcal{F}$, we have $\operatorname{diam}f(E)\le\omega(\operatorname{diam}E)$. So

$$\operatorname{diam}G_n(\overline{\Omega})\le\omega^n(\operatorname{diam}\overline{\Omega})=\omega^n(d).$$

By the continuity of $\omega$ and $\omega(r)<r$, it is easy to see $\lim\limits_{n\to\infty}\omega^n(r)=0,\quad\forall r\in[0,d]$, which implies $\operatorname{diam}G_n(\overline{\Omega})\to 0$, and so $G_n\to a\in\overline{\Omega}$.

**Remark 3.1.** Example 2.2 shows that condition (II) can not be dropped.

**Remark 3.2.** It is easily seen that the largest eigenvalue of $A$ is less than $\|A\|$ , so it is interesting to know whether or not Theorem 3.1 is true if we replace $\|A\|$ by the largest eigenvalue of $A$ in the conditions of Theorem 3.1.

## §4. More on Random Iterations

From the discussions in §2 and §3, we know that the random iteration $G_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z)$ converges to a point $a \in \overline{\Omega}$ under s ome conditions. Now it is natural to ask: What happens if we treat the random iteration $F_n(z) = f_n \circ f_{n-1} \circ \cdots \circ f_1(z)$ instead of $G_n$. Trivially, if all these $f_k$'s have no relations, nothing can be said about $F_n$. For example, let $\Delta \subset C$ be the unit disc, $f_{2n}(z) = \frac{1}{2}, f_{2n-1}(z) = 0$. They all are uniform contractions, but $F_n$ does not converge. However, if $f_n \to f$ for some $f \in H(\Omega, \Omega)$, we can give a complete answer to this question. The main results are

**Theorem 4.1.** *Let $\Omega \subset\subset C^N$ be a taut domain, $f \in H(\Omega, \Omega)$. Then, for any sequence $\{f_n\} \subset H(\Omega, \Omega)$ with $f_n \to f$, the full sequence $G_n = f_1 \circ f_2 \circ \cdots \circ f_n \to b$ for some $b \in \Omega$ when and only when $f$ has an attracting fixed point $a \in \Omega$.*

**Theorem 4.2.** *With the hypothesis as in Theorem 4.1, for any sequence $\{f_n\} \subset H(\Omega, \Omega)$ with $f_n \to f$, the full sequence $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$ converges to a point $b \in \Omega$ when and only when $f$ has an attracting fixed point $a \in \Omega$, and in this case we have $b = a$.*

The proofs of the above theorems will follow immediately from the following lemmas and propositions.

**Lemma 4.1.** *Let $A$ be an $N \times N$ upper triangular matrix. $\lambda$ is the largest modulus-of eigenvalue of $A, f(z) = Az$. If $\lambda < 1$, then for any $r > \lambda$ there are polydiscs $D^N(\beta)$ such that for any positive $\delta \leq 1$ we have $f(D^N(\delta\beta)) \subset D^N(r\delta\beta)$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_N)$, $\beta_N < \beta_{N-1} < \cdots < \beta_1 = K\beta_N, K$ is a constant, and*

$$D^N(\beta) = \{z \in C^N | \ |z_j| < \beta_j, \quad j = 1, 2, ..., N\}.$$

**Proof.** Let

$$A = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} & \ldots & a_{1N} \\ 0 & \lambda_2 & a_{23} & \ldots & a_{2N} \\ 0 & 0 & \lambda_3 & \ldots & a_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_N \end{pmatrix},$$

where $\{\lambda_j, j = 1, 2, ..., N\}$ are the eigenvalues of $A$. Denote

$$\lambda = \max_{1 \leq j \leq N} |\lambda_j| < 1, \quad a = \max_{1 \leq i,j \leq N} \{|a_{ij}|, 1\} \geq 1.$$

Choose $\varepsilon > 0$ so that $\varepsilon + \lambda \leq r$. Given $\beta_1 > 0$, we define $\beta_j = \frac{\varepsilon}{Na}\beta_{j-1} < \beta_{j-1}$, for $j = 2, 3, ..., N$. Then it is easy to see that $\beta_1 = (\frac{aN}{\varepsilon})^{N-1}\beta_N$.

Now note that

$$f(z) = \begin{pmatrix} \lambda_1 z_1 + a_{12} z_2 + \ldots + a_{1N} z_N \\ \lambda_2 z_2 + a_{23} z_3 + \ldots + a_{2N} z_N \\ \ldots\ldots \\ \lambda_{N-1} z_{N-1} + a_{N-1,N} z_N \\ \lambda_N z_N \end{pmatrix} \equiv (w_1, w_2, ..., w_N).$$

If $z \in D^N(\delta\beta)$, then

$$\begin{aligned}|w_j| &= |\lambda_j z_j + a_{j,j+1}z_{j+1} + \ldots + a_{jN}z_N| \\ &< \lambda\delta\beta_j + Na\delta\beta_{j+1} \\ &= \lambda\delta\beta_j + \varepsilon\delta\beta_j \le r\delta\beta_j, \quad j = 1,2,\ldots,N-1. \\ |w_N| &= |\lambda_N z_N| < \lambda|z_N| < \lambda\delta\beta_N < r\delta\beta_N.\end{aligned}$$

So we have proved $f(D^N(\delta\beta)) \subset D^N(r\delta\beta)$.

**Lemma 4.2.** *Let $\Omega \subset C^N$ be a pseudoconvex domain, $g \in H(\Omega), g(0) = 0$. Then there exist $g_j \in H(\Omega), j = 1,2,\ldots,N$, such that $g(z) = \sum\limits_{j=1}^{N} z_j g_j(z)$.*

**Proof.** This follows immediately from Theorem 5.3.1 of [8].

**Lemma 4.3.** *Let $\Omega \subset C^N$ be a pseudoconvex domain, $g \in H(\Omega)$. If the homogeneous expansion of $g$ at the point 0 has no terms with degree less than 2, then there exist $g_{jk} \in H(\Omega), j,k = 1,2,\ldots,N$, such that $g(z) = \sum\limits_{j,k=1}^{N} z_j z_k g_{j,k}(z)$.*

**Proof.** Since $g(0) = 0$, Lemma 4.2 is applicable, that is, $\exists g_j \in H(\Omega)$, such that

$$g(z) = \sum_{j=1}^{N} z_j g_j(z). \tag{4.1}$$

Now by the conditions on $g$ we know $\frac{\partial g}{\partial z_k}(0) = 0, \quad \forall k = 1,2,\ldots,N$. On the other hand, by (4.1) we have

$$\frac{\partial g}{\partial z_k}(z) = \sum_{j=1}^{N} z_j \frac{\partial g_j}{\partial z_k}(z) + g_k(z).$$

So $g_k(0) = 0$. Again applying Lemma 4.2 to $g_k, \exists g_{jk} \in H(\Omega)$, such that

$$g_k(z) = \sum_{j=1}^{N} z_j g_{kj}(z). \tag{4.2}$$

Combining (4.1) and (4.2) yields

$$g(z) = \sum_{j,k=1}^{N} z_j z_k g_{kj}(z).$$

**Corollary 4.1.** *Under the conditions of Lemma 4.3, for any $\Omega' \subset\subset \Omega, \exists C > 0$, such that $|g(z)| \le C|z|^2$.*

**Proof.** This is trivial, since all the $g_{jk}$'s are bounded on $\Omega'$.

**Proposition 4.1.** *Let $\Omega \subset C^N, f \in H(\Omega,\Omega), f^m(z) \to a \in \Omega$. Then there exists a neighbourhood $V$ of $a$, such that $f(V) \subset V' \subset\subset V$.*

**Proof.** Without loss of generality, we may assume $0 \in \Omega$ and $f^m(z) \to 0$. Then we have $f(0) = 0$, and

$$[Df(0)]^m = Df^m(0) \to 0.$$

So all the eigenvalues of $Df(0)$ are less than 1.

By Schur Theorem, there is a unitary matrix $Q$ such that

$$\overline{Q}'Df(0)Q = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} & \dots & a_{1N} \\ 0 & \lambda_2 & a_{23} & \dots & a_{2N} \\ 0 & 0 & \lambda_3 & \dots & a_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_N \end{pmatrix} \equiv M.$$

Let $U = \overline{Q}'(\Omega), \Omega = Q(U)$, and let $g(z) = \overline{Q}' \circ f \circ Q(z)$. Then $g \in H(U,U)$, and $Dg(0) = \overline{Q}'Df(0)Q = M, g(0) = 0$, so the expansion of $g$ at 0 is $g(z) = Mz + G(z)$, where $G(z)$ has only the terms with degree larger than 1. So by Lemma 4.1, for any $r > \lambda, r < 1, \exists\beta = (\beta_1, ..., \beta_N)$, $\beta_N < \beta_{N-1} < ... < \beta_1 = K\beta_N$, such that

$$M : D^N(\beta) \to D^N(r\beta),$$

where $\lambda = \max_{1\le j\le N} |\lambda_j| < 1$.

On the other hand, by Corollary 4.1, for some $B(0,R) \subset\subset U, \exists C > 0$, such that $|G(z)| < C|z|^2$. Now let $\varepsilon = \frac{1-r}{2}$, and let $\beta = (\beta_1, ..., \beta_N)$ be so chosen that $\beta_N < \frac{\varepsilon}{CNK^2}$ and $D^N(\beta) \subset B(0,R)$. Denote $G(z) = (g_1(z), g_2(z), ..., g_N(z))$. Then each $g_j$ is a holomorphic function with terms of degree 1 and 0 vanished. So for $z \in D^N(\beta)$, we have

$$\begin{aligned}|g_j(z)| &< |G(z)| < C|z|^2 < CN\beta_1^2 \\ &= CN(K\beta_N)^2 = CNK^2\beta_N\beta_N \\ &< \varepsilon\beta_N < \varepsilon\beta_j.\end{aligned}$$

So if we let $g(z) = Mz + G(z) = (w_1, w_2, ..., w_N)$, then

$$|w_j| < r\beta_j + \varepsilon\beta_j = (r+\varepsilon)\beta_j = \frac{1+r}{2}\beta_j < \beta_j.$$

So

$$g(D^N(\beta)) \subset D^N\Big(\frac{1+r}{2}\beta\Big) \subset\subset D^N(\beta).$$

Now let

$$V = Q(D^N(\beta)), \quad V' = Q\Big(D^N\Big(\frac{1+r}{2}\beta\Big)\Big) \subset\subset V.$$

Then

$$\begin{aligned}\overline{Q}' \circ f(V) &= \overline{Q}' \circ f \circ Q(D^N(\beta) = g(D^N(\beta)) \\ &\subset D^N\Big(\frac{1+r}{2}\beta\Big) \subset\subset D^N(\beta).\end{aligned}$$

So $f(V) \subset Q(D^N(\frac{1+r}{2}\beta)) = V' \subset\subset V$. This completes the proof.

Now we are in a position to prove our main Theorems 4.1 and 4.2.

**Proof of Theorem 4.1.** The *only when* part is trivial, so we only prove the *when* part. Suppose that $f$ has the point $a$ as its attracting fixed point, that is, $f^m \to a$. Then by Proposition 4.1 we have for some neighbourhood $V$ of $a$ that $f(V) \subset V' \subset\subset V$.

On the other hand, since $f_n \to f$, we can find an $M > 0$ and a $V''$ such that for all $n > M$, the following holds: $f_n(V) \subset V'' \subset\subset V$. Applying Theorem 2.1 to

$$G_{M,n} \equiv f_{M+1} \circ \cdots \circ f_n$$

on $V$ , one has $G_{M,n}(z) \to c \in V, \quad \forall z \in V$. Now

$$G_n = f_1 \circ \cdots \circ f_M \circ f_{M+1} \circ \cdots \circ f_n = G_M \circ G_{M,n}.$$

So

$$G_n(z) \to G_M(c) \equiv b \in \Omega, \quad \forall z \in V.$$

Finally, since $\Omega$ is taut, $\{G_n\}$ is a normal family. Let $G$ be any limit of convergent subsequence, $G \in H(\Omega, \overline{\Omega})$. Then the above discussion shows that $G|_V \equiv b$. So by the Uniqueness Theorem we have $G \equiv b$ .This completes the proof.

**Proof of Theorem 4.2.** For the same reason as in the proof of Theorem 4.1, we need only to prove the *when* part. All the same as in Theorem 4.1, $\exists V'' \subset\subset V, M > 0$, such that for all $n > M$ we have

$$f_n(V) \subset V'' \subset\subset V, \quad f(V) \subset V'' \subset\subset V.$$

As in the proof of Theorem 2.1, we can get for some $\varepsilon < 1$ that

$$K_V(f_n(z), f_n(w)) < \varepsilon K_V(z,w), \qquad \forall z, w \in V.$$

Let

$$F_{M,n} = f_n \circ f_{n-1} \circ \cdots \circ f_{M+1}.$$

Then

$$K_V(F_{M,n}(z), F_{M,n}(w)) \leq \varepsilon^{n-M} K_V(z,w) \to 0, \quad \forall z, w \in V.$$

This means $\operatorname{diam} F_{M,n}(V) \to 0$. So for any given convergent subsequence $F_{M,n_j}$, $\exists b \in V$ such that $F_{M,n_j}(z) \to b$, $\forall z \in V$. Since $\Omega$ is taut, $\{F_{M,n_j}\}$ is normal. The above result shows that $F_{M,n_j}(z) \to b$ for all $z \in \Omega$. Consequently, $F_{n_j}(z) = F_{M,n_j} \circ F_M(z) \to b$. Note that in the proof of Proposition 4.1 the neighbourhood $V$ of $a$ can be arbitrarily small, so we must have $b = a$. That is, $F_{n_j}(z) \to a$ for any convergent subsequence, so $F_n(z) \to a, \forall z \in \Omega$. The proof is finished.

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