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THE CAPACITY DENSITY AND THE HAUSDORFF DIMENSION OF FRACTAL SETS

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Abstract

This paper defines the upper capacity densities of the subsets of R^n, gets uniform lower bound of the upper capacity densities for $\mathcal{H}^s$-almost all points of the Hausdorff s-sets or the analytic sets with Hausdorff dimension s in R^n, which improves the results of Wen Zhiying and Zhang Yiping's paper in [1].

Keywords Capacity density, Hausdorff dimension, Fractal set.
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§1. Introduction

The classic Frostman's theorems show that the Hausdorff dimension and the capacity of a set are related. Wen Zhiying and Zhang Yiping[1] introduced capacity density and studied the relation between the capacity density and the Hausdorff dimension. They defined the upper t-capacity of E at a point x as

$$\overline{D}_t^E(x) = \limsup_{r\to 0}\frac{C_t(E\cap B_r(x))}{C_t(B_r(x))} \tag{1.1}$$

and prove that $\dim\{x\in E : \overline{D}_t^E(x) > 0\} = s$ for any $0 < t < s$ if E is an analytic set in R^n of Hausdorff dimension s. In this paper we first prove that $C_t(B_r(x)) = C(n,t)r^t$, where $C(n,t)$ is a positive constant, so we can define the upper t-capacity density of E at t as

$$\overline{D_t}(E,x) = \limsup_{r\to 0}\frac{C_t(E\cap B_r(x))}{(2r)^t} \tag{1.2}$$

which differs from (1.1) by only a constant coefficient. We prove that $\overline{D_t}(E,x) \geq 4^{-s}(s-t)/s$ at $\mathcal{H}^s$-almost all $x\in E$ for any $0<t<s$ if E is a Hausdorff s-set or an analytic subset of R^n with Hausdorff dimension s, which improves Wang Zhiying and Zhang Yiping's results in [1].

§2. Definitions and Notations

A Borel measure μ on R^n, of compact support and with $0<\mu(R^n)<\infty$, is called a mass distribution. The t-potential at a point x due to the mass distribution μ is defined as

$$\phi_t(x) = \int\frac{d\mu}{|x-y|^t}. \tag{2.1}$$

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The t-energy of μ is given by

$$I_t(\mu) = \int \phi_t(x) d\mu(x). \tag{2.2}$$

If E is a compact subset of R^n, the t-capacity of E, written as $C_t(E)$, is defined by

$$C_t(E) = \sup\{1/I_t(\mu) : E \text{ supports } \mu \text{ and } \mu(E) = 1\}. \tag{2.3}$$

For an arbitrary subset E of R^n, define

$$C_t(E) = \sup\{C_t(F) : F \text{ is compact, } F \subset E\}. \tag{2.4}$$

We define the upper t-capacity density of E at x as

$$\overline{D_t}(E, x) = \limsup_{r \to 0} \frac{C_t(E \cap B_r(x))}{(2r)^t}, \tag{2.5}$$

where $B_r(x)$ denotes the closed ball of radius r and centre x.

A subset E of R^n is called an s-set if E is $\mathcal{H}$-measurable and $0 < \mathcal{H}^s(E) < \infty$, where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure.

The upper s-dimensional Hausdorff densities of an s-set E at a point $x \in R^n$ are defined as

$$\overline{D}_s^h(E, x) = \limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s}. \tag{2.6}$$

We denote the Hausdorff dimension of E by $\dim E$.

§3. The Capacity Densities and the Hausdorff Dimension of the S-Sets and the Analytic Sets in R^n

Lemma 3.1. *$C_t(B_r(x)) = C_t(B_1(x)) \cdot r^t$, where the balls $B_r(x)$ and $B_1(x)$ are in R^n.*

Proof. Since t-capacities are translationally invariable, we only need to prove

$$C_t(B_r(0)) = C_t(B_1(0)) \cdot r^t.$$

For any mass distribution μ' supported by $B_1(0)$, we can define a mass distribution μ supported by $B_r(0)$ as follows:

$$\mu(E) = \mu'(E'), \tag{3.1}$$

where E is any Borel subset of R^n and

$$E' = \{x : x = y/r,\ y \in E\}. \tag{3.2}$$

Then

$$\begin{aligned}\phi(x) &= \int \frac{d\mu(y)}{|x-y|^t} = \frac{1}{r^t} \int_{B_r(0)} \frac{d\mu(y)}{\left|\frac{x}{r} - \frac{y}{r}\right|^t} \\ &= \frac{1}{r^t} \int_{B_r(0)} \frac{d\mu'(y')}{|x'-y'|^t} \\ &= \frac{1}{r^t} \phi_t'(x'),\end{aligned}$$

where $\phi_t'(x')$ is the t-potential at $x' = x/r$ due to the mass distribution μ'. Hence

$$\begin{aligned}I_t(\mu) &= \int \phi_t(x) d\mu(x) \\ &= r^{-t} \int \phi_t'(x') d\mu'(x') = r^{-t} I_t(\mu'),\end{aligned}$$

Using (2.3) we get

$$C_t(B_r(0)) \geq r^t \cdot C_t(B_1(0)).$$

Similarly we can prove that

$$C_t(B_1(0)) \geq r^t \cdot C_t(B_r(0)).$$

So

$$C_t(B_r(0)) = C_t(B_1(0)) \cdot r^t.$$

Lemma 3.2. *Let $B_1(0)$ be the unit closed ball of R^n, $0 < t < n$. Then $0 < C_t(B_1(0)) < \infty$.*

Proof. For any mass distribution μ on $B_1(0)$ with $\mu(B_1(0)) = 1$, we have

$$\phi_t(x) = \int \frac{d\mu(y)}{|x-y|^t} \geq \int \frac{d\mu(y)}{2^t} = 2^{-t}$$

at all $x \in B_1(0)$, so

$$I_t(\mu) = \int \phi_t(x) d\mu(x) \geq 2^{-t}.$$

Thus $C_t(B_1(0)) \leq 2^t$.

For any Borel subset E of R^n, we define

$$\mu(E) = \frac{m(E \cap B_1(0))}{m(B_1(0))}, \tag{3.4}$$

where m denotes the n-dimensional Lebesgue measure. Then μ is a mass distribution supported by $B_1(0)$ and $\mu(B_1(0)) = 1$. For any point $x_0 \in B_1(0)$, write $A = B_1(x_0) \cap B_1(0)$, $B = B_1(0) \backslash A$. Then

$$\begin{aligned}
\phi_t(x_0) &= \int \frac{d\mu(y)}{|x_0-y|^t} \\
&= \int_{B_1(0)} \frac{d\mu(y)}{|x_0-y|^t} \\
&= \int_A \frac{dm(y)}{|x_0-y|^t} + \int_B \frac{dm(y)}{|x_0-y|^t} \\
&\leq \int_A \frac{dm(x_0+y')}{|y'|^t} + \int_B dm(y') \\
&\leq \int_A \frac{dm(y')}{|y'|^t} + \int_A \frac{dm(y')}{|y'|^t} \\
&= \int_{B_1(0)} \frac{dm(y')}{|y'|^t} = \phi_t(0).
\end{aligned}$$

Since $m(B_r(0)) = a_n r^n$, where $a_n = \pi^{\frac{1}{2}n}/(\frac{1}{2}n)!$, we have

$$\begin{aligned}
\Phi_t(0) &= \int_0^1 \frac{dm(r)}{r^t} = \int_0^1 \frac{a_n n r^{n-1}}{r^t} dr \\
&= \frac{n}{n-t} \cdot a_n.
\end{aligned}$$

Hence

$$\begin{aligned}
I_t(\mu) &= \int \phi_t(x) d\mu(x) \leq \int \phi_t(0) d\mu(x) \\
&= n a_n/(n-t).
\end{aligned}$$

Finally we get

$$C_t(B_1(0)) \geq \frac{1}{I_t(\mu)} \geq \frac{n-t}{na_n} > 0.$$

Combining Lemma 3.1 and Lemma 3.2, we get

Lemma 3.3. *If $B_r(x)$ is the closed ball of R^n of radius r, $0 < t < n$, then $C_t(B_r(x)) = C(n,t)r^t$, where $C(n,t)$ is a positive constant depending only on n and t.*

Lemma 3.4. *Any s-set contains a closed subset differing from it by arbitrarily small measure.*

Proof. See Theorem 1.6 in [4].

Lemma 3.5. *If E is an s-set in R^n, then*

$$2^{-s} \leq \overline{D}_t^h(E,x) \leq 1 \tag{3.5}$$

at $\mathcal{H}^s$-almost all $x \in E$.

Proof. See Corollary 2.5 in [4].

Theorem 3.1. *If E is an s-set in R^n, then for any t such that $0 < t < s$,*

$$\overline{D_t}(E,x) \geq \frac{s-t}{s} \cdot 4^{-s} \tag{3.6}$$

at $\mathcal{H}^s$-almost all $x \in E$.

Proof. Since E is an s-set, $\overline{D}_s^h(E,x) \leq 1$ at $\mathcal{H}^s$-almost all $x \in E$. Let $\varepsilon_1 > 0$ and

$$F_k = \Big\{x : x \in E,\ \frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s} \leq 1 + \varepsilon_1 \ \text{ for all } r < k^{-1}\Big\}.$$

Then $\bigcup_{k=1}^{\infty} F_k$ consists of $\mathcal{H}^s$-almost all $x \in E$. Considering F_k such that $\mathcal{H}^s(F_k) > 0$, from Lemma 3.4 we know that there exists a sequence of closed sets $F_{ki} \subset F_k$ such that $\bigcup_{i=1}^{\infty} F_{ki}$ consists of $\mathcal{H}^s$-almost all $x \in F_k$. We can assume that $F_{ki} \subset F_{k(i+1)}$ and $\mathcal{H}^s(F_{ki}) > 0$ for all i. It is clear that

$$\frac{\mathcal{H}^s(F_{ki} \cap B_r(x))}{(2r)^s} \leq \frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s} \leq 1 + \varepsilon_1$$

at all $x \in F_{ki}$ for all $r < k^{-1}$. Using Lemma 2.5 we have

$$\overline{D}_s^h(F_{ki},x) \geq 2^{-s}$$

at $\mathcal{H}^s$-almost all $x \in F_{ki}$. Let $x \in F_{ki}$ and $\overline{D}_s^h(F_{ki},x) \geq 2^{-s}$. Then there exists a sequence of r_j decreasing to zero such that $r_j < (2k)^{-1}$ and $\mathcal{H}^s(F_{ki} \cap B_{r_j}(x)) \geq (2^{-s} - \varepsilon_2)(2r_j)^s$ for all $j = 1, 2, 3, \cdots$, where $\varepsilon_2 > 0$. For every Borel subset G of R^n, we define

$$\mu(G) = \frac{\mathcal{H}^s(G \cap F_{ki} \cap B_{r_j}(x))}{\mathcal{H}^s(F_{ki} \cap B_{r_j}(x))}. \tag{3.7}$$

It is easy to see that μ is a Borel measure supported by $F_{ki} \cap B_{r_j}(x)$ and $\mu(F_{ki} \cap B_{r_j}(x)) = 1$. Let $x' \in F_{ki} \cap B_{r_j}(x)$ and

$$m_{x'}(r) = \mu(B_r(x')).$$

Then

$$\begin{aligned} m_{x'}(r) &= \frac{\mathcal{H}^s(B_r(x') \cap F_{ki} \cap B_{r_j}(x))}{\mathcal{H}^s(F_{ki} \cap B_{r_j}(x))} \\ &\leq \frac{(1+\varepsilon_1)(2r)^s}{(2^{-s} - \varepsilon_2)(2r_j)^s} = br^s, \end{aligned} \tag{3.8}$$

where $r \le 2r_j$ and

$$b = \frac{1+\varepsilon_1}{2^{-s}-\varepsilon_2} \cdot \frac{1}{r_j^s}. \tag{3.9}$$

Hence

$$\begin{aligned}
\phi_t(x') &= \int \frac{d\mu(y)}{|x'-y'|^t} \\
&= \int r^{-t} dm_{x'}(r) \\
&= \Big[r^{-t} m_{x'}(r)\Big]_{0+}^{2r_j} + t \cdot \int_0^{2r_j} r^{-(t+1)} m_{x'}(r) dr \\
&\le b(2r_j)^{s-t} + tb \int_0^{2r_j} r^{-(t+1)+s} dr \\
&= b\frac{s}{s-t}(2r_j)^{s-t} = b' r_j^{-t},
\end{aligned} \tag{3.10}$$

where

$$b' = \frac{1+\varepsilon_1}{2^{-s}-\varepsilon_2} \cdot \frac{s}{s-t} 2^{s-t}. \tag{3.11}$$

We get

$$I_t(\mu) = \int \phi_t(x') d\mu(x') \le b' r_j^{-t}. \tag{3.12}$$

Hence

$$C_t(F_{ki} \cap B_{r_j}(x)) \ge \frac{1}{b'} \cdot r_j^t \tag{3.13}$$

and

$$\frac{C_t(F_{ki} \cap B_{r_j}(x))}{(2r_j)^t} \ge \frac{1}{b'} \cdot \frac{1}{2^t}.$$

Thus

$$\begin{aligned}
\limsup_{r\to 0} \frac{C_t(E \cap B_r(x))}{(2r)^t} &\ge \limsup_{r\to 0} \frac{C_t(F_{ki} \cap B_{r_j}(x))}{(2r_j)^t} \\
&\ge \frac{2^{-s}-\varepsilon_2}{1+\varepsilon_1} \cdot \frac{s-t}{s} \cdot 2^{-s}.
\end{aligned} \tag{3.14}$$

Since ε_1 and ε_2 can be arbitrarily small, we get

$$\overline{D_t}(E,x) \ge \frac{s-t}{s} \cdot \frac{1}{4^s}. \tag{3.15}$$

Since such a point x makes up $\mathcal{H}^s$-almost all points of E, we finish the proof.

Let $O_r(x)$ be the open ball of radius r and centre x. We define

$$\overline{D}'_t(E,x) = \limsup_{r\to 0} \frac{C_t(E \cap O_r(x))}{(2r)^t}. \tag{3.16}$$

Then we have

Lemma 3.6. *If E is a subset of R^n, then*

$$\overline{D}'_t(E,x) = \overline{D_t}(E,x) \tag{3.17}$$

at any $x \in R^n$.

Proof. It is apparent that $\overline{D_t}(E,x) \geq \overline{D}'_t(E,x)$. Let r_i be a sequence of radii decreasing to zero such that

$$\lim_{r\to 0} \frac{C_t(E\cap B_{r_j}(x))}{(2r_i)^t} = \overline{D_t}(E,x). \tag{3.18}$$

Let $r'_i = (1+\varepsilon)r_i$, where $\varepsilon > 0$. Then $O_{r'_i}(x) \supset B_{r_i}(x)$, so

$$\begin{aligned}\overline{D}'_t(E,x) &= \limsup_{r'_i\to 0} \frac{C_t(E\cap O_{r'_i}(x))}{(2r'_i)^t}\\ &\geq \lim_{r\to 0} \frac{C_t(E\cap B_{r_i}(x))}{(2r_i)^t(1+\varepsilon)^t}\\ &= \frac{1}{(1+\varepsilon)^t}\overline{D_t}(E,x).\end{aligned} \tag{3.19}$$

Letting $\varepsilon \to 0$, we have

$$\overline{D}'_t(E,x) \geq \overline{D_t}(E,x), \tag{3.20}$$

so

$$\overline{D}'_t(E,x) = \overline{D_t}(E,x). \tag{3.21}$$

Lemma 3.7. *Let E be a subset of R^n. Then*

(a) *$C_t(E\cap O_r(x))$ is a Borel-measurable function of x for each r;*

(b) *$\overline{D_t}(E,x)$, which equals $\overline{D}'_t(E,x)$, is a Borel-measurable function of x.*

Proof. (a) Given $r > 0$, $\alpha > 0$, write

$$F = \{x : C_t(E\cap O_r(x)) > \alpha\}. \tag{3.22}$$

Let $x \in F$. From (2.4) we know that there exists a compact set E_x such that $E_x \subset E\cap O_r(x)$ and $C_t(E_x) > \alpha$. Let

$$\delta_x = \sup\{|x_1 - x_2| : x_1 \in E_x,\ |x_2 - x| = r\}. \tag{3.23}$$

Since E_x and $\{x_2 : |x_2 - x| = r\}$ are two disjoint compact sets, we have $\delta_x > 0$. Let

$$F_x = \{y : |y - x| < \delta_x/2\}. \tag{3.24}$$

Then $O_r(y) \supset E_x$ for any $y \in F_x$, so $E_x \subset E\cap O_r(y)$ and

$$C_t(E\cap O_r(y)) \geq C_t(E_x) > \alpha. \tag{3.25}$$

Hence we have $y \in F$, so F is an open subset of R^n. This is true for all α, so we conclude that $C_t(E\cap O_r(x))$ is a Borel-measurable function of x for each r.

(b) Using part (a), we see that

$$\{x : C_t(E\cap O_r(x)) > \alpha(2r)^t\}$$

is open. Thus for two given positive integers m and n,

$$F_{mn} = \Big\{x : C_t(E, \cap O_r(x)) > \Big(\alpha + \frac{1}{n}\Big)(2r)^t \text{ for some } r < \frac{1}{m}\Big\} \tag{3.26}$$

is the union of such sets and so is open. Now

$$\{x : \overline{D}'_t(E,x) > \alpha\} = \bigcup_{n=1}^{\infty}\bigcap_{m=1}^{\infty} F_{mn}. \tag{3.27}$$

Hence $\{x : \overline{D}'_t > \alpha\}$ is a Borel set for each α, making $\overline{D}'_t(E, x)$ a Borel-measurable function of x.

In Euclidean space R^n, analytic sets are Souslin sets, so we have the following

Lemma 3.8. *If E is an analytic subset of R^n with $\mathcal{H}^s(E) = \infty$, then E contains an s-set.*

Proof. See Theorem 5.6 in [4].

About analytic sets we have the following

Lemma 3.9. *The analytic subsets of R^n are measurable under Hausdorff measures. If E is a Borel subset of R^n, then E is an analytic set. If $E_1, E_2, \cdots$ are analytic subsets of R^n, then $\bigcap\limits_{k=1}^{\infty} E_k$ is an analytic set.*

Proof. See [2].

Now we can prove the following result.

Corollary 3.1. *If E is an analytic subset of R^n with* $\dim E = s$, *then for any t such that $0 < t < s$,*

$$\overline{D_t}(E, x) \geq \frac{s-t}{s} \cdot \frac{1}{4^s} \tag{3.28}$$

at $\mathcal{H}^s$-almost all $x \in E$.

Proof. It is apparent that the corollary is true if $\mathcal{H}^4(E) = 0$. If $0 < \mathcal{H}^s(E) < \infty$, then it is the case of Theorem 3.1. We assume $\mathcal{H}^s(E) = \infty$. Let

$$A = \left\{x : x \in E \ \text{ and } \ \overline{D_t}(E, x) \geq \frac{s-t}{s} \cdot \frac{1}{4^s}\right\}, \tag{3.29}$$

$$B = E \backslash A \ \text{ and } \ B' = R^n \backslash A. \tag{3.30}$$

Using Lemma 3.7 we know that A is a Borel set, so B' is a Borel set. Thus from Lemma 3.9 and the fact that $B = B' \cap E$, we know that B is an analytic set. If $0 < \mathcal{H}^s(B) < \infty$, then from Theorem 3.1 we have

$$\overline{D_t}(E, x) \geq \overline{D_t}(B, x) \geq \frac{s-t}{s} \cdot \frac{1}{4^s} \tag{3.31}$$

at $\mathcal{H}^4$-almost all $x \in B$. If $\mathcal{H}^s(B) = \infty$, then using Lemma 3.8 we can get a subset F_B of B such that F_B is an s-set. So we get also

$$\overline{D_t}(E, x) \geq \overline{D_t}(F_B, x) \geq \frac{s-t}{s} \cdot \frac{1}{4^s} \tag{3.32}$$

at $\mathcal{H}^s$-almost all $x \in F_B$. Both the above cases contradict the definition of the set A, so we have $\mathcal{H}^s(B) = 0$, which completes the proof.

Using Theorem 3.1 and Corollary 3.1 and the fact that $C_t(E) = 0$, if $\mathcal{H}^t(E) < \infty$ we get immediately the following

Corollary 3.2. *If the subset E of R^n is an s-set or an analytic set, then*

$$\dim E = \sup\{t : \overline{D_t}(E, x) \not\equiv 0 \ \ \textit{for} \ \ x \in R^n\} \tag{3.33}$$

$$= \inf\{t : \overline{D_t}(E, x) \equiv 0 \ \ \textit{for} \ \ x \in R^n\}. \tag{3.34}$$

Corollary 3.1 and Corollary 3.2 improve the results of [1].

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