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OCCUPATION TIME PROCESSES OF FLEMING-VIOT PROCESSES**

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Abstract

Suppose that X_t is the Fleming-Viot process associated with fractional power Laplacian operator $-(-\triangle)^{\alpha/2}$ $0<\alpha\leq 2$, and $Y_t=\int_0^t X_s ds$ is the so-called occupation time process. In this paper, the asymptotic behavior at a large time and the absolute continuity of Y_t are investigated.

Keywords Fleming-Viot superprocess, Occupation time process, Asymptotic behavior, Absolute continuity.

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§1. Introduction

Fleming and Viot (1979) introduced a probability-measure-valued Markov process in a variation of a model for the distribution of allelic frequencies in a selectively neutral genetic population. Following previous authors (cf. Dawson, Dynkin), we refer to this kind of measure-valued processes as Fleming-Viot superprocesses (for short, FV-superprocesses). For the study of FV-superprocesses, Dawson and Hochberg[2] introduced useful techniques. They obtained some results on local structure and qualitative behaviors of a class of FV-superprocesses. In this paper, our interest is in the occupation time processes, the counterpart processes of DW-superprocesses have been extensively studied by many authors (for example, I. Iscoe[7,8], J. T. Cox and D. Griffeath[1], K. Fleischmann[5], and Sugitani[11]). We shall investigate the asymptotic behaviors at a large time, and motivated by Sugitani[11] we consider the absolute continuity of the occupation time processes.

In comparison with DW-superprocesses, FV-superprocesses are much more difficult to deal with. In fact, that DW-superprocesses are determined by Laplacian transforms enables us to use many existing results and tools in analysis, but most of them are invalid for FV-superprocesses. One of the main difficulties in this paper is to bound the high-order moments for the occupation time processes of FV-superprocesses, while for DW-superprocesses, it can be easily induced by Taylor's expansion into power-series (cf. [11]). This forces us to find a new way to approach our goals.

This paper is organized as follows. In Section 2, we will briefly review the foundations of theory of FV-superprocesses which is established by Fleming and Viot, Dawson and

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Hochberg. Section 3 introduces the related occupation time processes and presents our main results, their proof can be found in Section 4 and Section 5. In Section 6, we will give some remarks.

§2. Preliminary Notations and Results

We begin with considering $S \hat{=} R^d \cup \{\infty\}$, the one-point compactification of R^d, $\mathcal{B}(S)$, the σ-algebra of Borel subsets of S, and the space $M_1(S)$ of probability measures on S furnished with the topology of weak convergence of probability measures. $M_1(S)$ serves as the state-space for FV-superprocesses. Let $\Omega = C([0,\infty), M_1(S))$, the space of functions mapping $[0,\infty)$ into $M_1(S)$ that are continuous. We consider the canonical process $X : [0,\infty) \times \Omega \to M_1(S)$ defined by $X(t,\omega,A) \equiv \omega(t,A)$ for $A \in \mathcal{B}(S)$, $\omega \in \Omega, t \geq 0$. The distribution of FV-superprocesses X is determined by a mapping $\mu \to P^\mu$ from $M_1(S)$ into the space of probability measures on Ω, and $\{P^\mu, \mu \in M_1(S)\}$ satisfies the conditions

$$P^\mu(X(0) = \mu) = 1 \tag{2.1}$$

and for $\psi \in \mathcal{D}(L)$,

$$\psi(X(t)) - \int_0^t L\psi(X(s))ds \tag{2.2}$$

is a P^μ-martingale for each $\mu \in M_1(S)$, where L is a linear operator defined on the linear subspace $\mathcal{D}(L)$ of $C(M_1(S))$ and has the form

$$L\psi(\mu) \equiv \int_S A(\delta\psi(\mu)/\delta\mu(x))\mu(dx) + \int_S\int_S (\delta^2\psi(\mu)/\delta\mu(x)\delta\mu(y))Q(\mu : dx \times dy), \tag{2.3}$$

where $\delta\psi(\mu)/\delta\mu(x) = \lim_{\epsilon\downarrow 0}(\psi(\frac{\mu+\epsilon\delta_x}{1+\epsilon}) - \psi(\mu))/\epsilon$, $Q : M_1(S) \to M_1(S\times S)$ (quadratic fluctuation functional), A is the infinitesimal generator of a strongly continuous Markov semigroup on $C_0(R^d)$, the space of continuous functions on R^d which vanish at ∞, and δ_x represents a unit mass at the point $x \in R^d$ (see [6, 2, 10]). We note from [6] that X_t is a path continuous Feller process.

Suppose $Q(\mu : dx \times dy) = \frac{1}{2}(\mu(dx)\delta_x(dy) - \mu(dx)\mu(dy))$ and $A = -(-\triangle)^{\frac{\alpha}{2}} (0 < \alpha \leq 2)$, the fractional power of the Laplacian operator which generates the contraction semigroup P_t^α of a symmetric stable process. In this case, we can take the value space of FV-superprocesses to be $M_1(R^d)$, the family of probability measures on R^d. It is a convolution operator: $P_t^\alpha\phi(x) = \int_{R^d} p_t^\alpha(x-y)\phi(y)dy$, $\phi \in C(R^d)$.

Various properties of the density p_t^α which can be found in [4] or [8] are as follows:

Lemma 2.1.

1) *For* $0 < \alpha \leq 2$, $t > 0$, p_t^α *is smooth, symmetric and unimodal.*

2) *For* $0 < \alpha \leq 2$, $t > 0$, $x \in R^d$,

$$p_t^\alpha(t^{1/\alpha}x) = t^{-d/\alpha}p_1^\alpha(x). \tag{2.4}$$

3) *For* $0 < \alpha < 2$, $x \in R^d$, *with* $|x| \geq 1$

$$p_1^\alpha(x) \leq \frac{c}{|x|^{d+\alpha}}, \quad c \geq 0 \text{ a constant depending on } \alpha. \tag{2.5}$$

For $\alpha = 2$, $x \in R^d$, $p_1^2(x) = (4\pi)^{-d/2}\exp(-x^2/4)$.

Following Dawson and Hochberg, we define the nth moment measure $M_n(s,X(s);t: dx_1,\cdots,dx_n)$ which is a probability measure on S^n and satisfies the equality

$$P^{s,X(s)}\Big(\prod_{j=1}^n\langle\phi_j,X_t\rangle\Big)=\int_S\cdots\int_S\phi_1(x_1)\cdots\phi_n(x_n)M_n(s,X(s);t:dx_1\cdots dx_n),$$

where $\phi_i\in C(S)$, $i=1,2,\cdots n$; $\langle f,m\rangle\hat{=}\int fdm$ for any function f and measure m.

Let $\phi_1,\cdots,\phi_n\in C_K^2(R^d)$, $\psi(\mu)=\langle\phi_1,\mu\rangle\cdots\langle\phi_n,\mu\rangle$ (see [2]). Then, from the defined martingale problem for the Fleming-Viot superprocess, it follows that

$$\begin{aligned}\xi(t)=&\prod_{j=1}^n\langle\phi_j,X(t)\rangle-\prod_{j=1}^n\langle\phi_j,X(s)\rangle\\&-\int_s^t\Big(\sum_{j=1}^n\prod_{k=1,k\neq j}^n\langle\phi_k,X(u)\rangle\langle A\phi_j,X(u)\rangle\\&+\sum_{j=1}^n\sum_{i=1,i\neq j}^n\prod_{k=1,k\neq i,j}^n\langle\phi_k,X(u)\rangle[\langle\phi_j\phi_i,X(u)\rangle-\langle\phi_j,X(u)\rangle\langle\phi_i,X(u)\rangle]\Big)du\end{aligned}\tag{2.6}$$

is a P^μ-martingale for every μ and $0\le s\le t$. Then

$$\begin{aligned}&\int_S\cdots\int_S\prod_{j=1}^n\phi_j(x_j)M_n(s,X(s);t:dx_1\cdots dx_n)\\&=\prod_{j=1}^n\langle\phi_j,X(s)\rangle+\int_s^t\int_S\cdots\int_S\Big(\sum_{j=1}^n\Big\{\prod_{k=1,k\neq j}^n\phi_k(x_k)A\phi_j(x_j)\Big\}\\&-\frac12 n(n-1)\prod_{j=1}^n\phi_j(x_j)\Big)duM_n(s,X(s);u:dx_1\cdots dx_n)\\&+\frac12\int_s^t\int_S\cdots\int_S\Big(\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n\prod_{\substack{k=1\\k\neq i,j}}^n\phi_k(x_k)\delta(x_j-x_i)\Big)M_{n-1}\Big(s,X(s);u:\prod_{k=1,k\neq j}^n dx_k\Big)du.\end{aligned}\tag{2.7}$$

This implies that $M_n(s,\mu;.:.)$ satisfies the following system of partial differential equations in the weak sense:

$$\begin{aligned}&\partial M_n(t:dx_1,\cdots,dx_n)/\partial t\\&=\sum_{i=1}^n A_iM_n(t:dx_1,\cdots,dx_n)-\frac12 n(n-1)M_n(t:dx_1,\cdots,dx_n)\\&+\frac12\sum_{i=1}^n\sum_{j=1,j\neq i}^n M_{n-1}\Big(t:\prod_{p=1,p\neq i}^n dx_p\Big)\delta(x_i-x_j)\end{aligned}\tag{2.8}$$

with the initial condition $M_n(s;dx_1,\cdots,dx_n)=\prod_{i=1}^n X(s,dx_i)$. The initial-value problem given by (2.8) can be solved successively for $n=1,2,3,\cdots$ and $t>s$ as follows:

$$\begin{aligned}&M_n(s,X(s);t:dx_1,\cdots,dx_n)=k_{t-s}*\prod_{i=1}^n X(s,dx_i)\\&+\frac12\sum_{i=1}^n\sum_{j=1,j\neq i}^n\int_s^t k_{t-u}*[M_{n-1}(s,X(s);u:dx_1,\cdots,dx_{j-1},dx_{j+1},\cdots,dx_n)\delta(x_i-x_j)]du,\end{aligned}\tag{2.9}$$

where $*$ denotes the convolution and

$$k_t(x_1,\cdots,x_n)=\prod_{i=1}^{n}p_t^\alpha(x_i)\exp\Big(-\frac{n(n-1)}{2}t\Big).$$

Remark. The implication above is adopted from ([2], p.563), in which $A=\triangle$. Some typing errors in formulas (6.8), (6.9), and (6.11) in [2] are corrected in (2.6), (2.7), and (2.9) respectively. Particularly, from (2.9) we have

Lemma 2.2. *For $\phi,\phi_1,\phi_2\in b(R^d)$, the family of bounded Borel measurable functions on R^d, we have*

$$P^\mu\langle X_t,\phi\rangle=\langle\mu,P_t^\alpha\phi\rangle, \tag{2.10}$$

$$\begin{aligned}P^\mu(\langle X_t,\phi_1\rangle\langle X_t,\phi_2\rangle)&=e^{-t}\langle\mu,P_t^\alpha\phi_1\rangle\langle\mu,P_t^\alpha\phi_2\rangle\\&\quad+\int_0^t e^{-(t-s)}\langle\mu P_s^\alpha,P_{t-s}^\alpha\phi_1P_{t-s}^\alpha\phi_2\rangle ds.\end{aligned} \tag{2.11}$$

§3. Main Results

By the regularity of sample path of X_t, the integral $Y_t=\int_0^t X_s ds,\ t>0$ exists a.s. and defines a measure-valued process $\{Y_t,t\geq 0\}$, the occupation time process (associated to X). Obviously, Y_t has properties as follows:

Proposition 3.1. *Y_t, $P^\mu Y_t$ are path-continuous in t, and for any $t\geq 0$, $Y_t(S)=P^\mu Y_t(S)=t$. Moreover, $P^\mu Y_t$ is absolutely continuous with respect to Lebesgue measure (say λ). Denote by $n_t(x)$ the density of $P^\mu Y_t$. Then*

$$n_t(x)=\int_0^t ds\int_S p_s^\alpha(x,y)X(0,dy). \tag{3.1}$$

To the first and second moments of the random measure Y_t, we have

Proposition 3.2. *For any $\phi\in b\mathcal{B}$, $t\geq 0$, $\mu\in M_1(S)$ and $0\leq s\leq t$, we have*

$$P^\mu\langle Y_t-Y_s,\phi\rangle=\int_s^t\langle\mu,P_r^\alpha\phi\rangle dr, \tag{3.2}$$

$$\begin{aligned}P^\mu\langle Y_t-Y_s,\phi\rangle^2&=2\int_s^t dr\int_s^r du e^{-u}\langle\mu,P_r^\alpha\phi\rangle\langle\mu,P_u^\alpha\phi\rangle\\&\quad+2\int_s^t dr\int_s^r du\int_0^u dv e^{-(u-v)}\langle\mu P_v^\alpha,P_{u-v}^\alpha\phi P_{r-v}^\alpha\phi\rangle.\end{aligned} \tag{3.3}$$

Proof. 1) is obvious, we only need to prove 2). Consider

$$\begin{aligned}P^\mu\langle Y_t-Y_s,\phi\rangle^2&=2\int_s^t dr\int_s^r du P^\mu\langle X_r,\phi\rangle\langle X_u,\phi\rangle\\&=2\int_s^t dr\int_s^r du P^\mu\langle X_u,P_{r-u}^\alpha\phi\rangle\langle X_u,\phi\rangle\ \text{(by Markov property and (2.10))}.\end{aligned}$$

Combine with (2.11), this follows (3.3) as desired.

Proposition 3.1 shows that $Y_t(S)\to\infty$ as $t\to\infty$. Furthermore, by the increase of Y_t, we know that for any open set G, with probability one, $Y_t(G)$ converges as $t\to\infty$. So an interesting question arises, that is, what is the local behavior of Y_t as $t\to\infty$? The following theorem partially answers this question.

Theorem 3.1. (a) *For $d > \alpha$, with probability one, Y_t vaguely converges to some σ-finite measure Y_∞ on R^d, and $P^\mu Y_\infty$ also is a σ-finite measure which is absolutely continuous with respect to (w.r.t., for short) λ. Denote the density by n_∞. Then n_∞ is a.e. finite w.r.t. λ and given by*

$$n_\infty(x) = \int_S G_\alpha(x,y)X(0,dy), \tag{3.4}$$

where $G_\alpha(x,y) = \int_0^\infty p_t^\alpha(x,y)dt$ is the potential kernel of semigroup P_t^α. Especially when $\alpha = 2$, $G_2(x,y) = \frac{\Gamma(\frac{d}{2}-1)}{2\pi^{d/2}}|x-y|^{2-d}$.

(b) *For $d \le \alpha$, we have for any $\mu \in M_1(R^d)$*

$$\lim_{t\to\infty} \frac{P^\mu Y_t}{\gamma(t)} = p_1^\alpha(0)\lambda \quad \textit{vaguely}, \tag{3.5}$$

where

$$\gamma(t) = \begin{cases} \log t, & d/\alpha = 1, \\ \frac{\alpha}{\alpha-d} t^{1-d/\alpha}, & d/\alpha < 1. \end{cases} \tag{3.6}$$

We will prove this theorem in Section 4.

A typical interesting question for a measure-valued process is whether the random measure is absolutely continuous w.r.t. Lebesgue measure λ. In fact, this kind of questions has been studied by many authors (cf. Konno and Shiga[10], S. Sugitani[11], and K. Fleischmann[5], etc.) for DW-superprocesses and FV-superprocesses. In case of $d = 1$ and $1 < \alpha \le 2$, Konno and Shiga[10] proved that X_t $(t > 0)$ is absolutely continuous w.r.t. λ. On the other hand, Dawson and Hochberg[2] showed that if $\alpha = 2$, $d \ge 3$, Hausdorff-Besicovitch dimension of carrying sets of the corresponding FV-superprocess is less than or equal to 2, that is, the random measure is singular. Here, our interest is in the absolute continuity of Y_t. We have

Theorem 3.2. *Assume $d < 2\alpha$ and $\mu \in M_1(R^d)$ which satisfies*

$$\mu P_t^\alpha(x - \circledcirc) \quad \textit{is jointly continuous in } (t,x) \in [0,\infty) \times R^d. \tag{3.7}$$

Then there exists a family of nonnegative random variables $\{Y(t,x),\ t \ge 0,\ x \in R^d\}$ such that the following 1) *and* 2) *hold (P^μ-a.s.).*

1) *$Y(t,x)$ is jointly Hölder continuous in $t \ge 0$ and $x \in R^d$ with order less than $(2 - d/\alpha) \wedge (\alpha - d/2) \wedge 1/2$.*

2) *For every $\phi \in C_K(R^d)$ and $t > 0$, $\langle Y_t, \phi\rangle = \int_{R^d} Y(t,x)\phi(x)dx$.*

This theorem will be proved in Section 5.

§4. Proof of Theorem 3.1

We shall prove the theorem by the following lemmas. For simplicity, we will not distinguish constants if no confusion arises. Denote by $B(0,r)$ the ball of center at 0 with radius r.

Lemma 4.1.

$$P_t^\alpha 1_{B(0,r)}(x) \le \text{const.}\ t^{-d/\alpha}, \quad x \in R^d, \tag{4.1}$$

where the constant depends on r, d, and α. Moreover, $G_\alpha(x, B(0,r)) \triangleq \int_0^\infty P_t^\alpha 1_{B(0,r)} dt <$ const. $< \infty$ if $d > \alpha$.

Proof. By the Lemma 2.1, p_t^α is symmetric and unimodal. Hence

$$\begin{aligned}P_t^\alpha 1_{B(0,r)}(x) &= \int_{B(0,r)} p_t^\alpha(x-y)dy \\ &= \int_{B(0,r)} t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha}(x-y))dy \quad (\text{ by Lemma 2.1, 2)}) \\ &\le \text{const.} \int_0^r u^{d-1} t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha}u)du \le \text{const.} \int_0^{rt^{-1/\alpha}} u^{d-1} p_1^\alpha(u)du \\ &\le \text{const.}\, p_1^\alpha(0) \int_0^{rt^{-1/\alpha}} u^{d-1}du \ \le\ \text{const.}\, t^{-d/\alpha}.\end{aligned}$$

This immediately implies $G_\alpha(x, B(0,r)) < \text{const.} < \infty$ if $d > \alpha$.

Lemma 4.2. *Assume $d > \alpha$, with probability one, Y_t vaguely converges to a random σ-finite measure Y_∞ on R^d. Moreover, the formula* (3.4) *holds.*

Proof. Obviously, vague-$\lim\limits_{t\to\infty} Y_t$ exists from the increase of Y_t, $Y_\infty(S) = \infty$, and almost surely Y_∞ is a measure on $(R^d, \mathcal{B}(R^d))$ from the common arguments and measure extension theorem. Therefore, we only need to check the σ-finiteness of Y_∞. For $r > 0$, $\mu \in M_1(R^d)$, consider

$$\begin{aligned}P^\mu Y_t(B(0,r)) &= \int_0^t ds \int_S \mu(dx) \int_{B(0,r)} p_s^\alpha(x-y)dy = \int_S \mu(dx) \int_0^t ds \int_{B(0,r)} dy p_s^\alpha(x-y) \\ &\le \int_S \mu(dx) G_\alpha(x, B(0,r)) < \infty,\end{aligned}$$

that is, almost surely $Y_\infty(B(0,r))$ is finite. From this we can prove that Y_∞ is σ-finite. Repeating above computation, we can easily verify (3.4) by the dominated convergence theorem.

So far, we have proved assertion (a) in Theorem 3.1. The rest of this section is devoted to proving (b).

Lemma 4.3. *For $d \le \alpha$ and a fixed ball $B(0,r)$, we have*

$$P^\mu Y_t(B(0,r)) < \text{const.}\, \gamma(t), \ \textit{for } t > 0 \ \textit{large enough}, \tag{4.2}$$

where the constant depends on α, d and r, and γ is given by (3.6).

Proof. From Proposition 3.2 1), we have

$$\begin{aligned}P^\mu Y_t(B(0,r)) - P^\mu Y_2(B(0,r)) &= \int_2^t ds \int_{R^d} \mu(dx) \int_{B(0,r)} p_s^\alpha(x-y)dy \\ &\le \int_2^t \int_{B(0,r)} p_s^\alpha(x)dx \quad (\text{by the unimodality of } p_s^\alpha) \\ &\le \text{const.} \int_2^t s^{-d/\alpha}ds \quad (\text{by Lemma 4.1}) \\ &\le \text{const.} \begin{cases} \log t, & d/\alpha = 1, \\ t^{1-d/\alpha}, & d/\alpha < 1. \end{cases}\end{aligned}$$

Since Y_2 is finite, for $t > 0$ large enough formula (4.2) holds, and the proof is complete.

Lemma 4.4. *Assertion* (b) *in Theorem* 3.3 *is true.*

Proof. At first, we remark that

$$t^{d/\alpha} p_t^\alpha(x) = p_1^\alpha(xt^{-1/\alpha}) \le p_1^\alpha(0) < \infty. \tag{4.3}$$

Therefore, for any $f \in BC(R^d)$, by L'Hôpital's rule and the dominated convergence theorem we have

$$\begin{aligned}\lim_{t\to\infty}\frac{P^{\mu}\langle Y_t,f\rangle}{\gamma(t)} &= \lim_{t\to\infty}\frac{\int_0^t ds\int_{R^d}\mu(dx)\int_{R^d}p_s^{\alpha}(x-y)f(y)dy}{\gamma(t)}\\ &= \lim_{t\to\infty} t^{d/\alpha}\int_{R^d}\mu(dx)\int_{R^d}p_t^{\alpha}(x-y)f(y)dy\\ &= \int_{R^d}\mu(dx)\int_{R^d}\lim_{t\to\infty}p_1^{\alpha}((x-y)t^{-1/\alpha})f(y)dy\\ &= p_1^{\alpha}(0)\int_{R^d}f(y)dy = p_1^{\alpha}(0)\langle\lambda,f\rangle.\end{aligned}$$

Thus we complete the proof.

§5. Proof of Theorem 3.2

Throughout this section, we assume $d < 2\alpha$. Set $Y_h(t,x) = \langle Y_t, p_h^{\alpha}(x-\bullet)\rangle$.

Lemma 5.1. *If $\mu \in M_1(R^d)$ and (3.7) holds, then for $t > 0$ and $x \in R^d$,*

$$\sup_h P^{\mu}(Y_h(t,x))^2 < C(t,x) < \infty,$$

where $C(t,x)$ is bounded in $[0,T]\times K$ for any fixed $T>0$ and compact set K.

Proof. (3.3) shows that

$$\begin{aligned}P^{\mu}(Y_h(t,x))^2 = 2\int_0^t ds\int_0^s du\Big(&e^{-u}\langle\mu, P_{u+h}^{\alpha}(x-\bullet)\rangle\langle\mu,P_{s+h}^{\alpha}(x-\bullet)\rangle\\ &+\int_0^u dre^{-(u-r)}\langle\mu P_r^{\alpha}, P_{u-r+h}^{\alpha}(x-\bullet)P_{s-r+h}^{\alpha}(x-\bullet)\rangle\Big).\end{aligned}$$

We notice that $\int_0^t \mu p_s^{\alpha}(x-\bullet)ds$ is finite and $p_t^{\alpha} <$ const. $t^{-d/\alpha}$. Then

$$P^{\mu}(Y_h(t,x))^2 \le \mathrm{const}(t,x)\Big(1+\int_0^t (t-r)^{-d/\alpha}dr\Big) < C(t,x).$$

And it is easy to see that we can choose $C(t,x)$ to be locally bounded. So we have proved the lemma.

Lemma 5.2. *If $\mu \in M_1(R^d)$ and (3.7) holds, then for any $t \ge 0$ and $x \in R^d$,*

$$\lim_{h\to 0} Y_h(t,x) = Y(t,x) \text{ in the sense of } L^2 . \tag{5.1}$$

Proof. It is sufficient to prove that $\lim\limits_{h\to 0,\ h'\to 0} P^{\mu}|Y_h(t,x)-Y_{h'}(t,x)|^2 = 0$, and this can be easily verified by direct computation from Lemma 5.1 and the fact $\lim\limits_{h\to 0} P_{t+h}^{\alpha}(x) = P_t^{\alpha}(x)$.

From Lemma 5.1 and Lemma 5.2, we can easily verify 2) in Theorem 3.2 in the same manner as in [10]. We shall prove 1) in the remainder of this section.

Lemma 5.3. *For any $0 < \beta < (2\alpha - d)\wedge 1$,*

$$|p_t^{\alpha}(x) - p_t^{\alpha}(y)| \le \mathrm{const.}\,(\alpha,\beta)t^{-\frac{d+\beta}{\alpha}}|x-y|^{\beta}. \tag{5.2}$$

Proof. Recalling Lemma 2.1, we know that p_1^{α} is smooth and unimodal. Hence, $|p_1^{\alpha}(x) - p_1^{\alpha}(y)| \le$ const. $(\alpha)|x-y|$ and $p_1^{\alpha}(x) \le p_1^{\alpha}(0)$, $x \in R^d$. So if $|x-y| \le (2p_1^{\alpha}(0))^{1/\beta}$, then

$$|p_1^{\alpha}(x) - p_1^{\alpha}(y)| \le \mathrm{const}_1(\alpha,\beta)|x-y|^{\beta}; \tag{5.3}$$

if $|x-y| \geq (2p_1^\alpha(0))^{1/\beta}$, obviously, we also have

$$|p_1^\alpha(x) - p_1^\alpha(y)| \leq |x-y|^\beta. \tag{5.4}$$

Put const. $(\alpha,\beta) = \text{const}_1(\alpha,\beta) \vee 1$. (5.2) follows from (5.3), (5.4) and (2.4), and the proof is complete.

Lemma 5.4. *If $\mu \in M_1(R^d)$ and (3.7) holds, then for any fixed positive constant $K > 0$ and positive integer n, there exists a constant $C(K,n)$ such that for $0 < s \leq t \leq K$, $|x| \leq K$,*

$$P^\mu|Y(t,x) - Y(s,x)|^n \leq C(K,n)(t-s)^{((2-d/\alpha)\wedge 1/2)n}. \tag{5.5}$$

Proof. Let $\phi(\bullet) = p_h^\alpha(x - \bullet)$. Observe that

$$P^\mu|\langle Y_t,\phi\rangle - \langle Y_s,\phi\rangle|^n = n!\int_s^t dt_1 \int_s^{t_1} \cdots dt_{n-1}\int_s^{t_{n-1}} dt_n P^\mu\langle X_{t_1},\phi\rangle\cdots\langle X_{t_n},\phi\rangle.$$

Therefore, at first, we need to calculate

$$\begin{aligned}
&P^\mu\langle X_{t_1},\phi\rangle\cdots\langle X_{t_n},\phi\rangle\\
&= P^\mu\langle X_{t_2}, P^\alpha_{t_1-t_2}\phi\rangle\langle X_{t_2},\phi\rangle\cdots\langle X_{t_n},\phi\rangle\\
&\quad \text{(by Proposition 3.2 1) and Markov property)}\\
&= P^\mu(P^{X_{t_3}}\langle X_{t_2-t_3}, P^\alpha_{t_1-t_2}\phi\rangle\langle X_{t_2-t_3},\phi\rangle)\langle X_{t_3},\phi\rangle\cdots\langle X_{t_n},\phi\rangle\\
&= P^\mu\langle X_{t_n},\phi\rangle\langle X_{t_{n-1}},\phi\rangle\cdots\langle X_{t_3},\phi\rangle\cdot(e^{-(t_2-t_3)}\langle X_{t_3},P^\alpha_{t_1-t_3}\phi\rangle\langle X_{t_3},P^\alpha_{t_2-t_3}\phi\rangle\\
&\quad + \int_0^{t_2-t_3} dr e^{-(t_2-t_3-r)}\langle X_{t_3}, P_r^\alpha(P^\alpha_{t_1-t_3-r}\phi\, P^\alpha_{t_2-t_3-r}\phi)\rangle)\\
&\leq P^\mu\langle X_{t_n},\phi\rangle\langle X_{t_{n-1}},\phi\rangle\cdots\langle X_{t_3},P^\alpha_{t_1-t_3}\phi\rangle\langle X_{t_3},P^\alpha_{t_2-t_3}\phi\rangle\\
&\quad + \int_0^{t_2-t_3}(t_1-t_3-r+h)^{-d/\alpha}dr P^\mu\langle X_{t_n},\phi\rangle\langle X_{t_{n-1}},\phi\rangle\cdots\langle X_{t_3},\phi\rangle\langle X_{t_3},P^\alpha_{t_2-t_3}\phi\rangle\\
&\leq I_1 + \text{const.}(\alpha,d)I_2 \bullet \begin{cases}(t_1-t_2)^{-d/\alpha+1}, & \text{if } d/\alpha > 1,\\ t_1^{-d/\alpha+1}, & \text{if } d/\alpha < 1,\\ \log\frac{t_1-t_3}{t_1-t_2} & \text{if } d/\alpha = 1,\end{cases}
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= P^\mu\langle X_{t_n},\phi\rangle\langle X_{t_{n-1}},\phi\rangle\cdots\langle X_{t_3},P^\alpha_{t_1-t_3}\phi\rangle\langle X_{t_3},P^\alpha_{t_2-t_3}\phi\rangle,\\
I_2 &= P^\mu\langle X_{t_n},\phi\rangle\langle X_{t_{n-1}},\phi\rangle\cdots\langle X_{t_3},\phi\rangle\langle X_{t_3},P^\alpha_{t_2-t_3}\phi\rangle.
\end{aligned}$$

By formula (2.8), Markov property and repeating above computation for I_1 and I_2, in which we use trick as follows: whenever a term like $\int_0^{t_{i_2-1}-t_{i_2}} dr P^\alpha_{t_{i_1}-t_{i_2}-r}\phi P^\alpha_{t_{i_3}-t_{i_2}-r}\phi$ appears we pick out $P^\alpha_{t_{i_1}-t_{i_2}-r}\phi$ (resp. $P^\alpha_{t_{i_3}-t_{i_2}-r}\phi$) from the term if $i_2 \geq i_1+2$ (resp. $i_2 \geq i_3+2, i_2 = i_1+1$), and magnify $P^\alpha_{t_{i_1}-t_{i_2}-r}\phi$ (resp. $P^\alpha_{t_{i_3}-t_{i_2}-r}\phi$) to $(t_{i_1}-t_{i_2}-r)^{-d/\alpha}$ (resp. $(t_{i_3}-t_{i_2}-r)^{-d/\alpha}$), we can prove that $P^\mu\langle X_{t_1},\phi\rangle\cdots\langle X_{t_n},\phi\rangle \leq \text{const.}\sum_{i=1}^{n_0}\lambda_i$, where

n_0 is finite and λ_i is of forms as follows:

$$\begin{cases} \langle\mu, P^\alpha_{t_{\pi(1)}}\phi\rangle\cdots\langle\mu, P^\alpha_{t_{\pi(m)}}\phi\rangle t_{\pi(m+1)}^{-d/\alpha+1}\cdots t_{\pi(n)}^{-d/\alpha+1}, & d/\alpha<1 \\ \langle\mu, P^\alpha_{t_{\pi(1)}}\phi\rangle\cdots\langle\mu, P^\alpha_{t_{\pi(m)}}\phi\rangle\cdot & \\ \quad (t_{\pi(m+1)}-t_{\pi(m+1)+1})^{-d/\alpha+1}\cdots(t_{\pi(n)}-t_{\pi(n)+1})^{-d/\alpha+1}, & d/\alpha>1 \\ \langle\mu, P^\alpha_{t_{\pi(1)}}\phi\rangle\cdots\langle\mu, P^\alpha_{t_{\pi(m)}}\phi\rangle\cdot & \\ \quad \log\dfrac{t_{\pi(m+1)}-t_{\pi(m+1)+2}}{t_{\pi(m+1)}-t_{\pi(m+1)+1}}\cdots\log\dfrac{t_{\pi(n)}-t_{\pi(n)+2}}{t_{\pi(n)}-t_{\pi(n)+1}}, & d/\alpha=1. \end{cases}$$

Here π is a permutation of $\{1,2,\cdots,n\}$ satisfying $\pi(1)=n$, $1\le m\le n$, $t_{n+1}=0$.

Suppose that $\mu\in M_1(R^d)$ and (3.7) holds. We conclude that for $t\le K$, $|x|\le K$, and any λ_i described as above,

$$\begin{aligned} &\int_s^t dt_1\int_s^{t_1}dt_2\cdots\int_s^{t_{n-1}}dt_n\lambda_i \\ &\le\begin{cases} \text{const.}\,(K)(t-s)^m(t-s)^{(n-m)(2-d/\alpha)}, & d/\alpha\ne 1, \\ \text{const.}\,(K,\epsilon)(t-s)^{n(1-\epsilon)}, & d/\alpha=1,\ \ 0<\epsilon<1, \end{cases} \\ &\le\begin{cases} \text{const.}\,(K)(t-s)^{((2-d/\alpha)\wedge 1)n}, & d/\alpha\ne 1, \\ \text{const.}\,(K,\epsilon)(t-s)^{n(1-\epsilon)}, & d/\alpha=1,\ \ 0<\epsilon<1. \end{cases} \end{aligned}$$

In particular, choose $\epsilon=1/2$ in the case $d=\alpha$. Then

$$P^\mu|\langle Y_t,\phi\rangle-\langle Y_s,\phi\rangle|^n\le C(K,n)(t-s)^{((2-d/\alpha)\wedge 1/2)n} \tag{5.6}$$

holds for any $d<2\alpha$. At last, (5.1) and (5.6) yield (5.5) obviously.

In order to prove the joint continuity of $Y(t,x)$, the next lemma is also necessary.

Lemma 5.5. *If $\mu\in M_1(R^d)$ and (3.7) holds, then for any fixed positive constant $K>0$ and positive integer n, there exists a constant $C(K,n)$ such that for $t\le K$, $|x|,|y|\le K$,*

$$P^\mu|Y(t,x)-Y(t,y)|^{2n}\le C(K,n)|x-y|^{n\beta}, \tag{5.7}$$

where β is given in Lemma 5.3.

Proof. The proof of this lemma relies on sharper estimations. Let $\phi(\circledcirc)=p_h^\alpha(x-\circledcirc)-p_h^\alpha(y-\circledcirc)$ and notice that

$$P^\mu|\langle Y_t,p_h^\alpha(x-\circledcirc)\rangle-\langle Y_t,p_h^\alpha(y-\circledcirc)\rangle|^{2n}=P^\mu\langle Y_t,\phi\rangle^{2n} \tag{5.8}$$

is well defined since ϕ is bounded for $h>0$.

By an elementary (perhaps tedious) computation we claim that

$$P^\mu\langle Y_t,\phi\rangle^{2n}\le\sum_i\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{2n-1}}dt_{2n}\lambda_i \tag{5.9}$$

and λ_i is the product of the following three types of factors:

$T_1:\langle\mu,|P_s^\alpha\phi|\rangle$;

$T_2:\langle\mu,\int_0^{u_1}dr_{j_1}P^\alpha_{r_{j_1}}(|P^\alpha_{s_1+u_1-r_{j_1}}\phi||P^\alpha_{s_2+u_1-r_{j_1}}\phi|)\rangle$;

$T_3:\langle\mu,|\int_0^{u_1}dr_{j_1}P^\alpha_{r_{j_1}}(P^\alpha_{u_1-r_{j_1}}\phi_1P^\alpha_{u_1-r_{j_1}}\phi_2)|\rangle$, ϕ_i is of form $P_u^\alpha\phi$ or $\int_0^{u_2}dr_{j_2}P^\alpha_{s_i+r_{j_2}}(P^\alpha_{v_1+u_2-r_{j_2}}\phi_3P^\alpha_{v_2+u_2-r_{j_2}}\phi_4)$, $i=1,2$, ϕ_3 and ϕ_4 are of the same forms as ϕ_1, ϕ_2 and so on;

where

- $s=t_j$ for some $j=1,2,\cdots,2n$;

• s_1, s_2 are of form $t_j - t_{j+k}$, $k \geq 0$, $j = 1, 2, \cdots, 2n$, $|s_1 - s_2| \geq t_j - t_{j+1}$, $j_1 \neq j_2$, $t_{2n+1} \hat{=} 0$;

• $u_i = t_{j_i} - t_{j_i+1}$ for some $j_i : 1 \leq j_i \leq 2n$, $i = 1,\ 2$;

• $v_i = t_{k_i} - t_{k_i+k}$, $k \geq 1$ and $j_i = k_i + k$, or $v_i = r_{j_i}$ which is the integral variable with respect to the integral $\int_0^{t_{j_i} - t_{j_i+1}}$, $i = 1,\ 2$; $v_1 \neq v_2$, $k_1 \neq k_2$.

Moreover, in each λ, $P_u^\alpha \phi$-like term exactly appears $2n$ times.

Denote by $T_3(k)$, $k \geq 3$, the term that belongs to type T_3 and contains k $P_u^\alpha \phi$-like terms. Each λ_k can be represented by

$$\lambda_k = \underbrace{T_1(1)T_1(2)\cdots T_1(k_1)}_{T_1-\text{like terms}} \underbrace{T_2(1)\cdots T_2(k_2)}_{T_2-\text{like terms}} \underbrace{T_3(n_1)\cdots T_3(n_{k_3})}_{T_3-\text{like terms}}$$

where $k_1 + 2k_2 + \sum_{i=1}^{k_3} n_i = 2n$, all $P_u^\alpha \phi$-like terms in λ_k are of the forms $P_{t_i}^\alpha \phi, P_{t_i - r_j}^\alpha \phi$, or $P_{t_i - t_{j+1} - r_j}^\alpha \phi$, $j \geq i$ (here t_i in different $P_t^\alpha \phi$-like term is different) and r_i denotes the integral variable with respect to the integral $\int_0^{t_i - t_{i+1}}$, $i = 1, 2, \cdots, 2n$, $t_{2n+1} \hat{=} 0$.

On the other hand, from Lemma 5.3 and the argument in the case $n = 1$, it is trivial to prove that, for any T_2-like terms, it is less than

$$\text{const.}\,(K)(t_i - t_{i+1})^{-\frac{d+\beta}{\alpha}+1} |x - y|^\beta$$

for some $i = 1, 2, \cdots 2n$; for any $T_3(k)$, it is less than

$$\text{const.}\,(K)(t_{i_1} - t_{i_1+1})^{-\frac{d+\beta}{\alpha}+1} \cdots (t_{i_{k-1}} - t_{i_{k-1}+1})^{-\frac{d+\beta}{\alpha}+1} |x - y|^{(k-1)\beta} T_1$$

for some $\{i_1, \cdots, i_{k-1}\} \subset \{1, 2, \cdots, 2n\}$; and for any two different T_1-like terms, their product is less than

$$\text{const.}\,(K)(t_i - t_{i+1})^{-\frac{d+\beta}{\alpha}} |x - y|^\beta \quad \text{for some } i : 1 \leq i \leq 2n - 1.$$

To sum up, we have, for any possible k,

$$\lambda_k \leq \text{const.}\,(K) f_k(t_1, \cdots, t_{2n}) |x - y|^{(\frac{k_1+k_3}{2} + k_2 + \sum_{i=1}^{k_3}(n_i - 1))\beta},$$

where $f_k(t_1, \cdots, t_{2n})$ satisfies

$$\int_0^t dt_1 \int_0^{t_1} \cdots \int_0^{t_{2n-1}} dt_{2n} f_k(t_1, \cdots, t_{2n}) < \infty \quad \text{for} \quad t \leq K.$$

Obviously,

$$\frac{k_1 + k_3}{2} + k_2 + \sum_{i=1}^{k_3}(n_1 - 1) \geq n.$$

Then we have, for $|t| \leq K$, $|x|$, $|y| \leq K$,

$$P^\mu |\langle Y_t, \phi \rangle|^{2n} \leq \text{const.}\,(K)|x - y|^{n\beta},$$

this yields (5.7) from Lemma 5.2. So the proof of this lemma is complete.

From Lemma 5.4 and Lemma 5.5, the following lemma is obvious.

Lemma 5.6. *Under the same assumption on μ as previous lemmas, then for each integer $K \geq 1$*

$$P^\mu\Big(|Y(t,x) - Y(t,y)|^{2n} + |Y(t,x) - Y(s,x)|^{2n}\Big)$$
$$\leq \text{const.}\,(K)\Big(|x - y|^{2n((2-d/\alpha)\wedge\beta)} + |t - s|^{2n((2-d/\alpha)\wedge\beta)}\Big)$$

holds for $0 \le s,\ t \le K$ *and* $|x|,\ |y| \le K$.

Finally, we conclude that there exists a jointly Hölder continuous version of $Y(t,x)$ in t and x from Lemma 5.6 and [9, p.55]. The proof of Theorem 3.2 is complete.

§6. Some Remarks

In this section, we first comment on the condition (3.7). Clearly, (3.7) implies that μ is absolutely continuous w.r.t. λ. Particularly, when $t = 0$, we know that the Radon-Nikodym derivative is continuous. Therefore, we have

Proposition 6.1. *If* $\mu \in M_1(R^d)$ *such that* (3.7) *holds, then* μ *must be absolutely continuous and its Radon-Nikodym derivative is continuous.*

That is, the absolute continuity with a continuous Radon-Nikodym derivative is the necessary condition for a probability measure on R^d to satisfy (3.7). So we suppose that $\mu(dx) = g(x)dx$ and $g(x)$ is continuous. Next, we will give a sufficient condition:

Proposition 6.2. *The condition* (3.7) *holds if the continuous function* $g(x) \le$ *const.* $(1+|x|)^p$ *for some constant and some* $p \in R^1$ *when* $\alpha = 2$; $p < \alpha$ *when* $\alpha < 2$.

Proof. It suffices to verify (3.7) for $\alpha < 2,\ 0 \le t \le K,\ |x| \le K,\ K > 0$. Consider

$$\begin{aligned}\mu P_t^\alpha(x - \cdot) &= \int_{R^d} \mu(dy) p_t^\alpha(x - y) = \int_{R^d} dy g(y) t^{-d/\alpha} p_1^\alpha((x-y)t^{-1/\alpha}) \\ &= \int_{R^d} dy g(x - y t^{1/\alpha}) p_1^\alpha(y).\end{aligned}$$

Recall the subordination formula (see [5], p.288)

$$p_t^\alpha(x) = \int_0^\infty ds q_{\alpha/2}(t,s) p_s^2(x), \tag{6.1}$$

where $q_\eta(t,s),\ 0 < \eta < 1$, is the density function of a stable distribution on R_+ with Laplacian transform

$$\int_0^\infty ds q_\eta(t,s) e^{-s\theta} = \exp(-t\theta^\eta), \quad \theta \ge 0 \tag{6.2}$$

and satisfies

$$\int_0^\infty ds q_\eta(t,s) s^v < \infty, \quad v \in (-\infty, \eta). \tag{6.3}$$

We have

$$\mu P_t^\alpha(x - \bullet) = \int_0^\infty ds q_{\alpha/2}(1,s) \int_{R^d} dy p_s^2 g(x - y t^{1/\alpha}). \tag{6.4}$$

Set $f(s,t,x) = \int_{R^d} p_s^2 g(x - y t^{1/\alpha}) dy$. Then

$$f(s,t,x) \le \text{const.} \int_{R^d} p_s^2(y)(1 + |x - y t^{1/\alpha}|)^p dy \le \text{const.}\,(K)(s^{p/2} + 1).$$

Combining this with (6.3), (6.4), and the dominated convergence theorem, we see that the desired assertion follows easily, so the proof is complete.

In comparison with DW-superprocesses, there are much less literature concerning FV-superprocesses. Therefore, many interesting and important questions are still unknown, for example, the questions in two fields as follows.

(1) Hausdorff dimension and Hausdorff measure of the carrying set for X_t and Y_t. So far, to my best knowledge, only Dawson and Hochberg (1982) investigated the Hausdorff

dimension when $\alpha = 2$, and they only gave the upper bound of the Hausdorff dimension. Therefore, a further question is what the low bound is. For more general cases (i.e., $0 < \alpha \leq 2$), it is also interesting to investigate the Hausdorff dimensions of the carrying sets of X_t and Y_t. It seems that Hausdorff dimension for X_t is α, and that for Y_t is 4 when $\alpha = 2$ and d large enough. Another interesting questions is what the Hausdorff measures of X_t and Y_t are. The parallel questions for DW-superprocesses have been studied extensively by many authors (cf. Dawson, Iscoe, Perkins, Zäle, etc.).

(2) The asymptotic behavior of Y_t at a large time when $d \leq \alpha, 1 < \alpha \leq 2$. In the previous paragraphs, we have investigated the asymptotic behavior of Y_t at a large time when $d > \alpha$, and for $d \leq \alpha$ we have shown that, with probability one, $Y_t(E) \to \infty$ $(t \to \infty)$; for any open set $G \subset R^d$, $Y_t(G) \to \infty$ $(t \to \infty)$ with positive probability; and we have presented a limit result on the expectation processes $P^\mu Y_t$. But more precise description of the asymptotic behavior of Y_t in the case $d \leq \alpha$ is still unknown. So an interesting question is whether the following limit theorem is true for $1 \leq d \leq \alpha$, $1 < \alpha \leq 2$.

$$P^\mu\{\text{vague-}\lim_{t\to\infty} Y_t/\gamma(t) = \text{some nonzero } \sigma\text{-finite measure}\} = 1, \tag{6.5}$$

where $\gamma(t)$ is given by (3.6).

If not, what is the asymptotic rate? Or, does such limit not hold for any asymptotic rate?

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