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PURE STATE APPROACH TO $C(X)\times_\alpha \mathbb{Z}_n$**

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Abstract

Consider a C^*-system $(C(X), \mathbb{Z}_n, \alpha)$, where α is a homeomorphism of X such that $\alpha^n = id$. The authors characterize the pure state space of $C(X) \times_\alpha \mathbb{Z}_n$, the transition probability and orientation on it. Two special cases (free action and $n = 2$) are studied in detail.

Keywords C^*-crossed product, Pure state space, Transition probability, Orientation.
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The C^*-crossed product $C(X) \times_\alpha \mathbb{Z}_n$ has been studied for a long time. For instance, Effros and Hahn have studied the equivalence classes of the pure state of $C(X) \times_\alpha \mathbb{Z}_n$ (or primitive ideal space with Jacobson topology) (see [3]). When $\mathbb{Z}_n$ acts freely on X, their result showed $P_r(C(X)\times_\alpha\mathbb{Z}_n) \cong X/\alpha$, where X/α is the orbit space. But the understanding of this simple C^* algebra is still far from being complete. And it is not so clear why we should view $C(X) \times_\alpha \mathbb{Z}_n$ as a topological object. F. W. Shultz has shown that the pure state space carrying the W^*-topology, transition probability and orientation is dual (prefactly dual) to the C^*-algebra[1,9]. And for the C^*-algebra $C(X) \times_\alpha \mathbb{Z}_n$, it is not so hard to describe its dual (in the sense of Shultz). We feel that the C^*-algebra $C(X)\times_\alpha\mathbb{Z}_n$, especially the various topological phenomena on it, is better understood through its dual.

In this paper, based on X/α, we first explicitly characterize the pure state space of $C(X) \times_\alpha \mathbb{Z}_n$, specifying the W^*-topology (or W^*-closure), transition probability and orientation on it. Then we study two special cases in detail. One is when $\mathbb{Z}_n$ acts on X freely, the structure mentioned above on the pure state space agrees with the classical flat PU_n-bundle over X/α, where $PU_n = U_n/S^1$ and U_n is the $n\times n$ unitary matrix group. This gives a hand to study the structure of $C(X) \times_\alpha \mathbb{Z}_n$ through its dual, which is partly known by geometers and topologists. One interesting consequence of our work may be quoted here:

If X is a connected compact Hausdorff space, $\mathbb{Z}_n$ acts on X freely (the action is denoted by α) and $H^2(X/\alpha, \mathbb{Z})$ has no element annihilated by n (i.e., $na = 0 \Rightarrow a = 0$), then

$$C(Y) \times_\beta \mathbb{Z}_n \simeq C(X) \times_\alpha \mathbb{Z}_n \Leftrightarrow \mathbb{Z}_n$$

acts on Y freely and $X/\alpha \cong Y/\beta$.

Remember in this case $X/\alpha = P_r(C(X) \times_\alpha \mathbb{Z}_n)$.

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The second special case is when $n = 2$, the action of $\mathbb{Z}_2$ may not be free. In this case we may geometrically visualize the dual of $C(X) \times_\alpha \mathbb{Z}_n$ and we may think it to be picture of this nontrivial C^*-crossed product. An example of this case is given.

For simplicity, we shall assume X to be a connected compact Hausdorff space in this paper (some generalizations are obvious). And we identify a $\mathbb{Z}_n$-action $\alpha : \mathbb{Z}_n \to \mathrm{Aut}(C(X))$ with $\alpha(1)$, denoted still by the letter α. Thus $\alpha^n = id$. Also, a pure state of $C(X)$ wil be denoted by $ev_x, ev_x(f) = f(x)$. A vector state of $M_n(\mathbb{C})$ will be denoted by φ_λ,

$$\varphi_\lambda(T) = \langle T\lambda, \lambda\rangle / \langle \lambda, \lambda\rangle,$$

where λ is nonzero in $\mathbb{C}^n$.

Recall that $C(X) \times_\alpha \mathbb{Z}_n$ is the topological vector space $C(X), \times \cdots \times C(X)$ with the following $*$-algebraic operation:

$$f = (f_j)_{j\in\mathbb{Z}_n}, \quad g = (g_j)_{j\in\mathbb{Z}_n}, \quad f_j \text{ and } g_j \in C(X),$$

$$f \cdot g = \Big(\sum_{j\in\mathbb{Z}_n} f_j(g_{s-j} \circ \alpha^j)\Big)_{s\in\mathbb{Z}_n},$$

$$f^* = (\bar{f}_{n-j} \circ \alpha^j)_{j\in\mathbb{Z}_n}.$$

With these formulas in hand, it is easy to check

Proposition 1. *The map $\varepsilon : (f_j)_{j\in\mathbb{Z}_n} \mapsto (f_{i-j} \circ \alpha^{-i})_{i,j}$ is a $*$-algebraic embedding of $C(X) \times_\alpha \mathbb{Z}_n$ into $C(X) \otimes M_n$.*

Let $\varepsilon(f)(x_0)$ be denoted by $\varepsilon_{x_0}(f)$, $f \in C(X) \times_\alpha \mathbb{Z}_n$.

Corollary 1. *Any pure state of $C(X) \times_\alpha \mathbb{Z}_n$ is of the form $\varphi_\lambda \circ \varepsilon_{x_0}$, where φ_λ is a vector states on M_n.*

Proof. Any pure state of $\varepsilon(C(X) \times_\alpha \mathbb{Z}_n)$ can be extended to a pure state of $C(X) \otimes M_n$, which is the tensor product of two pure states $ev_{x_0} \otimes \varphi_\lambda$ (see [7,8]). Finally,

$$\varphi_{x_0,\lambda} = (ev_{x_0} \otimes \varphi_\lambda) \circ \varepsilon = \varphi_\lambda \circ \varepsilon_{x_0}.$$

Note that if $x_1 \notin \mathbb{Z}_n \cdot x_0$, then $\varphi_{x_1,\lambda_1} \ncong \varphi_{x_0,\lambda}$.

Now, the problem left to us is "for each x_0, which λ makes $\varphi_{x_0,\lambda}$ pure ?"

The following definition follows from the observation that $\varphi_{x_0,\lambda}(f)$ only depends on the values of f on the α-orbit of x_0.

Definition 1. *Let $x_0 \in X$, $X_0 = \alpha$-orbit of x_0, $r : C(X) \to C(X_0)$ the usual restriction map. Then*

$$C(X) \times_\alpha \mathbb{Z}_n \xrightarrow{r^*} C(X_0) \times_\alpha \mathbb{Z}_n$$

defined by

$$r^*((f_j)_{j\in\mathbb{Z}_n}) = (f_j|_{X_0})_{j\in\mathbb{Z}_n}$$

is an onto $*$-algebraic homomorphism, which is called the localization of $C(X) \times_\alpha \mathbb{Z}_n$ at x_0.

Note that

$$\varphi_{x_0,\lambda} = \psi_{x_0,\lambda} \circ r^*; \qquad (*)$$

where $\psi_{x_0,\lambda}$ is defined similarly on $C(X_0) \times_\alpha \mathbb{Z}_n$.

The advantage of this localization comes from two sides.

Te first side consists of two easy principles, which we state here without proof.

Lemma 1. *If $A \xrightarrow{\pi} B$ is an onto C^*-homomorphism, then any state φ on B is pure $\Leftrightarrow \varphi \circ \pi$ is pure on A. Moreover, if $\varphi_1 \not\simeq \varphi_2$ on B, then $\varphi_1 \circ \pi \not\simeq \varphi_2 \circ \pi$.*

Lemma 2. *If $\varphi_1 \simeq \varphi_2$ on B through a unitary in B of exponential form, then $\varphi_1 \circ \pi \simeq \varphi_2 \circ \pi$.*

Remark 1. The (*) formula above and Lemma 1 tell us that, for a given x_0, a λ makes $\varphi_{x_0,\lambda}$ pure if and only if it makes $\psi_{x_0,\lambda}$ pure.

The second side is the simplicity of $C(X_0) \times_\alpha \mathbb{Z}_n$.

We say $x_0 \in X$ is of p-degree, if $\alpha^p(x_0) = x_0$, but $\alpha^j(x_0) \neq x_0$ for all $1 \leq j < p$ (i.e., $\#\mathbb{Z}_n \cdot x_0 = p$). In this case, $pl = n$.

Case 1. x_0 is of n-degree.

In this case, $C(X_0) \times_\alpha \mathbb{Z}_n \overset{\varepsilon_{x_0}}{\underset{\sim}{\to}} M_n$ (ε_{x_0} is introduced as before).

Case 2. x_0 is of 1-degree.

In this case,

$$C(X_0) = \mathbb{C} \quad \text{and} \quad C(X_0) \times_\alpha \mathbb{Z}_n \overset{F}{\underset{\sim}{\to}} C(\widehat{\mathbb{Z}}_n),$$

where F is the classical Fourier transform defined by

$$F((a_j)_{j\in\mathbb{Z}_n})(z) = \sum_{j=0}^{n-1} a_j z^{-j},$$

where $z \in \widehat{\mathbb{Z}}_n = \{1, \gamma, \cdots, \gamma^{n-1}\}$ and $\gamma = e^{2\pi i/n}$.

Case 3. x_0 is of p-degree, $pl = n$.

This time, $\alpha^p = id$ on X_0 and each point of X_0 is p-degree. Moreover, we have

Proposition 2.

$$\begin{aligned} C(X_0) \times_\alpha \mathbb{Z}_n &\overset{\psi}{\underset{\sim}{\to}} (C(X_0) \times_\alpha \mathbb{Z}_p) \otimes C([0, l-1]) \\ &\overset{\varepsilon_{x_0} \times id}{\underset{\sim}{\to}} M_p \otimes C([0, l-1]), \end{aligned}$$

where $[0, l-1] \overset{\text{def.}}{=} \{0, 1, \cdots, l-1\}$, and if $f = (f_{tp+j})_{0\leq t\leq l-1,\ 0\leq j\leq p-1}$ is in $C(X_0) \times_\alpha \mathbb{Z}_n$, then

$$\psi(f)(k) = \Big(\sum_{t=0}^{l-1} f_{tp+j} \gamma^{-k(tp+j)}\Big)_{0\leq j\leq p-1}, \quad \forall k \in [0, l-1].$$

Since this result is a finite version of a well-known result about mapping torus that

$$C(X) \times_\alpha \mathbb{Z}_n \simeq M_{\hat{\alpha}}(C(X_0) \times_\alpha \mathbb{Z}_p),$$

we only sketch the main line of proof (see [2]).

Let $C(X_0)$ be denoted by $A, \widehat{\mathbb{Z}}_l = \{1, \delta, \cdots, \delta^{l-1}\}$, where $\delta = \gamma^p = e^{2\pi i/l}$. Then any function g on $\widehat{\mathbb{Z}}_l$ can be converted into the function $\tilde{g}$ on $[0, l-1]$ by $\tilde{g}(k) = g(\delta^k)$. Define a function β on $C([0, l-1])$ by $\beta(k) = \gamma^{-k}$. Write

$$\Omega = \{(f_t)_{t\in\mathbb{Z}_n} : f_t \in A,\ f_t = 0 \text{ if } t \not\equiv 0 \,(\text{mod}\, p)\}.$$

Clearly, Ω is a C^*-subalgebra of $A \times_\alpha \mathbb{Z}_n$ and $\Omega \overset{\sim}{\to} C(\widehat{\mathbb{Z}}_l, A)$ by Fourier transform

$$\Lambda : (f_{pt})_{0\leq t\leq l-1} \to \sum_{t=0}^{l-1} f_{pt} z^{-t}, \quad z \in \widehat{\mathbb{Z}}_l.$$

Let $(0,1,0,\cdots,0)$ in $A\times_\alpha \mathbb{Z}_n$ be denoted by λ, then

$$\lambda^{-1}=\lambda^*=(0,\cdots,0,1) \text{ and } \lambda^{\pm+p}\in\Omega,\quad \hat{\lambda}=\beta^{\pm+p}.$$

Now we can uniquely decompose an element of $A\times_\alpha \mathbb{Z}_n$ into $\sum_{j=0}^{p-1} a_j\lambda^j$ and the map $(a_0,\cdots,a_{p-1})\to\sum_{j=0}^{p-1} a_j\lambda^j$ is a topological linear isomorphism of Ω^p onto $A\times_\alpha \mathbb{Z}_n$.

Think of $A\subseteq A\times_\alpha \mathbb{Z}_p$ in the natural way, and let $L=(0,1,0,\cdots,0)$ be in $A\times_\alpha \mathbb{Z}_p$, then we may write

$$A\times_\alpha \mathbb{Z}_p=\Big\{\sum_{j=0}^{p-1} f_jL^j:\ f_j\in A\Big\}.$$

In fact, $(f_j)_{j\in\mathbb{Z}_p}=\sum_{j=0}^{p-1} f_jL^j$.

Then the composition of the following maps is a topological linear isomorphism:

$$\begin{array}{ccccccc}\sum_{j=0}^{p-1} a_j\lambda^j & \to & (a_j)_{j\in\mathbb{Z}_p} & \to & (\tilde{\hat{a}}_j)_{0\le j\le p-1} & \to & \sum_{j=0}^{p-1}\tilde{\hat{a}}_j\beta^jL^j\\ \pitchfork & & \pitchfork & & \pitchfork & & \pitchfork\\ A\times_\alpha \mathbb{Z}_n & & \Omega^p & & (C([0,l-1],A))^p & & C([0,l-1],A\times_\alpha \mathbb{Z}_p)\end{array}$$

Let it denoted by ψ. It is easy to check that this ψ is also a $*$-algebraic isomorphism of $A\times_\alpha \mathbb{Z}_n$ onto

$$C([0,l-1],A\times_\alpha \mathbb{Z}_p)=(A\times_\alpha \mathbb{Z}_p)\otimes C([0,l-1]).$$

Finally, if $f=(f_{tp+j})_{0\le t\le l-1,\ 0\le j\le p-1}$ is in $A\times_\alpha \mathbb{Z}_n$, then $f=\sum_{j=0}^{p-1} a_j\lambda^j$, where $a_j=(g_s^{(j)})_{0\le s\le n-1}$ with

$$g_s^{(j)}=\begin{cases} f_{s+j}, & s\equiv 0(\mathrm{mod}\,p),\\ 0, & \text{otherwise.}\end{cases}$$

So

$$\tilde{\hat{a}}_j(k)=\hat{a}_j(\delta^k)=\sum_{t=0}^{l-1} f_{tp+j}\delta^{-kt}$$

and then $\psi(f)(k)$ has the desired formula.

Now we can apply these results to the pure state problem we set before.

Caes 1. x_0 is of n-degree.

The following diagram obviously commutes:

$$\begin{array}{ccc} C(X)\times_\alpha \mathbb{Z}_n & \xrightarrow{r^*} & C(X_0)\times_\alpha \mathbb{Z}_n\\ & \mathcal{E}_{x_0}\searrow & \big\downarrow \mathcal{E}_{x_0}\\ & & M_n\end{array}$$

which shows the subjectivity of $\mathcal{E}_{x_0}$ from $C(X)\times_\alpha \mathbb{Z}_n$ into M_n. Since $\varphi_{x_0,\lambda}=\varphi_\lambda\circ\mathcal{E}_{x_0}$, Lemmas 1, 2 together with general relation of a pure state and its associated irreducible representation yield

Theorem 1. *If x_0 is of n-degree, then $\psi_{x_0,\lambda}$ is pure for any $[\lambda] \in \mathbb{P}(\mathbb{C}^n)$. Moreover, $\varphi_{x_0,\lambda}$'s are all mutually unitarily equivalent. The correspoding primitive ideal is*

$$J^{(n)}(x_0) = \ker r^* = \{f = (f_j)_{j\in \mathbb{Z}_n} : f_j(\alpha^i(x_0)) = 0, \quad \forall 0 \le i, j \le n-1\}.$$

Remark 2. It is clear that $C(X) \times_\alpha \mathbb{Z}_n / J^{(n)}(x_0) \simeq M_n$.

Case 2. x_0 is of 1-degree.

In this case, $\varphi_{x_0,\lambda}$ is pure if and only if it is one of

$$\varphi_k = ev_{\gamma^k} \circ F \circ r^*, \quad 0 \le k \le n-1,$$

where $\gamma = e^{2\pi i/n}$, $ev_{\gamma^k} : C(\widehat{\mathbb{Z}}_n) \to \mathbb{C}$, and $F : C(X_0) \times_\alpha \mathbb{Z}_n \xrightarrow{\sim} C(\widehat{\mathbb{Z}}_n)$ are defined as before. Note that $\varphi_k(f) = \sum_{j=0}^{n-1} f_j(x_0)\gamma^{-kj}$ are all multiplicative, thus the associated primitive ideal is

$$J_K^{(1)}(x_0) = \ker \varphi_k = \{f \in C(X) \times_\alpha \mathbb{Z}_n : (f_0(x_0), \cdots, f_{n-1}(x_0)) \perp \delta_k\},$$

$0 \le k \le n-1$, where $\delta_k = \{1, \gamma^k, \cdots, \gamma^{(n-1)k})/\sqrt{n}$.

Theorem 2. *If $\alpha(x_0) = x_0$, then $\varphi_{x_0,\lambda}$ is pure $\Leftrightarrow$ $[\lambda] = [\delta_k]$, and $\varphi_{x_0,\lambda} = \varphi_k$ for some $0 \le k \le n-1$. Moreover, $k_1 \ne k_2$ implies $\varphi_{x_0,\delta_{k_1}} \not\sim \varphi_{x_0,\delta_{k_2}}$.*

Proof. Let

$$A_n = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}_{n\times n},$$

then $A_n\delta_k = \gamma^{-k}\delta_k$. Since A_n is a unitary, $\{\delta_k\}_{0\le k\le n-1}$ forms an orthonormal basis of $\mathbb{C}^n$. In particular, if $k_1 \ne k_2$, then $J_{k_1}^{(1)}(x_0) \ne J_{k_2}^{(1)}(x_0)$, so $\varphi_{k_1} \not\sim \varphi_{k_2}$.

Since $(f_{i-j}(x_0))_{i,j} = \sum_{j=0}^{n-1} f_j(x_0)A_n^j$, if $\|\lambda\| = 1$,

$$\varphi_{x_0,\lambda}(f) = \sum_{j=0}^{n-1} f_j(x_0)\langle A_n^j\lambda, \lambda\rangle.$$

From this, it is easy to see that $\varphi_{x_0,\delta_k} = \varphi_k$, and if $\lambda = \sum \alpha_k\delta_k$ ($\sum|\alpha|^2 = 1$), then $\varphi_{x_0,\lambda} = \sum_{k=0}^{n-1} |\alpha_k|^2\varphi_k$. So if $|\lambda| \ne |\delta_k|$ for all k, then $\varphi_{x_0,\lambda}$ is not pure.

Case 3. x_0 is of p-degree, $1 < p < n$, $pl = n$.

In this case, the pure state on $C(X_0) \times_\alpha \mathbb{Z}_p$ returns to the case 1 discussed. Thus, $\varphi_{x_0,\lambda}$ is pure if and only if it is one of

$$\varphi_{x_0,\mu,k} = (\varphi_{x_0,\mu} \otimes ev_k) \circ \psi \circ r^*,$$

where $[\mu] \in \mathbb{P}(\mathbb{C}^p)$, $0 \le k \le l-1$, and ψ is given in Proposition 2 above.

We can picture the definition of $\varphi_{x_0,\mu,k}$ by the following diagram:

$$
\begin{array}{ccc}
C(X)\times_\alpha \mathbb{Z}_n & \xrightarrow{r^*} & C(X_0)\times_\alpha \mathbb{Z}_n \\
 & \xrightarrow[\sim]{\psi} & (C(X_0)\times_\alpha \mathbb{Z}_p)\otimes C([0,l-1]) \\
 & \xrightarrow{\varepsilon_{x_0}\otimes ev_k} & M_p \\
\varphi_{x_0,\mu,k}\searrow & & \downarrow \varphi_\mu \\
 & & \mathbb{C}
\end{array}
$$

The first thing coming out from this picture is the primitive ideal associated to $\varphi_{x_0,\mu,k}$ $(0\le k\le l-1)$:

$$
\begin{aligned}
J_k^{(p)}(x_0) &= \ker[(\varepsilon_{x_0}\otimes ev_k)\circ\psi\circ r^*] \\
&= \{f=(f_{tp+d})_{t,d}\in C(X)\times_\alpha \mathbb{Z}_n:\ (f_{tp+j}\circ\alpha^j(x_0))_{0\le t\le l-1}\perp\tilde{\delta}_k,\ 0\le i,j\le p-1\},
\end{aligned}
$$

where $\tilde{\delta}_k=(1,\delta^k,\cdots,\delta^{(l-1)k}/\sqrt{l}$. Clearly,

$$C(X)\times_\alpha \mathbb{Z}_n/J_k^{(p)}(x_0)\cong M_p.$$

Set

$$
A_l=\begin{pmatrix} 0 & & & 1\\ 1 & \ddots & & 0\\ & \ddots & \ddots & \\ 0 & & 1 & 0\end{pmatrix}_{l\times l},
$$

then $A_l\tilde{\delta}_k=\delta^{-k}\tilde{\delta}_k,\ 0\le k\le l-1$. By normality of A_l, $\{\tilde{\delta}_k\}_{0\le k\le l-1}$ forms an orthonormal basis of $\mathbb{C}^l$. Thus $k_1\ne k_2$, $J_{k_1}^{(p)}(x_0)\ne J_{k_2}^{(p)}(x_0)$, which implies $\varphi_{x_0,\mu,k_1}\not\simeq\varphi_{x_0,\mu,k_2}$.

Since $\varphi_{\mu_1}\simeq\varphi_{\mu_2}$ must be through a unitary of exponential form and $(\varepsilon_{x_0}\otimes ev_k)\circ\psi\circ r^*$ is onto M_p, $\varphi_{x_0,\mu_1,k}\simeq\varphi_{x_0,\mu_2,k}$.

Now for each pair $(\mu,\nu)\in\mathbb{C}^p\times\mathbb{C}^l$, we define $\mu\otimes\nu\in\otimes\mathbb{C}^{pl}$ by

$$(\mu\otimes\nu)_{pt+d}=\mu_d\nu_t,\ 0\le t\le l-1,\ \ 0\le d\le p-1.$$

Lemma 3. *If $\mu,\mu'\in\mathbb{C}^p$, $\nu,\nu'\in\mathbb{C}^l$, then for each $f\in C(X)\times_\alpha\mathbb{Z}_n$*

$$
\begin{aligned}
\langle\varepsilon_{x_0}(f)\mu\otimes\nu,\mu'\otimes\nu'\rangle &= \sum_{\substack{0\le t\le l-1\\ 0\le a\le p-1}}\Big[\sum_{d=0}^{a}\overline{\mu'_a}\mu_{a-d}\langle A_l^t\nu,\nu'\rangle(f_{tp+d}\circ\alpha^{-a}(x_0))\Big] \\
&\quad+\sum_{\substack{0\le t\le l-1\\ 0\le a\le p-1}}\Big[\sum_{d=a+1}^{p-1}\overline{\mu'_a}\mu_{p+a-d}\langle A_l^{t+1}\nu,\nu'\rangle(f_{tp+d}\circ\alpha^{-a}(x_0))\Big].
\end{aligned}
$$

Proof. For any pair $(\lambda,\lambda')\in S^{2n-1}\times S^{2n-1}$, we have

$$\begin{aligned}\langle\varepsilon_{x_0}(f)\lambda,\lambda'\rangle &= \sum_{i,j\in\mathbb{Z}_n} f_{i-j}\circ\alpha^{-i}(x_0)\overline{\lambda'_i}\lambda_j\\ &= \sum_{i,j\in\mathbb{Z}_n} f_j\circ\alpha^{-i}(x_0)\overline{\lambda'_i}\lambda_{i-j}\\ &= \sum_{\substack{0\le t\le l-1\\0\le a\le p-1}}\Big[\sum_{d=0}^{a} f_{tp+d}\circ\alpha^{-a}(x_0)\Big(\sum_{s=0}^{l-1}\overline{\lambda'}_{sp+a}\lambda_{(s-t)p+a-d}\Big)\Big]\\ &\quad+\sum_{\substack{0\le t\le l-1\\0\le a\le p-1}}\Big[\sum_{d=a+1}^{p-1} f_{tp+d}\circ\alpha^{-a}(x_0)\Big(\sum_{s=0}^{l-1}\overline{\lambda'}_{sp+a}\lambda_{(s-t-1)p+p+a-d}\Big)\Big].\end{aligned}$$

Substituting λ by $\mu\otimes\nu$ and λ' by $\mu'\otimes\nu'$, we get the desired formula.

Corollary 2. *For $\mu=(\mu_j)_{0\le j\le p-1}$, we define $\mu^{(k)}=(\mu_0,\mu_1\gamma^k,\cdots,\mu_{p-1}\gamma^{(p-1)k})$. Then $\varphi_{x_0,\mu,k}=\varphi_{x_0,\mu^{(k)}\otimes\tilde\delta_k}$, $0\le k\le l-1$. Moreover, since $\mathbb{C}^n=\bigoplus_{k=0}^{l-1}\mathbb{C}^p\otimes\tilde\delta_k$, a unit vector λ of $\mathbb{C}^n$ has a unique decomposition*

$$\lambda=\sum_{k=0}^{l-1}\alpha_k\mu_k\otimes\tilde\delta_k,\quad \alpha_k\ge 0,\quad \|\mu_k\|=1,\quad \sum|\alpha_k|^2=1.$$

Then $\varphi_{x_0,\lambda}=\sum_{k=0}^{l-1}|\alpha_k|^2\varphi_{x_0,\mu_k\otimes\tilde\delta_k}$.

Summarizing these works, we have

Theorem 3. *$\varphi_{x_0,\lambda}$ is pure if and only if $\lambda=\mu\otimes\tilde\delta_k$ for some unit vector $\mu\in\mathbb{C}^p$ and $0\le k\le l-1$. Moreover,*

$$\varphi_{x_0,\mu_1\otimes\tilde\delta_k}\simeq\varphi_{x_0,\mu_2\otimes\tilde\delta_k}\quad\text{and}\quad\varphi_{x_0,\mu\otimes\tilde\delta_{k_1}}\not\simeq\varphi_{x_0,\mu\otimes\tilde\delta_{k_2}}$$

if $k_1\ne k_2$.

Let the pure state space of the C^*-algebra A be denoted by $P(A)$. We shall now specify the topology and additional structure on $\overline{P(A)}$, where $A=C(X)\times_\alpha\mathbb{Z}_n$, and the closure of $P(A)$ relative to the W^*-topology.

Proposition 3. *If X is connected with a dense n-degree point (relative to α), then the map $\Psi: X\otimes\mathbb{P}(\mathbb{C}^n)\to\overline{P(C(X)\times_\alpha\mathbb{Z}_n)}$ defined by*

$$(x,|\lambda|)\mapsto\varphi_{x,\lambda}$$

is continuous and onto.

Recall the definition of A_n and $\varepsilon(\varepsilon(f)=(f_{i-j}\circ\alpha^{-i})_{i,j})$, then the following lemma can be directly verified.

Lemma 4. *For each $f\in C(X)\times_\alpha\mathbb{Z}_n$, $\varepsilon(f)\circ\alpha=A_n\varepsilon(f)A_n^*$.*

Thus $\varphi_{\alpha(x),\lambda}=\varphi_{x,A_n^*\lambda}$, $\lambda\in\mathbb{P}(\mathbb{C}^n)$ and $x\in X$.

Using $\tilde\delta_k$'s introduced before, we have an orthogonal decomposition $\mathbb{C}^n=\bigoplus_{k=0}^{l-1}\mathbb{C}^p\otimes\tilde\delta_k$. Let I_k be the identity map of $\mathbb{C}^p\otimes\tilde\delta_k$, then

$$U_p=\Big\{\bigoplus_{k=0}^{l-1}\theta_kI_k:\theta_k\in S^1\Big\}\cong\mathbb{T}_l(l-\text{torus})$$

is a subgroup of U_n. *It acts on* $\mathbb{P}(\mathbb{C}^n)$ *naturally. Let the* U_p*-orbit of* $[\lambda]$ *be denoted by* $[[\lambda]]$, *then we have*

Lemma 5. *Let* x_0 *be a p-degree point of* X, $pl = n$, $[\lambda]$ *and* $[\lambda']$ *be in* $\mathbb{P}(\mathbb{C}^n)$. *Then*

$$\varphi_{x_0,\lambda} = \varphi_{x_0,\lambda'} \Leftrightarrow [\lambda'] \in [[\lambda]].$$

Thus $[[\lambda]] \to \varphi_{x_0,\lambda}$ *provides a natural isomorphism of* $\mathbb{P}(\mathbb{C}^n)/U_p$ *with* $\{\varphi_{x_0,\lambda} : [\lambda] \in \mathbb{P}(\mathbb{C}^n)\}$.

Proof. Let $\lambda = \sum_{k=0}^{l-1} \alpha_k(\tilde{\mu}_k \otimes \tilde{\delta}_k)$ with $0 \le \alpha_k \le 1$ and $\|\tilde{\mu}_k\| = 1$, $\sum |\alpha_k|^2 = 1$. Then Corollary 2 tells us

$$\varphi_{x_0,\lambda} = \sum_{k=0}^{l-1} |\alpha_k|^2 \varphi_{x_0,\tilde{\mu}_k \otimes \tilde{\delta}_k}.$$

Thus the direction "$\Leftarrow$" is clearly true.

Note that $\varphi_{x_0,\mu,k} = \varphi_\mu \circ (id \otimes ev_k) \circ (\varepsilon_{x_0} \otimes id) \circ \psi \circ r^*$ and

$$\text{range}[(\varepsilon_{x_0} \otimes id) \circ \psi \circ r^*] = M_p \otimes C([0, l-1]).$$

So

$$\sum_{k=0}^{l-1} \alpha_k^2 \varphi_{x_0,\mu_k,k} = \sum_{k=0}^{l-1} \alpha'^2_k \varphi_{x_0,\mu'_k,k} \, (\alpha_k, \alpha'_k \ge 0)$$

implies

$$\sum_{k=0}^{l-1} \alpha_k^2 f(k) \langle T\mu_k, \mu_k \rangle = \sum_{k=0}^{l-1} \alpha'^2_k f(k) \langle T\mu'_k, \mu'_k \rangle$$

for each $T \in M_p$ and $f \in C([0, l-1])$. Taking $f = \chi_k$ the characteristic function supported at k, we have $\alpha_k = \alpha'_k$, and if $\alpha_k \ne 0$, $[\mu_k] = [\mu'_k]$. Combining this with Corollary 2, we get the desired result.

Remark 3. Let the standard n-simplex be denoted by

$$\Delta_n = \Big\{ \sum_{k=0}^{n-1} \alpha_k \delta_k : \alpha_k \ge 0, \ \sum \alpha_k = 1 \Big\}.$$

Then if x_0 is a fixed point of α, $\Delta_n \tilde{\to} \{\varphi_{x_0,\lambda} : \lambda \in \mathbb{P}(\mathbb{C}^n)\}$ provided by

$$\sum_{k=0}^{n-1} l_k \delta_k \to \varphi_{x_0, \sum_{k=0}^{n-1} \alpha_k \delta_k}.$$

If x_0 is of degree n, then $\{\varphi_{x_0,\lambda} : \lambda \in \mathbb{P}(\mathbb{C}^n)\} \cong \mathbb{P}(\mathbb{C}^n)$ by

$$\varphi_{x_0,\lambda} \overset{1-1}{\leftrightarrow} [\lambda].$$

We shall treat the above natural isomorphisms as the identification.

Proposition 4. *If* x_0, $[\lambda]$ *are all as in the lemma above and* Ψ *is the map defined in Proposition* 3, *then*

$$\Psi^{-1}(\varphi_{x_0,\lambda}) = \{(\alpha^j(x_0), [\lambda']) : [\lambda'] \in [[A_n^j \lambda]], \quad 0 \le j \le n-1\}.$$

Proof. Note that $A_n^p = I \otimes A_l = \bigotimes_{k=0}^{l-1} \delta^{-k} I_k \times \tilde{\delta}_k$ is in U_p. So

$$[[A_n^{pt+d}\lambda]] = [(A_n^p)^t [A_n^d \lambda]] = [[A_n^d \lambda]], \quad 0 \le d \le p-1, \ 0 \le t \le l-1.$$

Thus we may assume $[\lambda'] \in [[A_n^j\lambda]]$ for some $0 \le j \le p-1$. By Lemmas 4 and 5, we then have

$$\varphi_{\alpha^j(x_0),\lambda'} = \varphi_{\alpha^j(x_0),A_n^j\lambda} = \varphi_{x_0,\lambda}. \qquad (**)$$

Conversely, if $\varphi_{x,\lambda'} = \varphi_{x_0,\lambda}$, then $x \in \alpha$-orbit of x_0. Assume $x = \alpha^j(x_0)$, then, by the (**) formula and Lemma 5 above, we have $[\lambda'] \in [[A_n^j\lambda]]$.

This proposition induces an equivalent relation in $X \in \mathbb{P}(\mathbb{C}^n)$ by

$$(x,[\lambda]) \sim (x',[\lambda']) \Leftrightarrow x' = \alpha^j(x_0) \text{ and } [\lambda'] \in [[A_n^j\lambda]], \quad 0 \le j \le n-1,$$

where the meaning of $[[A_n^j\lambda]]$ has to depend on the degree of x.

Let π be the corresponding quotient map. Then the following theorem becomes clear:

Theorem 4. *If n-degree points are dense in X, then the map $\widetilde{\Psi} : X \times \mathbb{P}(\mathbb{C}^n)/_\sim \to \overline{P(C(X)\times_\alpha \mathbb{Z}_n)}$ given by*

$$\widetilde{\Psi}(\pi(x,[\lambda])) = \varphi_{x,\lambda}$$

is well-defined and is a homeomorphism.

Remark 4. Let σ be the natural map of $X \to X/\alpha$ and $p_1 : X \times \mathbb{P}(\mathbb{C}^n) \to X$ the projection onto the first component, then a projection from $X \times \mathbb{P}(\mathbb{C}^n)/_\sim$ onto X/α is induced by $p(\pi(x,[\lambda])) = \sigma(x) = \mathbb{Z}_n \cdot x$. Obviously the following diagram commutes:

$$\begin{array}{ccccc} X\times\mathbb{P}(\mathbb{C}^n) & \xrightarrow{\pi} & X\times\mathbb{P}(\mathbb{C}^n)/_\sim & \xrightarrow[\sim]{\widetilde{\Psi}} & \overline{P(C(X)\times_\alpha\mathbb{Z}_n)} \\ \sigma\circ p_1 \searrow & & \downarrow p & & \swarrow p\circ\widetilde{\Psi}^{-1} \\ & & X/\alpha & & \end{array}$$

Let $p\circ\widetilde{\Psi}^{-1}$ be denoted by p', then $p'(\varphi_{x,\lambda}) = \mathbb{Z}_n \cdot x = \sigma(x)$. Think $\overline{P(C(X)\times_\alpha\mathbb{Z}_n)}$ to be fibred space over X/α (not a fibre space, or a fibration in general) through p', then $\varphi_{x,\lambda} \cong \varphi_{x',\lambda'} \Leftrightarrow$ they are on the same fibre. Thus we may identify the fibred space $p' : \overline{P(C(X)\times_\alpha\mathbb{Z}_n)} \to X/\alpha$ with that of $p : X\times\mathbb{P}(\mathbb{C}^n)/_\sim \to X/\alpha$.

The transition probability and orientation on $P(C(X)\times_\alpha\mathbb{Z}_n)$ now may be identified with the fibred space $p : X\times\mathbb{P}(\mathbb{C}^n)/_\sim \to X/\alpha$ with specified structure groups on each fibre type through the natural identification (see [9]):

if x is of degree n, then this group is PU_n on $p^{-1}(\mathbb{Z}_n\cdot x) \cong \mathbb{P}(\mathbb{C}^n)$;

if x is of degree 1, then this group is the map of all simplicial automorphisms of $\Delta_n \cong p^{-1}(\mathbb{Z}_n\cdot x)$.

If x is of degree p, $1<p<n$, then we may think of

$$p^{-1}(\mathbb{Z}_n\cdot x) \cong \mathbb{P}(\mathbb{C}^n)/U_p = \Big\{[[\sum_{k=0}^{l-1}\alpha_k u_k \otimes \tilde{\delta}_k]] : (\alpha_k)_{k=0}^{l-1} \in \Delta_l \text{ and } u_k \in \mathbb{C}^p \text{ with } \|u_k\| = 1\Big\}$$

as a generalized l-simplex with various copies of $\mathbb{P}(\mathbb{C}^p)$ as vertices. Let S_l be the l-th permutation group, then $S_l \times (PU_p)^l$ acts on $\mathbb{P}(\mathbb{C}^n)/U_p$ faithfully (not freely) by

if $(\tau;[\varphi_0],\cdots,[\varphi_{l-1}]) \in S_l\times(PU_p)^l$, where $\varphi_j \in U_p$, then

$$(\tau;[\varphi_0],\cdots,[\varphi_{l-1}])\cdot[[\sum_{k=0}^{l-1}\alpha_k u_k\otimes\tilde{\delta}_k]] = [[\sum_{k=0}^{l-1}\alpha_{\tau(k)}\varphi_k(u_k)\otimes\tilde{\delta}_k]].$$

We may call $S_l \times (PU_p)^l$ the group of generalized simplicial automorphisms. The structure group on $\mathbb{P}(\mathbb{C}^n)/U_p \cong p^{-1}(\mathbb{Z}_n \cdot x)$ is $S_l \times (PU_p)^l$.

When every point of X is of n-degree, or $\mathbb{Z}_n$ acts on X freely, this fibred space (with specified structure group) is just a classical flat PU_n-bundle over X/α, which we shall describe now.

Let $\{U_i\}$ be an open cover of X/α with each U_i and $U_i \cap U_j$ connected s.t. for each i

$$\sigma^{-1}(U_i) = \widetilde{U}_i \cup \alpha(\widetilde{U}_i) \cup \cdots \cup \alpha^{n-1}(\widetilde{U}_i),$$

with $\widetilde{U}_i \cap \alpha^k(\widetilde{U}_i) = \emptyset, \ \forall k \not\equiv 0 \,(\mathrm{mod} n)$.

The choice of $\widetilde{U}_i$ is not unique.

Lemma 6. *If $U_i \cap U_j \neq \emptyset$, then there is an integer k_{ji} (unique up to a multiple of n) s.t.*

$$\alpha^{k_{ji}}(\widetilde{U}_i \cap \sigma^{-1}(U_i \cap U_j)) = \widetilde{U}_j \cap \sigma^{-1}(U_i \cap U_j).$$

Proof. From the assumption, σ gives a homeomorphism of $\widetilde{U}_i$ onto U_i. Note that $\sigma(\widetilde{U}_i \cap \sigma^{-1}(U_i \cap U_j)) = U_i \cap U_j$. So

$$\widetilde{U}_i \cap \sigma^{-1}(U_i \cap U_j) = \widetilde{U}_i \cap \sigma^{-1}(U_j) = \bigcup_{t \in \mathbb{Z}_n} \widetilde{U}_i \cap \alpha^t(\widetilde{U}_j)$$

is connected, which forces for all t but one $-k$

$$\widetilde{U}_i \cap \alpha^t(U_j) = \emptyset,$$
$$\widetilde{U}_i \cap \sigma^{-1}(U_i \cap U_j) = \widetilde{U}_i \cap \alpha^{-k}(\widetilde{U}_j).$$

Or

$$\alpha^k(\widetilde{U}_i \cap \sigma^{-1}(U_i \cap U_j)) = \alpha^k(\widetilde{U}_i) \cap \widetilde{U}_j.$$

Similarly, $\widetilde{U}_j \cap \sigma^{-1}(U_i \cap U_j) = \alpha^l(\widetilde{U}_i) \cap \widetilde{U}_j$, and if $t \neq l$, $\alpha^t(\widetilde{U}_i) \cap \widetilde{U}_j = \emptyset$. Thus $k = l = k_{ji}$ is unique (mod n) and has the desired property.

With $\widetilde{U}_i$ as above, we have $p^{-1}(U_i) = \pi(\widetilde{U}_i \times \mathbb{P}(\mathbb{C}^n))$ and thus

$$\phi_i = (\sigma \times id) \circ (\pi|_{\widetilde{U}_i \times \mathbb{P}(\mathbb{C}^n)})^{-1}$$

is a well-defined homeomorphism $p^{-1}(U_i)$ onto $U_i \times \mathbb{P}(\mathbb{C}^n)$. The following calcultion shows that $(U_i, \phi_i)_i$ gives a coordinate system of $X \times \mathbb{P}(\mathbb{C}^n)/_\sim$ as PU_n-bundle over X/α:

$$p^{-1}(U_i \cap U_j) \xrightarrow{\ \phi_j\ } (U_i \cap U_j) \times \mathbb{P}(\mathbb{C}^n)$$
$$\pi(x, A_n^{k_{ij}}[\lambda]) = \pi(y, [\lambda]) \longrightarrow (\sigma(y), [\lambda])$$
$$\phi_i \downarrow \qquad \downarrow \qquad \swarrow \ \phi_i \circ \phi_j^{-1}$$
$$(\sigma(y), A_n^{k_{ij}}[\lambda])$$
$$(U_i \cap U_j) \times \mathbb{P}(\mathbb{C}^n)$$

where $x \in \widetilde{U}_i \cap p^{-1}(U_i \cap U_j)$ and $y = \alpha^{k_{ji}}(x) \in \widetilde{U}_j \cap p^{-1}(U_i \cap U_j)$.

Thus $\phi_i \phi_j^{-1} = g_{ij} = [A^{k_{ij}}] \in PU_n$. We may identify the PU_n-bundle with the 1-cocycle $(U_i \cap U_j, [A_n^{k_{ij}}])_{i,j}$, which is clearly flat.

Let E be the U_n-vector bundle corresponding to the 1-cocycle $(U_i \cap U_j, A_n^{k_{ij}})_{i,j}$, then its projectivilization $\mathbb{P}(E) \cong X \times \mathbb{P}(\mathbb{C}^n)/_\sim$ (see [6]). The equivalence of $E_1 \cong E_2$ clearly implies

$\mathbb{P}(E_1) \cong \mathbb{P}(E_2)$. Now $E \cong \bigoplus_{k=0}^{n-1} L^k$, where L is the U_1-bundle $(U_i \cap U_j, \gamma^{k_{ij}})_{i,j}$ (remember $\gamma = e^{2\pi i/n}$) and L^k is the k-th tensor power of L.

Proposition 5. *If $\mathbb{Z}_n$ acts on X freely (generated by α), and $H^2(X/\alpha, \mathbb{Z})$ has no element annihilated by n, then*

$$C(Y) \times_\beta \mathbb{Z}_n \cong C(X) \times_\alpha \mathbb{Z}_n \Leftrightarrow \mathbb{Z}$$

acts on Y freely and $X/\alpha \cong Y/\beta$.

Proof. In this case, since $[L^n] = n[L] = 0$ in $H^2(X/\alpha, \mathbb{Z})$ (L is described as above), we have $[L] = 0$, i.e., L is trivial. Thus $X \times \mathbb{P}(\mathbb{C}^n)/_\sim$ as PU_n-bundle over X/α is trivial. By the consideration of the dimension of irreducible representations, the above statement is clearly true.

E.g. Let $X_0 = T^2$, then $H^1(X_0, \mathbb{Z})$ and $H^2(X_0, \mathbb{Z})$ are both free abelian groups.

$$H^1(X_0, \mathbb{Z}_n) \cong \mathbb{Z}_n^2 \neq 0.$$

Thus any nonzero element in $H^1(X_0, \mathbb{Z}_n)$ will give a n-fold regular covering X of X_0 with $\mathbb{Z}_n$ as its deck transformation group. Of course this $\mathbb{Z}_n$ action on X is free with the obrit space X_0.

When $n = 2$, we may visualize the fibred space $\overline{P(C(X) \times_\alpha \mathbb{Z}_n)}$ over X/α, which may be thought to be the dual of $C(X) \times_\alpha \mathbb{Z}_2$. Let X^α = fixed point set of α. We still assume it to be nowhere dense in X. We may put a metric d on X if X is 2nd countable.

At first, we may identify $X \times \mathbb{P}(\mathbb{C}^2)$ with $X \times S^2$ (thus the PU_2 action becomes SO_3 action on S^2) by

$$(x, [\lambda, \mu]) \underset{\sim}{\overset{\Phi}{\to}} (x; 2\mathrm{Re}\bar{\lambda}\mu, 2\mathrm{Im}\bar{\lambda}\mu, |\mu|^2 - |\lambda|^2),$$

where $(\lambda, \mu) \in \mathbb{C}^2$ with $|\lambda|^2 + |\mu|^2 = 1$. The equivalence relation introduced after Proposition 4 on $X \times \mathbb{P}(\mathbb{C}^2)$ is now translated to $X \times S^2$ by

if $x \neq \alpha(x)$, $(x; x_1, x_2, x_3) \sim (x'; x_1', x_2', x_3') \Leftrightarrow$ either $x = x'$, $(x_1, x_2, x_3) = (x_1', x_2', x_3')$ or $x' = \alpha(x)$, $(x_1', x_2', x_3') = (x_1, -x_2, -x_3)$;

if $x = \alpha(x)$, $(x; x_1, x_2, x_3) \sim (x'; x_1', x_2', x_3') \Leftrightarrow$ $x = x'$ and $x_1 = x_1'$.

Define a continuous map $T : X \times S^2 \to X \times \mathbb{R}^3$ by

$$T(x; x_1, x_2, x_3) = (x; x_1, d(x, X^\alpha)x_2, d(x, X^\alpha)x_3).$$

Let the range T be denoted by Y. Let p_1 be the natural projection of $X \times \mathbb{R}^3$ onto X. Then if $x \notin X^\alpha$, $(p_1|Y)^{-1}(x)$ carries the metric and orientation induced from S^2 (by using T). We shall call it the natural metric on $(p_1|Y)^{-1}(x)$. Identify the point

$$(x; x_1, d(x, X^\alpha)x_2, d(x, X^\alpha)x_3) \text{ with } (\alpha(x); x_1, -d(x, X^\alpha)x_2, -d(x, X^\alpha)x_3),$$

and let the resulted quotient space be denoted by $Y/\sim$. Then clearly $Y/\sim \cong X \times S^2/\sim$ and the following diagram commutes:

$$\begin{array}{ccccc} Y & \longrightarrow & Y/\sim & \xrightarrow{\sim} & X \times S^2/\sim \\ \sigma \circ p_1 \searrow & & \big\downarrow \sigma\circ p_1 & & \swarrow p' \\ & & X/\alpha & & \end{array}$$

Thus the fibred space $Y/\sim$ over X/α may be identified with that of $X \times S^2/\sim$ or $\overline{(P(C(X) \times_\alpha \mathbb{Z}_2)}, p')$. This $Y/\sim$ is visualizable. It is dual to $C(X) \times_\alpha \mathbb{Z}_2$.

E.g. Let $X = \{(t,0) : |t| \leq 1\} \cup \{(0,y) : |y| \leq 1\}$. The automorphism $\alpha : X \to X$ is given by $\alpha(t,0) = (-t,0)$, $\alpha(0,y) = (0,y)$. Thus $\alpha^2 = id$, and

$$X^\alpha = \{(0,y) : |y| \leq 1\}, \quad X/\alpha = \{(t,0) : 0 \leq t \leq 1\} \cup \{(0,y) : |y| \leq 1\}.$$

From the provious work, we have (the W^*-topology on the pure state space)

$$P(C(X) \times_\alpha \mathbb{Z}_2) \cong [\{(t,0) : 0 < t < 1\} \times I\!P(\mathbb{C}^2)] \cup \{((0,y),[\lambda]) : |y| \leq 1 \text{ and } \lambda = (0,1) \text{ or } (1,0)\}.$$

Using Φ and the induced topology from $X/\alpha \times I\!R^3$, we have

$$\overline{P(C(X) \times_\alpha \mathbb{Z}_2)} \cong \{((t,0); x_1, tx_2, tx_3) : 0 \leq t \leq 1 \text{ and } (x_1,x_2,x_3) \in S^2\} \cup \{((0,y); \mu, 0, 0) : 0 < |y| < 1,\ \mu = \pm 1\} \subseteq X/\alpha \times I\!R^3.$$

The projection of this fibres space is the natural one onto the first component (to $t-y$ plane) and the metric and orientation on fibred are all the natural ones.

This gives a geometric picture of the "dual" of $C(X) \times_\alpha \mathbb{Z}_2$.

Remark 5. We have extended this "dual study" of $C(X) \times_\alpha \mathbb{Z}_n$ to $C(X) \times_\alpha \mathbb{Z}$ with $\alpha^n = id$ (see [5]). Along this line, a classification of rational rotation C^*-algebras on unit circle has been reproduced (see [4]).

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