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SCALAR CURVATURES ON NONCOMPACT RIEMANN MANIFOLDS**

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Abstract

The author obtains some theorems for a function to be the scalar curvature of some complete conformal metric of a noncompact complete Riemann manifold, and also presents a kind of manifolds on which Yamabe problem is unsolvable.

Keywords Riemann manifold, Scalar curvature, Laplacian
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§1. Introduction

A basic problem in Riemann geometry is studying the set of curvature functions that a manifold possesses. Let (M,g) be an n-dimensional Riemann manifold with scalar curvature $K(x)$ and let $\widetilde{K}(x)$ be a given function on M. Does there exist a conformal metric $\widetilde{g}$ such that the scalar curvature of $\widetilde{g}$ is $\widetilde{K}$? When $n \geq 3$, the problem is equivalent to finding a positive solution to the equation

$$c_n \Delta_g u + \widetilde{K}(x) u^{\frac{n+2}{n-2}} = K(x) u, \tag{1.1}$$

where

$$c_n = \frac{4(n-1)}{n-2}, \quad \widetilde{g} = u^{\frac{4}{n-2}} g.$$

This problem has been extensively studied for compact manifolds with or without boundary (see [1, 5, 7, 11, 14]). The special case of deforming to constant scalar curvature is known as the Yamabe problem and has recently been completely resolved for compact manifolds by Schoen[14].

If M is a complete noncompact Riemann manifold, very little is known. In the special case $M = R^n$ with Eucildean metric this problem has been studied by [8]. Recently Aviles and McOwen[2] discussed this problem on noncompact Riemann manifolds in case $\widetilde{K}(x) = -1$. The present paper is concerned with the general $\widetilde{K}(x)$. This paper is organized as follows: In §2 we give a completeness theorem for conformal metrics and the existence of the complete metrics with the prescribed scalar curvature and in §3 we obtain a class of manifolds on which the Yamabe problem is unsolvable. Unless otherwise stated, (M,g) is always assumed to be a complete noncompact simply-connected Riemann manifold.

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§2. Existence Theorems

For a complete Riemann manifold (M,g) and a conformal metric $\widetilde{g}=fg$, where $f>0$ is a continuous function on M. A natural problem is under which conditions on f, $\widetilde{g}$ is also complete. If M is compact, the continuity of f can guarantee the completeness of $\widetilde{g}$. So this problem is essential for noncompact Riemann manifolds. In what follows we give a completeness criterion which is almost optimal[3]. We shall omit the proof since it is straightforward by using the Hopf-Rinow Theorem.

Theorem 2.1. *Let (M,g) be a complete noncompact Riemann manifold. Suppose that f is a positive continuous function on M and satisfies $f^{\frac{1}{2}}(x)\geq c(r(x))$ for $r(x)\geq r_0$, where r_0 is a positive constant and x_0 is a fixed point in M and $r(x)=d(x,x_0)$ and $c(t)$ is a real function subject to*

$$\int_0^{+\infty} c(t)\,dt=+\infty.$$

Then $\widetilde{g}=fg$ is a complete metric on M.

Consider the following inequality

$$\Delta_g u\geq -\widetilde{K}(x)u^{\alpha}+K(x)u, \tag{2.1}$$

where $\alpha>1$, $\widetilde{K}(x)$, $K(x)$ are continuous functions on M. Suppose that $\bar{\Omega}$ is a compact C^∞ Riemann manifold with boundary $\partial\Omega$ and interior $\Omega=\bar{\Omega}\backslash\partial\Omega$. Then we have

Lemma 2.1. *Let $\widetilde{K}(x)<0$, $K(x)\geq -K_0$, $\forall x\in\Omega$ for some constant K_0. Then for any compact subset $X\subset\Omega$, there exists a constant $C_0\geq 0$ such that any nonnegative continuous weak solution $u\in H_1^2(\Omega)$ of (2.1) satisfies*

$$\max_{x\in X} u(x)\leq C_0.$$

Proof. Since X is compact, there exists a positive constant $R>0$ and y_1, y_2 ,$\cdots$,y_N $\in X$ such that

$$\cup\{B_R(y_i): i=1,2,\cdots,N\}\supset X,\quad \overline{B}_{2R}(y_i)\subset\Omega.$$

From Theorem 8.17 in [6] we have for some i

$$\sup_{x\in X} u(x)\leq \sup_{x\in B_R(y_i)} u(x)\leq CR^{-\frac{n}{p}}\|u\|_{L^p(B_{2R}(y_i))},$$

where $p=\alpha+1$ and the constant C depends on n ,p, Ω and K_0.

Let

$$\varphi\in C_o^\infty(\Omega),\quad \varphi|_{B_{2R}(y_i)}\equiv 1 \ \text{ and } \ \varphi\geq 0.$$

Multiply both the sides of (2.1) by $u\varphi^q$ $(q=\frac{2(\alpha+1)}{\alpha-1})$ and integrate to obtain

$$\int_\Omega u\varphi^q\Delta_g u\,dv\geq -\int_\Omega \widetilde{K}(x)u^{\alpha+1}\varphi^q\,dv+\int_\Omega K(x)u^2\varphi^q\,dv.$$

Similar to the proof of Theorem 1.1 in [2] we have

$$-\int_\Omega \widetilde{K}(x)u^{\alpha+1}\varphi^q\,dv\leq C_1,$$

where C_1 is a constant independent of u. Then

$$\int_\Omega u^{\alpha+1}\varphi^q\,dv\leq \max_{x\in\bar{\Omega}}\{(-\widetilde{K}(x))^{-1}\}\int_\Omega(-\widetilde{K}(x))u^{\alpha+1}\varphi^q\,dv\leq C_2.$$

Thus

$$\|u\|^p_{L^p(B_{2R}(y_i))} \le \int_\Omega u^{\alpha+1}\varphi^q \, dv \le C_2$$

and this proves our lemma.

Lemma 2.2. *Assume that $\widetilde{K}(x)$, $K(x)$ are continuous functions and $\widetilde{K}(x) < 0$. Then there is a positive C^∞ solution u of (1.1) if and only if there is a nonnegative continuous function $u_- \in (H_1^2)_{\rm loc}$ satisfying $u_- \not\equiv 0$ and*

$$c_n\Delta_g u_- + \widetilde{K}(x)u_-^{\frac{n+2}{n-2}} \ge K(x)u_-. \tag{2.2}$$

Moreover $u \ge u_-$.

Proof. The necessity is obvious, we shall prove the sufficiency. Let $\Omega \subset M$ be a bounded subset. Then $K(x)$ is bounded below and we can define $\bar{u}$ to be a constant such that $\bar{u} \ge \sup\{u_-(x) : x \in \Omega\}$ and

$$0 = c_n\Delta_g\bar{u} \le -\widetilde{K}(x)\bar{u}^{\frac{n+2}{n-2}} + K(x)\bar{u}. \tag{2.3}$$

One can find a positive constant A such that

$$f(x,u) = \widetilde{K}(x)u^{\frac{n+2}{n-2}} - K(x)u + Au$$

is monotone increasing with respect to $u \in [u_-, \bar{u}]$. Let u_1 be the solution of following problem

$$-c_n\Delta_g u_1 + Au_1 = f(x, u_-), \tag{2.4}$$

$$u_1|_{\partial\Omega} = u_-|_{\partial\Omega}. \tag{2.5}$$

Then

$$-c_n\Delta_g(u_1 - u_-) + A(u_1 - u_-) \ge 0, \tag{2.6}$$

$$-c_n\Delta_g(\bar{u} - u_1) + A(\bar{u} - u_1) \ge f(x,\bar{u}) - f(x,u_-) \ge 0, \tag{2.7}$$

$$\bar{u} - u_1|_{\partial\Omega} \ge u_1 - u_-|_{\partial\Omega} = 0. \tag{2.8}$$

It follows from (2.6), (2.7), (2.8) and the maximum principle that

$$0 \le u_- \le u_1 \le \bar{u}.$$

Similarly we define u_m for $m = 1, 2, \cdots$,

$$-c_n\Delta_g u_m + Au_m = f(x, u_{m-1}),$$
$$u_m|_{\partial\Omega} = u_{m-1}|_{\partial\Omega}.$$

Then

$$0 \le u_- \le u_1 \le u_2 \le \cdots \le u_m \le \bar{u}.$$

From the monotone iteration scheme[1] we know that there exists a weak solution u of (1.1) on Ω such that $u_- \le u \le \bar{u}$. The standard elliptic theory shows that $u \in C^\infty(\Omega)$. Then exhausting M by a sequence of bounded domains Ω_k we can show that the sequence of solutions $\{u_k\}$ of (1.1) on each Ω_k has a subsequence which converges to the solution of (1.1) on M. The idea is similar to that of [8], so we omit the details.

Now let us prove the existence theorems.

Theorem 2.2. *Suppose that $\widetilde{K}(x)$, $K(x)$ are continuous function on M and $K(x) \le 0$ $\widetilde{K}(x) < 0$, $\forall x \in M$. If there exists a constant $\epsilon > 0$ such that*

$$K(x)/\widetilde{K}(x) \ge \epsilon, \quad x \in M \backslash M_0,$$

where M_0 is a compact subset of M, then there exists a conformal complete metric $\widetilde{g}$ with scalar curvature $\widetilde{K}(x)$.

Proof. By Lemma 2.2 it suffices to find a positive subsolution of (1.1). From [2] we can suppose that the conditions hold for all $x \in M$. Set $u_- = c > 0$ such that

$$c^{\frac{4}{n-2}} \le \epsilon \le \frac{K(x)}{\widetilde{K}(x)}.$$

Then

$$0 = c_n \Delta_g u_- \ge -\widetilde{K}(x) u_-^{\frac{n+2}{n-2}} + K(x) u_-,$$

namely, u_- is a subsolution of (1.1); this proves our theorem.

In order to prove our next two theorems we need the following lemma.

Lemma 2.3.[12] *Let (M, g) be a complete Riemann manifold. Let us denote by $r(x) = d(x, x_0)$ the distance function to some fixed point $x_0 \in M$. If*

$$\mathrm{Ric}\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) \ge -c(r),$$

then in the distributional sense

$$\Delta_g r \le \frac{n-1}{r} + \frac{1}{r^2} \int_0^r t^2 c(t)\, dt.$$

Lemma 2.4. *Let (M, g) be a complete Riemann manifold with nonnegative scalar curvature $K(x)$, $r(x) = d(x, x_0)$. If there exists a constant $R > 0$ such that in distributional sense*

$$2(n-1)\Delta_g r + \frac{2(n-1) - \widetilde{K}(x)}{r} \le -rK(x) \quad \text{as } r(x) \ge R, \tag{2.9}$$

then there exists a conformal complete metric $\widetilde{g}$ whose scalar curvature is $\widetilde{K}(x)$.

Proof. Let

$$u_-(x) = (r^2 + b)^{-\frac{n-2}{4}},$$

where b is a positive constant. Then

$$u_-^{\frac{n+2}{n-2}}(x) = (r^2 + b)^{-\frac{n+2}{4}},$$

$$\Delta_g u = \frac{n^2 - 4}{4} r^2 (r^2 + b)^{-\frac{n+6}{4}} - \frac{n-2}{2}(r^2 + b)^{-\frac{n+2}{4}} r \Delta_g r - \frac{n-2}{2}(r^2 + b)^{-\frac{n+2}{4}},$$

$$c_n \frac{\Delta_g u_-}{u_-} = \frac{(n-1)(n+2)r^2}{(r^2 + b)^2} - \frac{2(n-1) r \Delta_g r}{r^2 + b} - \frac{2(n-1)}{r^2 + b}.$$

(2.9) implies

$$\frac{(n-1)(n+2)r^2}{r^2 + b} - 2(n-1) r \Delta_g r - 2(n-1) \ge -\widetilde{K}(x) + K(x)(r^2 + b).$$

Then we have

$$c_n \frac{\Delta_g u_-}{u_-} \geq -\widetilde{K}(x) u_-^{\frac{4}{n-2}} + K(x) u_-,$$

namely, we have obtained a subsolution of (1.1) and from Lemma 2.2 we know that (1.1) possesses a positive solution $u \geq u_-$.

Since

$$u^{\frac{4}{n-2}}(x) \geq u_-^{\frac{4}{n-2}}(x) \geq \frac{c}{r^2 + b},$$

by Theorem 2.1 we know that the metric $\widetilde{g} = u^{\frac{4}{n-2}} g$ is a complete metric on M.

Theorem 2.3. *Let (M, g) be a noncompact complete Riemann manifold with the scalar curvature $K(x) < 0$. For a fixed point $x_0 \in M$ and $r(x) = d(x, x_0)$*

$$\mathrm{Ric}\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) \geq -c(r)$$

for some function $c(r)$. If there exists a constant $R > 0$ such that

$$0 > \widetilde{K}(x) \geq 2n(n-1) + r^2 K(x) - \frac{2(n-1)}{r(x)} \int_0^{r(x)} t^2 c(t) dt \ \ as\ r(x) \geq R,$$

then there exists a conformal complete metric $\widetilde{g}$ whose scalar curvature is $\widetilde{K}(x)$.

The proof of this theorem is straightforward from Lemma 2.3 and Lemma 2.4.

§3. Nonexistence Results

Definition 3.1. *Let M be an n-dimensional complete Riemann manifold*

$$V(r) = \mathrm{vol}(B(x_0, r)).$$

The order of M is defined to be

$$O(M) = \inf\{k | \varliminf_{r \to +\infty} V(r)/r^k < \infty\}.$$

Theorem 3.1. *Let (M, g) be a complete Riemann manifold with $O(M) \leq 2$ and scalar curvature $K(x) < 0$. Then any continuous function $\widetilde{K}(x) \geq 0$ cannot be the scalar curvature of a conformal metric of g.*

Proof. Assume on the contrary that equation (1.1) possesses a positive solution u. Let $f = -u$. Then $f(x) < 0$ for any $x \in M$ and

$$-c_n \Delta_g f = -\widetilde{K}(x)(-f)^{\frac{n+2}{n-2}} - K(x) f.$$

So $-\Delta_g f < 0$. Hence f is a negative subharmonic function on M. This contradicts a theorem of Cheng and Yau in [4].

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