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ON CONNECTIVITY OF THE ESSENTIAL SPECTRA OF TOEPLITZ OPERATORS WITH SYMBOLS IN $H^\infty + C$

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Abstract

This paper discusses the connectivity of the essential spectra of Toeplitz operators with symbols in $H^\infty + C$ on Hardy spaces and weighted Bergman spaces for several complex variables.

Keywords Hardy space, Weighted Bergman space, Toeplitz operator, Essential spectrum.

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§1. Introduction

It is well known that the spectra and the essential spectra of Toeplitz operators on Hardy spaces of one complex variable are connected. But in the case of Hardy spaces of several complex variables and Bergman spaces, the similar results fail. For example, $T_{1-|x|^2}$ is a compact operator on Bergman space $L_a^2(D)$, where D is the unit disk in $\mathbb{C}$, and its spectrum is total disconnected. In the case of Hardy space of several complex variables, Davie and Jewell[4] give the following examples:

(1) There exists a symbol $\varphi \in C(S)$ such that $\sigma(T_\varphi)$ is disconnected.

(2) There exists a symbol $\varphi \in L^\infty(S)$ such that $\sigma_e(T_\varphi)$ is disconnected.

Hence, the following question is naturally interesting: for which $\varphi \in L^\infty$, is the spectrum or the essential spectrum of T_φ connected? In the case of Bergman space of one complex variable, C. Sundberg[7] conjectured that the spectra of Toeplitz operators with harmonic symbols are connected. In 1988, D. H. Yu, S. H. Sun and Z. G. Dai[8] proved that the essential spectrum of Toeplitz operator with symbol $\hat{\varphi}$ is connected, where $\hat{\varphi}$ is the harmonic extension of $\varphi \in H^\infty(\mathbb{T}) + C(\mathbb{T})$, and $\mathbb{T}$ is the unit circle in $\mathbb{C}$. In the case of Hardy space for several complex variables, Davie and Jewell[4] conjectured that the essential spectrum of Toeplitz operator with symbol φ in $H^\infty(S) + C(S)$ equals the spectrum of φ in the algebra $H^\infty(S) + C(S)$, i.e., $\sigma_e(T_\varphi) = \sigma_{H^\infty(S)+C(S)}(\varphi)$ for $\varphi \subset H^\infty(S) + C(S)$. In this paper, we will prove that the conjecture is true, and discuss the connectivity of the essential spectra of Toeplitz operators with symbols of type $H^\infty + C$ on weighted Bergman space of several complex variables, which extends the result in [8].

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In the paper, B denotes the unit ball in $\mathbb{C}^n$, S the boundary of B, i.e., $S = \{z \in \mathbb{C}^n \| z| = 1\}$. φ_z is the automorphism of B:

$$\varphi_z(w) = \frac{z - \frac{\langle w,z\rangle}{|z|^2} z - (1-|z|^2)(w - \frac{\langle w,z\rangle}{|z|^2} z)}{1 - \langle w, z\rangle}.$$

Let v be Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$, so normalized that $v(B) = 1$, and σ the rotation-invariant positive Borel measure on S for which $\sigma(S) = 1$ (the positive measure of total mass 1 is often called probability measure). $L^\infty(B)$ and $L^\infty(S)$ denote the spaces of essential bounded measurable functions with respect to v and σ, respectively; $H^\infty(B)$ denotes the space of bounded analytic functions on B. For $1 \le p < \infty$, $L^p(S)$ denotes the space of integrable functions of power p with respect to σ, and $H^p(S)$ is the Hardy space on S.

Write

$$C(z,\zeta) = \frac{1}{(1-\langle z,\zeta\rangle)^n} \quad (z \in B,\ \zeta \in S),$$

$$C_z(\zeta) = \frac{(1-|z|^2)^{2/n}}{(1-\langle z,\zeta\rangle)^n} \quad (z \in B,\ \zeta \in S),$$

$$p(z,\zeta) = \frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}} \quad (z \in B,\ \zeta \in S),$$

$C(z,\zeta)$ is said to be Cauchy kernel, $p(z,\zeta)$ the invariant Poisson kernel, $C_z(\zeta)$ the normalized reproducing kernel of $H^\infty(S)$. It is known that $C_z \to 0$ weakly as $|z| \to 1^-$, and $|C_z(\zeta)|^2 = p(z,\zeta)$. Let $P[f]$ denote the invariant Poisson integral of f. Then we have

$$\int_S f \circ \varphi_z(\zeta) d\sigma(\zeta) = \int_S f(\zeta)|C_z(\zeta)|^2 d\sigma(\zeta) = P[f](z) \quad (f \in L^2(S)),$$

by the invariant mean value property of Poisson integral (consult Theorems 3.3.7 and 3.3.8 in [5]).

Let $P : L^2(S) \to H^2(S)$ be the orthogonal projection. Clearly, $Pf = k - \lim C[f]$ for $f \in L^2(S)$, where k-$\lim F$ is the k-limit of F (consult [5]), $C[f]$ is the Cauchy integral of f, i.e., $C[f](z) = \int_S f(\zeta)C(z,\zeta)d\sigma(\zeta)$. For $\varphi \in L^\infty(S)$, define

$$T_\varphi f = P(\varphi f) = k\text{-}\lim C[\varphi f], \quad f \in H^2(S).$$

T_φ is called Toeplitz operator with symbol φ. In §2, we will discuss the spectrum of T_φ and the algebraic spectrum of φ for φ in $H^\infty(S) + C(S)$.

For $a > -1$, set

$$K_a(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+a}},$$

$$dV_a(w) = \frac{\Gamma(n+1+a)}{\Gamma(n+1)\cdot\Gamma(a+1)}(1-|w|^2)^a d\nu(w).$$

We know that dV_a is a probability measure on B from §7.1 in [5]. For a fixed $a > -1$, $L^p(B, dV_a)$ (or $L^p(dV_a)$) denotes the space of integrable functions of power p with respect to dV_a, $L^p_a(B, dV_a)$ (or $L^p_a(dV_a)$) denotes the Bergman space. For $f \in L^p(B, dV_a)$, write

$$P^{(a)} f(z) = \int_B K_a(z,w) f(w) dV_a(w).$$

From §7,1 in [5], we know that $K_a(z,w)$ is the reproducing kernel of $H^\infty(B)$, and when $a>0$, $p^{(a)}$ is a bounded linear operator from $L^p(B,dV_a)$ to $L^p_a(B,dV_a)$, where $1\le p<\infty$. When $a>-\frac{1}{2}$, $P^{(a)}$ is the orthogonal projection from $L^2(B,dV_a)$ to $L^2_a(B,dV_a)$. If $\varphi\in L^\infty(B)$, define a bounded linear operator $T^{(a)}_\varphi$ on $L^2_a(B,dV_a)$ as

$$T^{(a)}_\varphi f(z)=P^{(a)}(\varphi f)(z)=\int_B K_a(z,w)\varphi(w)f(w)dV_a(w),\quad f\in L^2_a(B,dV_a).$$

Also define a bounded linear operator $H^{(a)}_\varphi$ from $L^2_a(B,dV_a)$ to $(L^2_a(B,dV_a))^\perp$ as

$$H^{(a)}_\varphi f(z)=(1-P^{(a)})(\varphi f)(z).$$

$T^{(a)}_\varphi$ and $H^{(a)}_\varphi$ are called Toeplitz operator and Hankel operator with symbol φ, respectively.

For $f\in L^2(B,dV_a)$, it is easily seen that

$$\int_B f\circ\varphi_z(w)dV_a(w)=\int_B f(w)|K^{(a)}_z(w)|^2dV_a(w),$$

where

$$K^{(a)}_z(w)=\frac{(1-|z|^2)^{(n+1+a)/2}}{(1-\langle w,z\rangle)^{n+1+a}}$$

is called the normalized reproducing kernel of $L^2_a(B,dV_a)$. When $|z|\to 1^-$, $K^{(a)}_z\overset{w}{\to}0$ in $L^2_a(B,dV_a)$. (Hereafter, the weak convergence is denoted by $\overset{w}{\to}$.) K. H. Zhu[9] and L. Stroethoff[6] discussed the compactness and Stchatten C_p-class of $T^{(a)}_\varphi$ and $H^{(a)}_\varphi$. In §3, we will discuss the connectivity of the essential spectra of $T^{(a)}_\varphi$ for $\varphi\in H^\infty(B)+C(B)$.

§2. The Essential Spectra of Toeplitz Operators with Symbols in $H^\infty(S)+C(S)$

Lemma 2.1.[2] *Let $f,g\in H^\infty(S)+C(S)$. Then*

$$\|P_r[f]P_r[g]-P_r[fg]\|_\infty\to 0\quad (r\to 1^-),$$

where $P_r[f](z)=P[f](rz)$ $(z\in B)$.

Lemma 2.2. *Let $f\in H^\infty(S)+C(S)$. Then*

$$\sigma_{H^\infty(S)+C(S)}(f)=\bigcap_{\delta>0}\overline{\{P[f](z)|\delta<|z|<1\}},$$

where $\sigma_{H^\infty(S)+C(S)}(f)$ denotes the spectrum of f in the algebra $H^\infty(S)+C(S)$.

Proof. If $n=1$, the result is classical, so we can suppose $n\ge 2$. Without loss of generality, we assume $0\notin\bigcap_{\delta>0}\overline{\{P[f](z)|\delta<|z|<1\}}$. Then there are $\delta,\varepsilon>0$ such that $|P[f](z)|\ge\varepsilon$ wherever $\delta<|z|<1$. Let $f=f_1+f_2$, $f_1\in H^\infty(S)$, $f_2\in C(S)$. Then

$$P[f]=P[f_1]+P[f_2],\quad P[f_1]\in H^\infty(B),\quad P[f_2]\in C(\overline{B}).$$

$P[f_1]^*=f_1$, a.e. $[\sigma]$, $P[f_2]|_S=f_2$, where $P[f_1]^*$ is the k-limit of $P[f_1]$. Suppose that $\{\tilde f_k\}$ is a sequence of polynomials of z_j, $\bar z_j$ such that $\tilde f_k$ converges to $P[f_2]$ in $C(\overline{B})$, and $F_k(z)=P[f_1](z)+\tilde f_k(z)$. Then there is an $\varepsilon_0>0$ and a large $K>0$. When $k>K$, we have

$$|F_k(z)|\ge\varepsilon_0\quad(\delta<|z|<1).$$

Since $P[f_1]\in H^\infty(B)$, $\bar\partial_jP[f_1]=0$ $(j=1,\cdots,n)$, we have $\bar\partial_jF_k=\bar\partial_j\tilde f_k\in C(\overline{B})$. Furthermore, $\bar\partial_jF_k$ is bounded on $\Omega_\delta=\{z\in B|\delta<|z|<1\}$.

For any $z_0 \in \overline{\Omega}_\delta$, there is an open neighborhood $V(z_0)$ of z_0 such that $|\tilde{f}_k(z) - \tilde{f}_k(z_0)| < \varepsilon_0/4$ when $z \in V(z_0) \cap \Omega_\delta$. So we have

$$|P[f_1](z) + \tilde{f}_k(z_0)| \geq |P[f_1] + \tilde{f}_k(z)| - |\tilde{f}_k(z) - \tilde{f}_k(z_0)| \geq 3\varepsilon_0/4 \quad (z \in V(z_0) \cap \Omega_\delta).$$

Consequently, $P[f_1] + \tilde{f}_k(z_0)$ is invertible in $H^\infty(V(z_0) \cap \Omega_\delta)$ (here $H^\infty(W)$ denotes the space of bounded analytic functions on W). Write

$$\widetilde{G}_k(z) = [P[f_1] + \tilde{f}_k(z_0)]^{-1}(z), \quad z \in V(z_0) \cap \Omega_\delta;$$
$$G_k(z) = \widetilde{G}_k(z)(\tilde{f}_k(z) - \tilde{f}_k(z_0)), \quad z \in V(z_0) \cap \Omega_\delta.$$

If $z \in V(z_0) \cap \Omega_\delta$, we have

$$|G_k(z)| = |\widetilde{G}_k(z)| \cdot |\tilde{f}_k(z) - \tilde{f}_k(z_0)| \leq 1/3.$$

Hence, $\log(1 + G_k)$ is a bounded smooth function on $V(z_0) \cap \Omega_\delta$, and

$$F_k(z) = (P[f_1](z) + \tilde{f}_k(z_0))e^{\log(1+G_k(z))}, \quad z \in V(z_0) \cap \Omega_\delta.$$

Note

$$\begin{aligned}\bar{\partial}_j \log(1 + G_k(z)) &= \frac{1}{1 + G_k(z)} \bar{\partial}_j G_k(z) \\ &= \frac{\widetilde{G}_k(z)}{1 + G_k(z)} \bar{\partial}_j \tilde{f}_k(z), \quad z \in V(z_0) \cap \Omega_\delta.\end{aligned}$$

So $\bar{\partial}_j \log(1 + G_k)$ is also bounded on $V(z_0) \cap \Omega_\delta$. Let V be an open subset of $V(z_0)$ which contains z_0 such that $\overline{V} \subset V(z_0)$ and $V \subset \{z \in \mathbb{C}^n \big| |z - z_0| < (1-\delta)/4\}$, let h be a smooth function in $\mathbb{C}^n$, such that $h|_V = 1$ and the support of h is contained in $V(z_0)$. We dfine

$$H_k(z) = \begin{cases} h(z) \log(1 + G_k(z)), & z \in V(z_0) \cap \Omega_\delta, \\ 0, & z \in B \backslash [V(z_0) \cap \Omega_\delta]. \end{cases}$$

Then H_k is a smooth function on B, and

$$\bar{\partial}_j H_k(z) = \begin{cases} [\bar{\partial}_j h(z)] \log(1 + G_k(z)) + h(z) \bar{\partial}_j \log(1 + G_k(z)), & z \in V(z_0) \cap \Omega_\delta, \\ 0, & z \in B \backslash [V(z_0) \cap \Omega_\delta]. \end{cases}$$

Clearly, $\bar{\partial}_j H_k$ and H_k are both bounded smooth functions on B. Hence, there is a $u_k \in C^1(B) \cap (\mathrm{Lip}\frac{1}{2})(\overline{B})$ such that $\bar{\partial} u_k = \bar{\partial} H_k$ by Theorem 16.7.2 in [5]. Thus $\bar{\partial}(u_k - H_k) = 0$, so $u_k - H_k$ is holomorphic in B, and

$$u_k(z) - H_k(z) = u_k(z) - \log(1 + G_k(z)), \quad z \in V \cap \Omega_\delta.$$

Let $\widetilde{F}_k = (P[f_1] + \tilde{f}_k(z_0))e^{H_k - u_k} \cdot e^{u_k}$. Then

$$(P[f_1] + \tilde{f}_k(z_0))e^{H_k - u_k} \in H^\infty(B), \quad e^{u_k} \in C(\overline{B}), \quad \text{and } \widetilde{F}_k(z) = F_k(z) \text{ if } x \in V \cap \Omega_\delta.$$

For each $z \in \overline{\Omega}_\delta$, we can give the construction as above, so $\overline{\Omega}_\delta$ is covered by finite open sets $\{V_j\}_{j=1}^m$ such that

$$\widetilde{F}_k^j = (P[f_1] + \tilde{f}_k(z_j))e^{H_k^j - u_k^j} e^{u_k^j} \text{ for } z_j \in V_j \subset V(z_j) \cap \{z \in \mathbb{C}^n \big| |z - z_j| < (1-\delta)/4\},$$

where $H_k^j - u_k^j \in H^\infty(B)$, $e^{u_k^j} \in C(\overline{B})$, and $\widetilde{F}_k^j(z) = F_k(z)$ on $V_j \cap \Omega_\delta$. Let

$$\Omega_{\delta'} = \{z \in B \big| \delta' < |z| < 1\} \ (1 > \delta' > (1+\delta)/2).$$

Then there is a subfamily $\{V_{j_i}\}$ of $\{V_j\}_{j=1}^m$ which covers $\overline{\Omega}_{\delta'}$, and $V_{j_i} \cap \Omega_\delta = V_{j_i} \cap B$. For convenience, we still denote $\{V_{j_i}\}$ by $\{V_j\}_{j=1}^m$. Let $V_0 = B \backslash \overline{\Omega}_{\delta'} = \{z \big| |z| < \delta'\}$. Then $\{V_j\}_{j=0}^m$

covers $\overline{B}$. Set

$$L_k^j = (P[f_1] + \tilde{f}_k(z_j))e^{H_k^j - u_k^j} \quad (j = 1, \cdots, m).$$

Then $L_k^j \in H^\infty(B)$. When $j, l \neq 0$, and $z \in V_j \cap V_l \cap \Omega_\delta$, we have

$$L_k^j(z)e^{u_k^j(z)} = L_k^l(z)e^{u_k^l(z)} = F_k(z).$$

For $j = 0,\ l \neq 0,\ V_j \cap V_l \cap \Omega_\delta = V_l \cap V_0 = V_l \cap \{z||z| < \delta'\}$, so

$$L_k^l \in A(V_l \cap V_0) = A(V_0 \cap V_l \cap \Omega_\delta) = A(V_0 \cap V_l \cap B)$$

since $L_k^l \in H^\infty(B)$, and L_k^l is invertible in $A(V_0 \cap V_l \cap B)$. On $V_j \cap V_l \cap B$ we define

$$\Phi_{jl} = L_k^j L_k^{l^{-1}} \quad (j, l = 1, 2, \cdots, m).$$

Then

$$\begin{aligned}\Phi_{jl} = L_k^j L_k^{l^{-1}} = e^{u_k^l - u_k^j} &\in H^\infty(V_j \cap V_l \cap B) \cap C(\overline{V_j \cap V_l \cap B}) \\ &= A(V_j \cap V_l \cap B).\end{aligned}$$

For $l = 0,\ j \neq 0$, we define Φ_{j0}, Φ_{0j} on $V_j \cap V_0 \cap B$ as

$$\Phi_{j0} = L_k^j, \quad \Phi_{0j} = L_k^{j-1}.$$

Then $\Phi_{j0},\ \Phi_{0j} \in A(V_j \cap V_0 \cap B)$. It is easy to see that

$$\Phi_{jl} \cdot \Phi_{li} \cdot \Phi_{ij} = 1 \quad \text{on} \quad V_j \cap V_l \cap V_i \cap B,$$

and

$$\Phi_{jl} \cdot \Phi_{lj} = 1 \quad \text{on } V_j \cap V_l \cap B.$$

Then the second Cousin problem yields Φ_i invertible in $A(V_i \cap B)$ with $\Phi_{ij} = \Phi_i \cdot \Phi_j^{-1}$ (consult [4] p.363-364).

Let

$$P_j^k = \Phi_j^{-1} \cdot L_k^j, \quad Q_j^k = \Phi_j e^{u_k^j} \quad (j = 1, \cdots, m).$$

Then $P_j^k \in H^\infty(V_j \cap B),\ Q_j^k \in C(\overline{V_j \cap B})$. For $z \in V_j \cap V_l \cap B$, we have

$$F_k(z) = L_k^j(z)e^{u_k^j(z)} = P_j^k(z)Q_j^k(z) = P_l^k(z)Q_l^k(z),$$

and

$$\begin{aligned}P_j^k(z)P_l^{k^{-1}}(z) &= \Phi_j^{-1}(z)L_k^j(z)(\Phi_l^{-1}(z)L_k^l(z))^{-1} \\ &= (\Phi_l\Phi_j^{-1})(z)(L_k^j \cdot L_k^{l^{-1}})(z) = 1.\end{aligned}$$

Hence

$$P_j^k|_{V_j \cap V_l \cap B} = P_l^k|_{V_j \cap V_l \cap B}.$$

Similarly,

$$Q_j^k|_{V_j \cap V_l \cap B} = Q_l^k|_{V_j \cap V_l \cap B}.$$

Thus, if we set $P_k(z) = P_j^k(z),\ z \in V_j \cap B(j = 1, \cdots, m)$, and $Q_k(z) = Q_j^k(z),\ z \in V_j \cap B\,(j = 1, \cdots, m)$, then $P_k \in H^\infty(\Omega_\delta),\ Q_k \in C(\overline{\Omega}_\delta)$, and $F_k = P_k Q_k$ on Ω_δ. Note that each P_j^k is invertible in $H^\infty(V_j \cap B)\,(j = 1, \cdots, m)$. Hence P_k is invertible in $H^\infty(\Omega_\delta)$, i.e., $P_k^{-1} \in H^\infty(\Omega_\delta)$. Clearly, Q_k is invertible in $C(\overline{\Omega}_\delta)$. By Hartogs theorem, we know that P_k and P_k^{-1} have analytic extensions on B. Write their extensions by $\widetilde{P}_k$ and $\widetilde{P}_k^{-1}$, repectively.

Then $\widetilde{P}_k$ and $\widetilde{P}_k^{-1}$ belong to $H^\infty(B)$ by the maximal model principle. We can continuously extend Q_k and Q_k^{-1} onto B, and denote their extensions by $\widetilde{Q}_k$ and $\widetilde{Q}_k^{-1}$, respectively. Thus

$$\widetilde{Q}_k\widetilde{Q}_k^{-1}\big|_S = (Q_kQ_k^{-1})\big|_S = 1, \quad (\widetilde{P}_k\widetilde{P}_k^{-1})^* = (P_k^*P_k^{-1*}) = 1.$$

Consequently

$$[(\widetilde{P}_k\widetilde{Q}_k)(\widetilde{P}_k^{-1}\widetilde{Q}_k^{-1})]^* = P_k^*Q_k\big|_S P_k^{-1*}Q_k^{-1}\Big|_S = P_k^*P_k^{-1*}Q_k\big|_S Q_k^{-1}\big|_S = 1.$$

Note $F_k^* = \widetilde{P}_k^*\widetilde{Q}_k\big|_S = P_k^*Q_k\big|_S$, and

$$P_k^{-1*}Q_k^{-1}\big|_S = (\widetilde{P}_k^{-1})^*\widetilde{Q}_k^{-1}\big|_S \in H^\infty(S) + C(S).$$

Hence F_k^* is invertible in $H^\infty(S) + C(S)$, i.e., $0 \notin \sigma_{H^\infty(S)+C(S)}(F_k^*)$. From $\|F_k^* - f\|_\infty \to 0$ $(k \to \infty)$, we easily know that $f^{-1} \in H^\infty(S) + C(S)$ since $H^\infty(S) + C(S)$ is closed and f is invertible in $L^\infty(S)$.

Conversely, if f is invertible in $H^\infty(S) + C(S)$, then $\|P_r[f]P_r[f^{-1}] - 1\|_\infty \to 0$ $(r \to 1^-)$ by Lemma 2.1. Thus there are $\varepsilon, \delta > 0$ such that

$$|P_r[f](\zeta)P_r[f^{-1}](\zeta)| \geq 1 - \|P_r[f]P_r[f^{-1}] - 1\|_\infty > 1 - \varepsilon \,(0 < \varepsilon < 1)$$

holds for any $\zeta \in S$ and $\delta < r < 1$. Furthermore $|P[f](z)| > (1-\varepsilon)/M$ (where $M = \|P[f^{-1}]\|_\infty$), $\delta < |z| < 1$, i.e., $P[f]$ is below bounded on some $\Omega_\delta = \{z \in B | \delta < |z| < 1\}$; in other words, $0 \notin \bigcap\limits_{\delta>0} \overline{\{P[f](z) | \delta < |z| < 1\}}$. We complete the proof.

Lemma 2.3. *Suppose $f \in H^\infty(S)$ and $g \in C(S)$. Then*

(i) $T_{\bar f}C_z = P[\bar f](z)C_z, \quad (z \in B)$;

(ii) $\|T_gC_z - P[g](z)C_z\|_2 \to 0 \quad (|z| \to 1^-)$.

Proof. Since C_z is the normalized reproducing kernel of $H^\infty(S)$, for any $\varphi \in H^2(S)$, we have

$$\begin{aligned}\langle\varphi, T_{\bar f}C_z\rangle &= \langle f\varphi, C_z\rangle = (1-|z|^2)^{\frac{n}{2}}P[f](z)\cdot P[\varphi](z)\\ &= P[f](z)\langle\varphi, C_z\rangle = \langle\varphi, \overline{P[f]}(z)C_z\rangle = \langle\varphi, P[\bar f](z)C_z\rangle.\end{aligned}$$

Hence $T_{\bar f}C_z = P[\bar f](z)C_z$, it shows (i).

In the sequel, we always denote the ideal of compact operators on $H^2(S)$ by $\mathcal{K}$. By L. A. Coburn[3], we have $T_{\bar g}T_g - T_{|g|^2} \in \mathcal{K}$; on the other hand,

$$\begin{aligned}\langle T_{|g|^2}C_z, C_z\rangle &= \langle |g|^2C_z, C_z\rangle = \int_S |g|^2|C_z|^2d\sigma(\zeta)\\ &= \int_S |g|^2p(z,\zeta)d\sigma(\zeta) = P[|g|^2](z).\end{aligned}$$

Similarly,

$$\langle T_gC_z, C_z\rangle = P[g](z), \quad \langle T_{\bar g}C_z, C_z\rangle = P[\bar g](z).$$

Thus

$$\begin{aligned}&\|T_gC_z - P[g](z)C_z\|_2^2\\ &= \langle T_gC_z, T_gC_z\rangle - \langle T_gC_z, P[g](z)C_z\rangle - \langle P[g](z)C_z, T_gC_z\rangle + \langle P[g](z)C_z, P[g](z)C_z\rangle\\ &= \langle T_{\bar g}T_gC_z, C_z\rangle - P[g](z)P[\bar g](z) - P[g](z)P[\bar g](z) + P[\bar g](z)P[g](z)\\ &= \langle T_{|g|^2}C_z, C_z\rangle - |P[g](z)|^2 + \langle KC_z, C_z\rangle\\ &= P[|g|^2](z) - |P[g](z)|^2 + \langle KC_z, C_z\rangle,\end{aligned}$$

where $K = T_{\bar{g}}T_g - T_{|g|^2} \in \mathcal{K}$. Note $C_z \xrightarrow{w} 0\,(|z| \to 1^-)$, so $\lim\limits_{|z|\to 1^-} \langle KC_z, C_z\rangle = 0$. By Lemma 2.1,

$$\lim_{|z|\to 1^-} (P[|g|^2](z) - |P[g](z)|^2) = 0.$$

Hence

$$\lim_{|z|\to 1^-} (\|T_g C_z - P[g](z)C_z\|_2) = 0,$$

that is, (ii) follows.

Theorem 2.1. *Let $f \in H^\infty(S) + C(S)$. Then*

$$\sigma_e(T_f) = \sigma_{H^\infty(S)+C(S)}(f).$$

In particular, $\sigma_e(T_f)$ is a connected set.

Proof. Suppose $f = f_1 + f_2,\ f_1 \in H^\infty(S),\ f_2 \in C(S)$. First, we prove that

$$\sigma_{H^\infty(S)+C(S)}(f) \subset \sigma_e(T_f).$$

Without loss of generality we assume $0 \notin \sigma_e(T_f)$, i.e., T_f is a Fredholm operator. Then there is an $S \in L(H^2(S)),\ K \in \mathcal{K}$, such that $ST_f^* = I + K$. By Lemma 2.3, we have

$$T_{\bar{f}_1} C_z = P[\bar{f}_1](z)C_z, \quad \text{and} \quad \|T_{\bar{f}_2}C_z - P[\bar{f}_2](z)C_z\|_2 \to 0\,(|z| \to 1^-).$$

Thus

$$\|(T_{\bar{f}} - P[\bar{f}](z))C_z\|_2 = \|T_{\bar{f}_2}C_z - P[\bar{f}_2](z)C_z\|_2 \to 0 \quad (|z| \to 1^-).$$

By Lemma 2.2, we know

$$\sigma_{H^\infty(S)+C(S)}(f) = \bigcap_{\delta>0} \overline{\{P[f](z) | \delta < |z| < 1\}}.$$

Hence it is sufficient to prove

$$0 \notin \bigcap_{\delta>0} \overline{\{P[f](z) | \delta < |z| < 1\}}.$$

If $0 \in \bigcap\limits_{\delta>0} \overline{\{P[f](z) | \delta < |z| < 1\}}$, then there is a sequence $\{z_k\}$ of points in B, such that $|z_k| \to 1^-$ as $k \to \infty$, and $P[f](z_k) \to 0 (k \to \infty)$. Since $\|(T_{\bar{f}} - P[\bar{f}](z))C_z\|_2 \to 0\,(|z| \to 1^-)$,

$$\|T_{\bar{f}}C_{z_k}\|_2 \to 0\,(k \to \infty),$$

we have

$$\|ST_{\bar{f}}C_{z_k}\|_2 \to 0\,(k \to \infty).$$

Note

$$ST_{\bar{f}} = I + K, \quad K \in \mathcal{K},$$

so $\|C_{z_k} + KC_{z_k}\|_2 \to 0$, but $\|C_{z_k}\|_2 = 1,\ \ \|KC_{z_k}\|_2 \to 0$; we get a contradiction. It shows $\sigma_{H^\infty(S)+C(S)}(f) \subset \sigma_e(T_f)$.

The inversive inclusion follows from [4]. The proof is complete.

Corollary 2.1. *Suppose $n > 1$, $f \in H^\infty(S) + C(S)$. Then*

$$\sigma_w(T_f) = \sigma_e(T_f),$$

where $\sigma_w(T_f)$ denotes the Weyl spectrum of T_f.

Proof. Since $\sigma_w(T_f) = \sigma_e(T_f) \cup \{\lambda \in \rho_e(T_f) | \mathrm{Ind}(T_f - \lambda) \neq 0\}$, where $\rho_e(T_f)$ denotes the Fredholm domain of T_f, it is sufficient to prove that $\mathrm{Ind}(T_f - \lambda) = 0$ if $\lambda \in \rho_e(T_f)$. And this is a direct consequence of Lemma 2.2 and Theorem 3.1 in [4].

§3. Toeplitz Operators on Weighted Bergman Space

Lemma 3.1. *Let $a > -1/2$, $T \in L(L_a^2(B, dV_a))$ commute with $T_{z_j}^{(a)}$ $(j = 1, \cdots, n)$, where z_j is the coordinate function. Then there is an $h \in H^\infty(B)$ such that $T = T_h^{(a)}$.*

Proof. For any $\varphi \in H^\infty(B)$, let $\varphi_r(z) = \varphi(rz)$, $0 < r < 1$. Then $\lim\limits_{r\to 1^-} \varphi_r(z) = \varphi(z)$ $(z \in B)$, $\varphi_r \in A(B)$, and $\|\varphi_r\|_\infty \leq \|\varphi\|_\infty$. Since $A(B)$ is a uniformly closed subalgebra of $H^\infty(B)$ generated by z_j $(j = 1, \cdots, n)$, we know that $\{T_\varphi^{(a)} | \varphi \in H^\infty(B)\}$ is a w-closed algebra generated by $T_{z_j}^{(a)}$ $(j = 1, \cdots, n)$. Hence, for any $\varphi \in H^\infty(B)$, T commutes with $T_\varphi^{(a)}$. Let $h = T1$. Then $h \in L_a^2(dV_a)$, and for any $\varphi \in H^\infty(B)$,

$$T\varphi = TT_\varphi^{(a)}1 = T_\varphi^{(a)}T1 = T_\varphi^{(a)}h = \varphi h.$$

Since $H^\infty(B)$ is dense in $L_a^2(B, dV_a)$, for any $\psi \in L_a^2(B, dV_a)$, there are $\varphi_n \in H^\infty(B)$ such that $\|\varphi_n - \psi\|_{L_a^2(dV_a)} \to 0\,(n \to \infty)$. Note $T \in L(L_a^2(dV_a))$, so $T\psi \in L_a^2(dV_a)$, and

$$\|T\varphi_n - T\psi\|_{L_a^2(dV_a)} \leq \|T\| \cdot \|\varphi_n - \psi\|_{L_a^2(dV_a)} \to 0,$$

i.e., $\|\varphi_n h - T\psi\|_{L_a^2(dV_a)} \to 0$. Thus

$$\begin{aligned}\|h\psi - T\psi\|_{L_a^1(dV_a)} &\leq \|h\psi - \varphi_n h\|_{L_a^1(dV_a)} + \|\varphi_n h - T\psi\|_{L_a^1(dV_a)} \\ &\leq \|h\|_{L_a^2(dV_a)}\|\psi - \varphi_n\|_{L_a^2(dV_a)} + \|\varphi_n h - T\psi\|_{L_a^2(dV_a)} \to 0.\end{aligned}$$

Hence $T\psi = h\psi$. Now we prove that h is bounded. Since $h = T1$, we have

$$h(z) = \langle hK_z^{(a)}, K_z^{(a)}\rangle_a = \langle K_z^{(a)}T1, K_z^{(a)}\rangle_a = \langle TK_z^{(a)}, K_z^{(a)}\rangle_a,$$

($\langle\cdot,\cdot\rangle_a$ denotes the inner product in $L_a^2(dV_a)$). Consequently $|h(z)| \leq \|T\| \cdot \|K_z^{(a)}\|^2_{L_a^2(dV_a)} = \|T\|$. Hence h is bounded and $T = T_h^{(a)}$.

Theorem 3.1. *Let $\mathcal{A}$ be any closed subalgebra of $L(L_a^2(dV_a))$ containing $T_{z_j}^{(a)}$ $(j = 1, \cdots, n)$. Then $\mathcal{A}$ is an irreducible algebra.*

Proof. It is sufficient to prove that there is no nontrivial orthogonal projection in the commutant $\mathcal{A}'$ of $\mathcal{A}$. Let $P \in \mathcal{A}'$ be an orthogonal projection. Then there is an $h \in H^\infty(B)$ such that $P = T_h^{(a)}$ by Lemma 3.1, since $T_{z_j}^{(a)} \in \mathcal{A}(j = 1, \cdots, n)$. Furthermore $T_{h^2}^{(a)} = T_h^{(a)^2} = T_h^{(a)}$, i.e., $h^2 = h$. Hence $h = 1$ or 0, i.e., $P = I$ or 0. The proof is complete.

Let $\mathcal{L}^{(a)}(C(\overline{B}))$ and $\mathcal{L}^{(a)}(H^\infty(B) + C(\overline{B}))$ be the closed algebras generated by

$$\{T_\varphi^{(a)} | \varphi \in C(\overline{B})\} \quad \text{and} \quad \{T_\varphi^{(a)} | \varphi \in H^\infty(B) + C(\overline{B})\},$$

respectively. We know that $\mathcal{L}^{(a)}(C(\overline{B}))$ and $\mathcal{L}^{(a)}(H^\infty(B)+C(\overline{B}))$ are irreducible by Theorem 3.1.

Let $\mathcal{C}_0^{(a)}$ and $\mathcal{C}^{(a)}$ denote the commutator ideal of $\mathcal{L}^{(a)}(C(\overline{B}))$ and $\mathcal{L}^{(a)}(H^\infty(B)+C(\overline{B}))$, respectively, $\mathcal{K}^{(a)} = \mathcal{K}^{(a)}(L_a^2(dV_a))$ the ideal of compact operators in $L(L_a^2(dV_a))$. Then we have the following

Lemma 3.2. $\mathcal{C}_0^{(a)} = \mathcal{C}^{(a)} = \mathcal{K}^{(a)}$.

Proof. First, we prove that $H_\varphi^{(a)}$ is a compact operator if $\varphi \in C(\overline{B})$. Since $\varphi_z(\omega) \to \zeta (z \to \zeta \in S)$ for any $w \in B$, $\varphi(\varphi_z(w)) \to \varphi(\zeta)$ $(z \to \zeta)$. So

$$\|\varphi \circ \varphi_z - P^{(a)}(\varphi \circ \varphi_z)\|_{L^2(dV_a)} \to 0\,(|z| \to 1^-).$$

Hence $H_\varphi^{(a)}$ is compact by Theorem 16 in [6]. For any $h, k \in L_a^2(dV_a)$, we have

$$\begin{aligned}\langle H_f^{(a)*} H_g^{(a)} h, k\rangle_a &= \langle H_g^{(a)} h, H_f^{(a)} k\rangle_a = \langle (I - P^{(a)})gh, (I - P^{(a)})fk\rangle_a \\ &= \langle (I - P^{(a)})gh, fk\rangle_a = \langle gh, fk\rangle_a - \langle P^{(a)}gh, fk\rangle_a \\ &= \langle T_{\bar{f}g}^{(a)} h, k\rangle_a - \langle T_{\bar{f}} T_g h, k\rangle_a \quad (f, g \in L^\infty(B)).\end{aligned}$$

Hence $H_f^{(a)*} H_g^{(a)} = T_{\bar{f}g}^{(a)} - T_{\bar{f}}^{(a)} T_g^{(a)}$. Furthermore, $T_{\bar{f}g}^{(a)} - T_{\bar{f}}^{(a)} T_g^{(a)}$ is a compact operator for $g \in C(\overline{B})$, $f \in L^\infty(B)$. Consequently, $[T_f^{(a)}, T_g^{(a)}] = T_f^{(a)} T_g^{(a)} - T_g^{(a)} T_f^{(a)}$ is also a compact operator for any $f, g \in C(\overline{B})$. Hence $\mathcal{C}_0^{(a)} \subset \mathcal{K}^{(a)}$. Clearly, $\mathcal{C}_0^{(a)} \neq \{0\}$, since $\mathcal{L}^{(a)}(C(\overline{B}))$ is an irreducible C^*-algebra, $\mathcal{K}^{(a)} \subset \mathcal{L}^{(a)}(C(\overline{B}))$ by Corollary 2 of Theorem 1.4.2 in [1]. Therefore $\mathcal{C}_0^{(a)} = \mathcal{K}^{(a)}$.

To complete the proof, let $f, g \in H^\infty(B) + C(\overline{B})$, $f = f_1 + f_2$, $g = g_1 + g_2$, $f_1, g_1 \in H^\infty(B)$, $f_2, g_2 \in C(\overline{B})$. Then

$$\begin{aligned}&T_f^{(a)} T_g^{(a)} - T_{fg}^{(a)} \\ &= (T_{f_1}^{(a)} + T_{f_2}^{(a)})(T_{g_1}^{(a)} + T_{g_2}^{(a)}) - T_{(f_1+f_2)(g_1+g_2)}^{(a)} \\ &= T_{f_1}^{(a)} T_{g_1}^{(a)} + T_{f_2}^{(a)} T_{g_1}^{(a)} + T_{f_1}^{(a)} T_{g_2}^{(a)} + T_{f_2}^{(a)} T_{g_2}^{(a)} \\ &\quad - T_{f_1 g_1}^{(a)} - T_{f_2 g_1}^{(a)} - T_{f_1 g_2}^{(a)} - T_{f_2 g_2}^{(a)} \\ &= (T_{f_1}^{(a)} T_{g_2}^{(a)} - T_{f_1 g_2}^{(a)}) + (T_{f_2}^{(a)} T_{g_2}^{(a)} - T_{f_2 g_2}^{(a)}) \in \mathcal{K}^{(a)}.\end{aligned}$$

Hence $\mathcal{K}^{(a)} = \mathcal{C}_0^{(a)} \subset \mathcal{C}^{(a)} \subset \mathcal{K}^{(a)}$, i.e., $\mathcal{C}_0^{(a)} = \mathcal{C}^{(a)} = \mathcal{K}^{(a)}$.

Write

$$Z(B) = \{f \in C(\overline{B}) |\, f|_S = 0\}; \; N(B) = \{f \in H^\infty(B) + C(\overline{B}) | f^* = 0\}.$$

Lemma 3.3. (i) $Z(B) = N(B)$;

(ii) *Let $f \in H^\infty(B) + C(\overline{B})$. Then $T_f^{(a)}$ is compact if and only if $f \in N(B)$.*

Proof. The proof of (i) is easy. We prove only (ii). First, we assume $f \in N(B)$. Then $f \in C(\overline{B})$ and $f|_s = 0$ by (i). For any $\varepsilon > 0$, choose $g \in C(\overline{B})$ such that $\|f - g\|_\infty < \varepsilon$, and g vanishes on some neighborhood of S. Suppose that h_m is a sequence of unit vectors in $L_a^2(dV_a)$ converging weakly to 0. Set $K = \overline{\{z \in B | g(z) \neq 0\}}$. Then K is a compact subset of B. Since $h_m \xrightarrow{w} 0$ in $L_a^2(dV_a)$, we have $h_m|_K \to 0$ uniformly. Hence

$$\begin{aligned}\|T_g^{(a)} h_m\|_{L_a^2(dV_a)} &= \|P^{(a)} g h_m\|_{L_a^2(dV_a)} \le \|g h_m\|_{L^2(dV_a)} \\ &\le \|g\|_\infty \|h_m|_K\|_{L^2(dV_a)} \to 0 \; (m \to \infty).\end{aligned}$$

So $T_g^{(a)}$ is compact, and $T_f^{(a)}$ is also compact by

$$\|T_f^{(a)} - T_g^{(a)}\| \le \|f - g\|_\infty < \varepsilon.$$

Conversely, if $T_f^{(a)}$ is a compact operator, then for $h_z = K_z^{(a)}$ we have

$$\|T_f^{(a)} h_z\|_{L_a^2(dV_a)} \to 0\,(|z| \to 1^-)$$

since $h_z \xrightarrow{w} 0$ in $L^2_a(dV_a)$. Furthermore,

$$\langle fK_z^{(a)}, K_z^{(a)}\rangle_a = \langle T_f^{(a)}K_z^{(a)}, K_z^{(a)}\rangle_a \to 0\,(|z| \to 1^-).$$

Note $(\langle fK_z^{(a)}, K_z^{(a)}\rangle_a)^* = f^*$. Hence $f^* = 0$, i.e., $f \in N(B)$. The proof is complete.

Lemma 3.4. *Let* $f \in H^\infty(S)$, $g \in C(S)$. *Then*

(i) $T_{P[\bar{f}]}^{(a)}K_z^{(a)} = P[\bar{f}](z)K_z^{(a)}$;

(ii) *if* $z \to \zeta \in S$, *then* $\|T_{P[g]}^{(a)}K_z^{(a)} - g(\zeta)K_z^{(a)}\|_{L^2(dV_a)} \to 0$.

The proof is similar to that of Lemma 2.3.

Theorem 3.2. *Let* $f \in H^\infty(B) + C(\overline{B})$. *Then*

$$\sigma_e(T_f^{(a)}) = \sigma_{H^\infty(S)+C(S)}(f^*)$$

for any $a > -1/2$. *In particular,* $\sigma_e(T_f^{(a)})$ *is a connected set.*

Proof. Since $f^* = (P[f^*])^*$, a. e. $T_f^{(a)} - T_{P[f^*]}^{(a)}$ is compact by Lemma 3.3. Hence

$$\sigma_e(T_f^{(a)}) = \sigma_e(T_{P[f^*]}^{(a)}).$$

It is sufficient to prove that

$$\sigma_e(T_{P[f]}^{(a)}) = \sigma_{H^\infty(S)+C(S)}(f)$$

if $f \in H^\infty(S) + C(S)$. By Lemma 2.2 and Lemma 3.4, the proof of $\sigma_{H^\infty(S)+C(S)}(f) \subset \sigma_e(T_{p[f]}^{(a)})$ is similar to that of Theorem 2.1. Now we prove that

$$\sigma_{H^\infty(S)+C(S)}(f) \supset \sigma_e(T_{p[f]}^{(a)}).$$

Without loss of generality, we assume $0 \notin \sigma_{H^\infty(S)+C(S)}(f)$. Then $f^{-1} \in H^\infty(S) + C(S)$. Note $(P[f]P[f^{-1}])^* = 1$ a.e. Hence $I - T_{P[f]P[f^{-1}]}^{(a)} \in \mathcal{K}^{(a)}$. Since

$$P[f],\ \ P[f^{-1}] \in H^\infty(B) + C(\overline{B}),\ \ T_{P[f]}^{(a)}T_{P[f^{-1}]}^{(a)} - T_{P[f]P[f^{-1}]}^{(a)} \in \mathcal{K}^{(a)}$$

by Lemma 3.2, we have $I - T_{P[f]}^{(a)}T_{P[f^{-1}]}^{(a)} \in \mathcal{K}^{(a)}$. Similarly, $I - T_{P[f^{-1}]}^{(a)}T_{P[f]}^{(a)} \in \mathcal{K}^{(a)}$. It shows that $T_{P[f]}^{(a)}$ is a Fredholm operator, i.e., $0 \notin \sigma_e(T_{P[f]}^{(a)})$. So $\sigma_e(T_{P[f]}^{(a)}) \subset \sigma_{H^\infty(S)+C(S)}(f)$, the proof is complete.

References

[1] Arveson, W., An invitation to C^*-algebra, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[2] Cao, G. F. & Zou, C. Z., On stable convergence of operator value functions, *Acta Math. Sinica,* **35** (1992), 585-591.

[3] Coburn, L. A., Toeplitz operators on odd spaheres, *Lecture Notes in Math.,* **345**, 7-12, Springer-Verlag, Berlin.

[4] Davie, A. M. & Jewell, N. P., Toeplitz operators in several complex variables, *J. Functional Analysis,* **26** (1977), 356-368.

[5] Rudin, W., Function theory in the unit ball of $\mathbb{C}^n$, Springer-Verlag, New York, Heidelberg, Berlin, 1980.

[6] Stroethoff, L., Compact Hankel operators on the Bergman spaces of the unit ball and polydisk in $\mathbb{C}^n$, *J. Operator Theory,* **23:1** (1990), 153-170.

[7] Sundberg, C., Toeplitz operators on the Bergman space, *Lecture Notes in Math.,* **1043**, 274-275.

[8] Yu, D. H., Sun, S. H. & Dai, Z. G., The Fredholm theory of Toeplitz operators on Bergman spaces, *Northeneast Math.,* **4:4** (1988), 405-411.

[9] Zhu, K. H., Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, *J. Operator Theory,* **20** (1988), 329-357.