Chin. Ann. of Math.
16B: 1(1995),103-108.

HÖRMANDER MULTIPLIER THEOREM ON $SU(2)$

FAN DASHAN* XU ZENGFU*

Abstract

A boundedness criterion is set up for some convolution operators on a compact Lie group. By this criterion a Hörmander multiplier theorem is proved in the Hardy spaces on $SU(2)$.

Keywords Hörmander multiplier theorem, H^p spaces, Atomic decomposition, Special unitary group.
1991 MR Subject Classification 43A15, 43A77.

§1. Notation

Let G be a connected, simply connected, compact semisimple Lie group of dimension n, and $\mathbf{g}$ be the Lie algebra of G . Then $\mathbf{g}$ can be identified with $T_e(G)$, the tangent space of G at the identity element e of G. Let d be the bi-invariant metric on G and denote $d(x,e)$ by $|\ x\ |$ for $x \in G$. Let exp be the exponential map from $\mathbf{g}$ into G. Then exp is an analytic diffeomorphism on an open neighborhood of the origin of $\mathbf{g}$. Choose ϵ and δ to be the maximal positive numbers so that $\exp^{-1} \cdot L_{x^{-1}}$ is such a diffeomorphism from $B(x,\epsilon)$ onto $B(0,\delta)$. For a positive integer k we define

$\mathbf{P}_k = \{P : P(x) = q(\exp^{-1}(x_0^{-1}x)), q$ is a polynomial on $T_e(G)$ with degree $\le k\}$
as the set of polynomials on $B(x_0,\epsilon)$ with degree less than or equal to k.

Let $Y_1, Y_2, \cdots, Y_n$ be an orthonormal basis of $\mathbf{g}$ and define the differential operators on space $C^{|J|}(G)$ by

$$Y^J f(y) = (Y_1^{j_1} Y_2^{j_2} \cdots Y_n^{j_n}) f(x), \quad f \in C^{|J|},$$

where $J = (j_1, \cdots, j_n)$ is a multi-index and $|\ J\ | = j_1 + j_2 + \cdots + j_n$.

We now introduce the atomic Hardy spaces H^p on compact Lie groups. An exceptional atom is an L^∞ function bounded by 1. A regular $(p,2,N)$ atom for $0 < p \le 1$ is a function $a(x)$ supported in some ball $B(y,\rho)$ satisfying

$$\|\ a\ \|_2 \le \rho^{n/2-n/p} \quad \text{and} \quad \int_G a(x)P(x)dx = 0,$$

where $\rho < \epsilon$ and P is any polynomial on $B(y,\epsilon)$ of degree less than or equal to $N = [n(1/p-1)]$.

The atomic H^p space $H^p_a(G)$, $0 < p \le 1$, now is the space of all $f \in \mathcal{S}'(G)$ ($\mathcal{S}'(G)$ is the Schwartz distribution space on $\mathcal{S}(G)$) of the form

$$f = \sum c_k a_k \quad \text{with} \quad \sum |c_k|^p < \infty,$$

Manuscript received August 4, 1992.
*Department of Mathematics, Auhui University, Hefei 230039, China.

where each $a(x)$ is either a regular atom or an exceptional atom. The "norm" $\| f \|_{H^p_a}$ is the infimum of all expressions $(\sum | c_k |)^{1/p}$ for which we have such a representation of f. Various characterizations of Hardy spaces on compact Lie groups were studied in [2, 3, 4].

Finally, throughout this paper, the letter "C" will denote (possibly different) constants that are independent of the essential variables in the argument; this independence will be clear from the context.

§2. A Boundedness Criterion

Let L be a positive number. A function $K(x)$ on G is said to be in class $\widetilde{M}(2, L)$ if K satisfies the following conditions:

$$\int_{R<|x|<2R} |Y^J K(y)|^2 dy \le C R^{-(n+2|J|)}, \tag{2.1}$$

where $R > 0$, and $| J |< L$, and for $L = \widetilde{L} + \nu$, $0 < \nu \le 1$

$$\Big\{ \int_{R<|y|<2R} |Y^J K(y) - Y^J K(yx^{-1})|^2 dy \Big\}^{1/2}$$
$$\le \begin{cases} C(|x|R^{-1})^{\nu} R^{-(n/2+\widetilde{L})}, & 0 < \nu < 1, \\ C\frac{|x|}{R} \ln(\frac{|x|}{R}) R^{-(n/2+\widetilde{L})}, & \nu = 1, \end{cases} \tag{2.2}$$

for all $| x |< R/2$, where $| J |= \widetilde{L}$.

Theorem 2.1. *Suppose that $0 < p \le 1, L > n/p - n$. If $K \in \widetilde{M}(2, L)$ and the convolution operator $Tf = K * f$ is bounded in $L^2(G)$, then T can be extended to a bounded operator in H^p spaces:*

$$\| Tf \|_{H^p_a(G)} \le C \| f \|_{H^p_a(G)} .$$

To prove this theorem, we need the following lemma.

Lmma 2.1. *Suppose that T is an operator as in Theorem 2.1. Then $\| Ta \|_p \le C$ for any atom $a(x)$. Here the constant C is independent of atoms.*

Proof. It is enough to prove this lemma for any regular atom. For any regular atom $a(x)$ having support $B(x_0, \rho)$, $A(x) = a(x_0, x)$ is an atom with support $B(e, \rho)$. Thus, without loss of generality, we can assume that $a(x)$ is supported in $B(e, \rho)$. Let ϕ be a non-negative C^∞ radial function which satisfies

$$\text{supp}(\phi) \subseteq \{1/2 \le |x| \le 2\} \quad \text{and} \quad \sum_{j=-\infty}^{\infty} \phi(2^j|y|) = 1 \quad \text{for} \quad y \ne 0.$$

Let $\eta(x) = 1 - \sum_{j=1}^{\infty} \phi(2^{-j-2}\rho^{-1} | x |)$. Then

$$K(x) = \eta(x)K(x) + \sum_{j=-\infty}^{\infty} K(x)\phi(2^{-j-2}\rho^{-1}|x|)$$
$$= K_0(x) + \Sigma K_j(x).$$

Thus

$$K * a(x) = K_0 * a(x) + \Sigma K_j * a(x).$$

Clearly,

$$\text{supp}(K_0 * a) \subseteq B(e, 8\rho), \quad \text{supp}(K_j * a) \subseteq B(e, 2^{j+4}\rho), \ j = 1, 2, \cdots .$$

Therefore using the condition $K \in \widetilde{M}(2, L)$ and mimicking the proofs on $\mathbf{R}^n$ (see [6], p.359, also see [3] for some special techniques on G), we can easily obtain

$$\| K_0 * a \|_2 \leq C\rho^{n/2-n/p}, \quad \text{and} \quad \| K_j * a \|_2 \leq C(2^j\rho)^{n/2-n/p}2^j.$$

Now the lemma easily follows by using Hölder's inequality.

We are now ready to prove Theorem 2.1. It is enough to prove that

$$\| Ta \|_{H^p} \leq C \tag{$*$}$$

uniformly for all regular $(p, 2, N)$ atom $a(x)$.

In order to prove $(*)$ we will introduce the generalized Riesz transforms on compact Lie groups which were studied in [1]. For an integer $L \geq 0$ and a multi-index $J = (j_1, \cdots, j_L) \in \{0, 1, \cdots, n\}^L$ let $R_J(f)$ denote the generalized Riesz transform $R_J(f) = R_{j_1} \cdots R_{j_L} f$, where $R_j(f)$ is the j-th Riesz transform of G if $j \neq 0$ and $R_0 f = f$. It is proved in [1] that for $p > (n-1)/(n-1+L)$ and all $f \in L^2 \cap H^p$

$$\sum_J \| R_J(f) \|_p \cong \| f \|_{H^p}, \quad \| R_J(f) \|_{H^p} \leq C \| f \|_{H^p}.$$

The proof of Theorem 2.1 is now easily obtained from the inequalities above and Lemma 2.1. For any $p \in (0, 1]$ take $L > n/p - n$ so that $p > (n-1)/(n-1+L)$. For any $(p, 2, N)$ atom $a(x)$ we have the atomic decomposition of $R_J(a)$

$$R_J(a) = \sum \lambda_i b_i,$$

where b_i's are atoms and $\sum | \lambda_j |^p \sim \| R_J(a) \|_{H^p}^p \leq C$. Now notice that both T and R_J are convolution operators. Hence

$$\| Ta \|_{H^p}^p \leq C \sum_J \| T(R_J(a)) \|_p^p \leq C \sum_i | \lambda_i |^p \| Tb_i \|_p^p \leq C$$

and Theorem 2.1 is proved.

§3. A Hörmander Multiplier Theorem on $H_a^p(SU(2))$

Consider the special unitary group $G = SU(2)$. Any $f \in H_a^p(SU(2))$ has a Fourier series in the distribution sense:

$$f(y) \sim \sum_{n=1}^{\infty} f * n\chi_n(y)$$

where $\chi_n(y) = \sin(n\theta)/\sin(\theta)$ and y is conjugate to a diagonal matrix $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ (see [5]).

Let $\{\lambda(n)\}$ be a bounded sequence and define a Fourier multiplier associated with $\{\lambda(n)\}$ by

$$T_\lambda : f \to T_\lambda f \sim \sum_{n=1}^{\infty} \lambda(n) n\chi_n * f(y).$$

Clearly, T_λ is a convolution operator and $T_\lambda f = K * f$. Here

$$K(y) \sim \sum n\lambda(n)\chi_n(y)$$

is a distribution function.

Recall the difference operators defined on $\{\lambda(n)\}$ by

$$\delta^0(\lambda(n))=\lambda(n),\delta^1(\lambda(n))=\lambda(n+1)-\lambda(n),\cdots,\delta^{j+1}(\lambda(n))=\delta^1(\delta^j(\lambda(n))).$$

If the sequence $\{\lambda(n)\}$ satisfies

$$\sum_{R\le|n|<2R}|\delta^j(\lambda(n))|^2\le CR^{1-2j}$$

for any $R>0$ and integer j between 0 and L, then the $\{\lambda(n)\}$ is said to be in Hörmander multiplier class $M(2,L)$.

Theorem 3.1. *Suppose that* $\{\lambda(n)\}\in M(2,L)$ *and* $L>3/p-3$. *Then* T_λ *can be extended to a bounded operator in the space* $H^p(SU(2))$.

Proof. By Theorem 2.1, we should prove that $K_\lambda\in\widetilde{M}(2,L)$ for some $L>3/p-3$. For the sake of simplicity, we will prove only the case $2/3<p\le 1$. The proof for the case of general p will be clear after proving this special case together with an explaination.

For $2/3<p\le 1$, we need to prove that for small $R>0$,

(i) $\int_{R<|y|<2R}|Y_jK(y)|^2dy\le CR^{-5}$,

(ii) $\int_{R<|y|<2R}|K(y)|^2dy\le CR^{-3}$,

(iii) $\int_{R<|y|<2R}|Y_jK(y)-Y_jK(yx^{-1})|^2dy\le C|x|R^{-6}$.

For simplicity, we denote $|y|\sim R$ if $C_1R<|y|<C_2R$ for some constants C_1 and C_2.

Notice that $|\theta|\sim|y|$, and $dy=\sin^2(\theta)d\theta d\Sigma$, where $d\Sigma$ is the Harr measure of $(SU(2)/T)$. Clearly,

$$\begin{aligned}\Big(\int_{|y|\sim R}|K(y)|^2dy\Big)^{1/2}&=O\Big(\int_{|\theta|\sim R}\Big|\frac{d}{d\theta}\sum_{|n|\le 1/R}\lambda(|n|)e^{in\theta}\Big|^2d\theta\Big)^{1/2}\\&\quad+O\Big(\int_{|\theta|\sim R}\Big|\frac{d}{d\theta}\sum_{|n|>1/R}\lambda(|n|)e^{in\theta}\Big|^2d\theta\Big)^{1/2}\\&=E_1+E_2.\end{aligned}$$

We observe

$$\begin{aligned}E_2&=O(R^{-2})\Big\{\int_{|\theta|\sim R}\Big|\frac{d}{d\theta}\Big\{\sum_{|n|\ge 1/R}\lambda(|n|)e^{in\theta}(e^{i\theta}-1)^2\Big\}\Big|^2d\theta\Big\}^{1/2}\\&\quad+O(R^{-2})\Big\{\int_{|\theta|\sim R}\Big|\sum_{|n|\ge 1/R}\lambda(|n|)e^{in\theta}(e^{i\theta}-1)\Big|^2d\theta\Big\}^{1/2}\\&=U+V\end{aligned}$$

and

$$\begin{aligned}U&=O(R^{-3/2})+O(R^{-2})\Big\{\int_{|\theta|\sim R}\Big|\sum_{|n|\ge 1/R}\delta^2(\lambda(|n|))ne^{in\theta}\Big|^2d\theta\Big\}^{1/2}\\&=O(R^{-3/2})+O(R^{-2})\Big\{\sum_{k=-lnR}^{\infty}\sum_{2^k\le|n|\le 2^{k+1}}n^2|\delta^2(\lambda(|n|))|^2\Big\}^{1/2}\\&=O(R^{-3/2})+O(R^{-2})\Big\{\sum_{k=-lnR}^{\infty}2^{-k}\Big\}^{1/2}=O(R^{-3/2}).\end{aligned}$$

Similarly,

$$V = O(R^{-2})\Big\{ \sum_{k=-\ln R} \sum_{2^k \le |n| \le 2^{k+1}} |\delta(\lambda(|n|))|^2 \Big\}^{1/2} + O(R^{-3/2}) = O(R^{-3/2}).$$

This shows that $E_2 = O(R^{-3/2})$.

Using the same argument, we have

$$\begin{aligned} E_1 &= O(R^{-1})\Big\{ \int_{|\theta|\sim R} \Big| \frac{d}{d\theta} \Big(\sum_{|n|\le 1/R} \lambda(|n|) e^{in\theta}(e^{i\theta}-1) \Big) \Big|^2 d\theta \Big\}^{1/2} \\ &\quad + O(R^{-1})\Big\{ \int_{|\theta|\sim R} \Big| \sum_{|n|\le 1/R} \lambda(|n|) e^{in\theta} \Big|^2 d\theta \Big\}^{1/2} \\ &= O(R^{-3/2}) + O(R^{-1})\Big(\sum_{0\le k \le -\ln R} 2^k \Big) = O(R^{-3/2}). \end{aligned}$$

This proves (ii).

To prove (i) and (iii), we need the following formula proved by Mayer in [5]:

$$Y_j \chi_n(y) = \{(n+1)\chi_{n-1}(y) - (n-1)\chi_{n+1}\} E(y),$$

where

$$E(y) = Y_j \chi_2(y)(3-\chi_3(y))^{-1} \simeq |\theta|^{-1}, \quad Y^J E(y) \simeq |\theta|^{-1-|J|}.$$

By this Mayer's formula and a simple calculation,

$$\begin{aligned} Y_j K(y) &= \Big\{ \sum_{n\ge 0} \lambda(n+1)(n^2+3n+2)\chi_n(y) - \sum_{n\ge 2} \lambda(n-1)(n^2-3n+2)\chi_n(y) \Big\} E(y) \\ &= \Big\{ \sum_{n\ge 2} \delta^1(\lambda(n))(n^2+2)\chi_n(y) \Big\} E(y) + \Big\{ \sum_{n\ge 0} \delta^1(\lambda(n-1))(n^2+2)\chi_n(y) \Big\} E(y) \\ &\quad + \Big\{ \sum_{n\ge 2} 3n(\lambda(n+1)+\lambda(n-1))\chi_n(y) \Big\} E(y) + O(|\theta|^{-1}). \end{aligned}$$

Using the same argument that we used in proving (ii), we easily obtain (i).

Now we turn to the proof of (iii). We first observe

$$\begin{aligned} &\Big\{ \int_{R<|y|<2R} |Y_j K(y) - Y_j K(yx^{-1})|^2 dy \Big\}^{1/2} \\ &= \sup_{\substack{f\in C^\infty \\ \mathrm{supp}(f)\subseteq B(0,2R)\setminus B(0,R) \\ \|f\|_2\le 1}} \Big| \int_G (Y_j K(y) - Y_j K(yx^{-1}) f(y) dy \Big| \\ &= O(|x|) \sup_{\|f\|_2\le 1} \sup_{|z|\le R/2} \Big| \int_G Y_j K(y) \sum_{i=1}^3 Y_i f(yz) dy \Big| \\ &= O(|x|) \Big\{ \int_{|y|\sim R} \sum_{i=1}^3 |Y_i Y_j K(y)|^2 dy \Big\}^{1/2}. \end{aligned}$$

By Mayer's formula again, after a careful calculation, we have

$$\begin{aligned}
&Y_iY_jK(y)\\
&= Y_jK(y)Y_iE(y)/E(y) + \Big(\sum_n (n^3+7n)\{\delta^2(\lambda(n)) - \delta^2(\lambda(n-1))\}\chi_n(y)\Big)E^2(y)\\
&\quad + \Big(\sum_n (7n^2+6)\{\delta^1(\lambda(n+1)) + \delta^1(\lambda(n)) + \delta^1(\lambda(n-1)) + \delta^1(\lambda(n-2))\}\chi_n(y)\Big)E^2(y)\\
&\quad + \Big(\sum_n 9n\{\lambda(n+2) + 2\lambda(n) + \lambda(n-2)\}\Big)E^2(y) + O(|\,\theta\,|^{-2}).
\end{aligned}$$

Thus we easily obtain (iii) by mimicking the proof of (i). This completes the case of $2/3 < p \le 1$. From the above estimate of $Y_iY_jK(y)$, it is not difficult to see that

$$Y^JK(y) = \sum_n \sum_{i=0}^{|J|} \sum_{k=-i}^{i} P_{i+1}^K(n)\delta^i(\lambda(n+k))\chi_n(y)O(|\theta|^{-|J|}) + O(|\theta|^{-|J|}),$$

where $P_{i+1}^k(x)$ is a polynomial dependent of k with degree $\le i$. For the reason of simplicity, many $P_{i+1}^k(n)$ may be zero in the above formula. Thus the case $0 < p \le 2/3$ is easily proved by mimicking the proofs of (i), (ii) and (iii). Therefore the proof of Theorem 3.1 is complete.

Note. The Hörmander multiplier theorem on compact Lie group G was first set up in $L^p(G)$ by N. Weiss[7]. Recently, in a way very different to this paper, we proved this Hörmander multiplier theorem on $H^p(G)$(see [4]). But both Weiss' result and ours need a restriction $\{\lambda_n\} \in M(2,L)$ with $L > n/p - n/2$ ($L > n/2$ for the case $p > 1$) and L being an even integer. In this paper, we have no this restriction of even integer on $SU(2)$.

References

[1] Bloom, W. R. & Xu Zengfu, Approximation of H^p functions by Bochner-Riesz means on compact Lie groups, *Math. Z.,* **216** (1994), 131-145.

[2] Clerc, J. L., Bochner-Riesz means of H^p functions ($0 < p < 1$) on compact Lie groups, *Lecture Notes in Math.,* **1243** (1987), 86-107.

[3] Fan, D., Hardy spaces on compact Lie groups, Ph. D. Thesis, Washington University, St. Louis, 1990.

[4] Fan, D. & Xu, Z., H^p estimate for bi-invariant operators on compact Lie groups, *Proc. Amer. Math. Soc.,* **122:2** (1994), 437-447.

[5] Mayer, R. A., Fourier series of differentiable functions on $SU(2)$, *Duke Math.,* **34** (1967), 93-101.

[6] Torchinsky, A., Real-variable methods in harmonic analysis, Academic Press, 1986.

[7] Weiss, N., L^p estimate for bi-invariant operators on compact Lie groups, *Amer. J. of Math.,* **94** (1972), 103-118.