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FEASIBILITY OF THE REICH PROCEDURE IN THE DECOMPOSITION OF PLANE QUASICONFORMAL MAPPINGS**

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Abstract

In the decomposition problems, studied by Reich, of quasiconformal self mappings of the unit disc which keep the boundary points fixed, the construction of the first one requires the application of the Hahn-Banach theorem (so it is abstract) and it is only a variational decomposition (a small weight one), and that of the second one avoids the Hahn-Banach theorem and gets rid of the restriction to the variational decomposition. But the success of the second decomposition procedure (the Reich procedure) is guaranteed only when minimal maximal dilatation $K(f)$ is sufficiently small. Therefore, it can not guarantee even a variational decomposition. Huang Xinzhong then proved that the inverse Reich procedure was successful for any $K(f)$. But the inverse Reich procedure is not so natural as the Reich procedure and the corresponding decomposition can not replace the first one. It is still an open problem whether the Reich procedure is successful for any $K(f)$. The present paper gives an affirmative answer to this problem.

Keywords Quasiconformal mapping, Decomposition, Reich's procedure.
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§1. Introduction

Let Q be the class of quasiconformal self mappings of the unit disc $U = \{|z| < 1\}$. For $f \in Q$, Q_f denotes all mappings in Q that agree with f on the boundary ∂U. Write $Q_I = Q_z$. Let f^μ denote the mapping in Q with complex dilatation μ, normalized so that $f^\mu(\pm 1) = \pm 1$, $f^\mu(i) = i$.

Set

$$\mu_f = \frac{f_{\bar z}}{f_z}, \qquad k(f) = \|\mu_f\|_\infty = \operatorname*{ess\,sup}_{z\in U}|\mu_f(z)|,$$

$$K(f) = \frac{1+k(f)}{1-k(f)}, \qquad F = \{\mu_f \mid f \in Q_I\}.$$

To avoid triviality we assume $k(f) > 0$.

Let B be the Banach space of functions $\phi(z)$ holomorphic in U, with

$$\|\phi\| = \iint_U |\phi(z)|dxdy < \infty, \qquad z = x + iy.$$

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For $g^* \in Q_g$, g^* is called an extremal mapping in Q_g if

$$K(g^*) = \inf_{f \in Q_g} K(f).$$

N will be the class of all complex valued measurable functions $v(z)$, $z \in U$, such that

$$\|v\|_\infty < \infty, \qquad \iint\limits_U v(z)\phi(z)dxdy = 0, \qquad \text{for all } \phi \in B.$$

For $\mu \in F$, the expression

$$L_\mu(\phi) = \iint\limits_U \frac{\mu}{1-|\mu|^2}\phi dxdy$$

defines a bounded linear functional over B. By the Hahn-Banach theorem and the Riesz representation theorem there exists a complex measurable function $\tau(z)$, $z \in U$, such that

$$\|\tau\|_\infty = \sup_{x \in U} |\tau(z)| = \|L_\mu\|$$

and

$$\iint\limits_U \frac{\mu}{1-|\mu|^2}\phi dxdy = \iint\limits_U \tau\phi dxdy, \qquad \text{for all } \phi \in B.$$

Hence

$$v(z) = \frac{\mu(z)}{1-|\mu(z)|^2} - \tau(z) \in N. \tag{1.1}$$

We are concerned with the question of the possibility of decomposing a given $f \in Q_I$ into factors

$$f = f_2 \circ f_1, \qquad f_i \in Q_I, \qquad K(f_i) < K(f), \qquad i = 1,2.$$

We further require that a decomposition should have a step length, i.e., a decomposition procedure $\{P_1, P_2\}$ is said to be feasible (or successful), if for $h \in Q_I$, $P_i h = h_i \in Q_I$, $i = 1,2$, we have $h = h_2 \circ h_1$ and

$$\sup_{\substack{h \in Q_I \\ K(h) \le K(f)}} \max\{K(P_1 h), K(P_2 h)\} < K(f).$$

The procedure of the first decomposition is as follows: For $f \in Q_I$ and its complex dilatation μ,

i) Find $v \in N$ in (1.1) as above;

ii) For $0 < t < 1/\|v\|_\infty$, let $g_1 = f^{tv}$ and $g_2 = f \circ g_1^{-1}$, we have $f = g_2 \circ g_1$, $g_i \in Q$, $i = 1,2$;

iii) Let g^* be an extremal mapping in Q_{g_1}, set $f_1 = g^{*-1} \circ g_1$ and $f_2 = g_2 \circ g^*$, then $f = f_2 \circ f_1$, $f_i \in Q_I$, $i = 1,2$.

There are two weaknesses in the above decomposition. One is that the construction of the decomposition depends on the Hahn-Banach theorem, thus it is abstract. And the other is that $\|v\|_\infty \le k(f)/(1-k(f))$ (see (2.11) of [1]), and the decomposing weight t is restricted by $t < 2/(K(f)-1)$. Thus $t \to 0$ as $K(f) \to \infty$, and hence it is only a variational decomposition (a small weight one).

The procedure of the second decomposition is as follows: For $f \in Q_I$ and its complex dilatation μ,

1) For $0 < t < 1$, let $g_1 = f^{t\mu}$ and $g_2 = f \circ g_1^{-1}$, we have $f = g_2 \circ g_1$, $g_i \in Q$, $i = 1, 2$;

2) Let g^* be an extremal mapping in Q_{g_1}, set $f_1 = g^{*-1} \circ g_1$ and $f_2 = g_2 \circ g^*$, then $f = f_2 \circ f_1$, $f_i \in Q_I$, $i = 1, 2$.

In what follows we call the above procedure the Reich procedure. The inverse Reich procedure is as follows:

1') For $0 < t < 1$, let $g_1 = f^{(1-t)\mu/(1-t|\mu|^2)}$ and $g_2 = f \circ g_1^{-1}$, we have $f = g_2 \circ g_1$, $g_i \in Q$, $i = 1, 2$;

2') Let g^* be an extremal mapping in Q_{g_1}, set $f_1 = g^{*-1} \circ g_1$ and $f_2 = g_2 \circ g^*$, then $f = f_2 \circ f_1$, $f_i \in Q_I$, $i = 1, 2$.

The construction of the second decomposition avoids the Hahn-Banach theorem and gets rid of the restriction to the variational decomposition. But Reich[1] proved that the success of the second procedure is guaranteed only when minimal maximal dilatation $K(f)$ is sufficiently small. Therefore, it can not guarantee even a variational decomposition. Huang Xinzhong[2] then proved that the inverse Reich procedure was successful for any $K(f)$. But the inverse procedure is not so natural as the Reich procedure and the corresponding decomposition can not replace the first one. It is still an open problem whether the Reich procedure is successful for any $K(f)$. Our present paper gives an affirmative answer to this problem.

We need two known results[3,4] for reference later:

Lemma 1.1. *If $\mu \in F$, then for any function $\phi \in B$,*

$$\left| \iint\limits_U \frac{\mu(z)\phi(z)}{1-|\mu(z)|^2} dxdy \right| \le \iint\limits_U \frac{|\mu(z)|^2|\phi(z)|}{1-|\mu(z)|^2} dxdy.$$

Lemma 1.2. *Suppose $g \in Q$, with complex dilatation $\kappa(z)$. If g^* is an extremal mapping in Q_g, $K(g^*) = (1+k^*)/(1-k^*)$, then*

$$\frac{k^*}{1-k^*} \le I(\kappa) + \Delta(\kappa),$$

where

$$I(\kappa) = \sup_{\substack{\phi\in B\\ \|\phi\|\le 1}} \left| \iint\limits_U \frac{\kappa(z)\phi(z)}{1-|\kappa(z)|^2} dxdy \right|,$$

and

$$\Delta(\kappa) = \sup_{\substack{\phi\in B\\ \|\phi\|\le 1}} \iint\limits_U \frac{|\kappa(z)|^2}{1-|\kappa(z)|^2} |\phi(z)| dxdy.$$

§2. Feasibility of the Inverse Reich Procedure

We will first point out the fact that "the inverse Reich procedure is feasible for $t = (K(f)+1)/(\sqrt{K(f)}+1)^2$ and any $K(f)$" is easy to prove.

Indeed, for $f \in Q_I$ and its complex dilatation μ, write $K = K(f)$, $t = (K+1)/(\sqrt{K}+1)^2$, $\mu_1(z) = (1-t)\mu(z)/(1-t|\mu(z)|^2)$. For f_i in 2'), $i = 1, 2$, we have

$$K(f_1) = K(g^{*-1} \circ g_1) \le K(g^*)K(g_1),$$
$$K(f_2) = K(g_2 \circ g^*) \le K(g^*)K(g_2).$$

Since the complex dilatation of g_1 is μ_1, $k(g_1) = \frac{(1-t)k}{1-tk^2}$,

$$K(g_1) = \frac{1+k(g_1)}{1-k(g_1)} = \frac{(1+k)(1-tk)}{(1-k)(1+tk)} = \sqrt{K}.$$

If we set $\mu_2(z) = g_{2\bar{z}}/g_{2z}$, then

$$\mu_2(g_1(z)) = t\mu(z)\frac{\overline{g_{1z}}}{g_{1z}}, \qquad K(g_2) = \frac{1+tk}{1-tk} = \sqrt{K}.$$

Thus

$$K(f_i) \le \sqrt{K}K(g^*), \qquad i = 1, 2.$$

We now want to estimate $K(g^*)$ from above.

By the definition of μ_1,

$$\iint_U \frac{\mu_1\phi}{1-|\mu_1|^2}dxdy = (1-t)\iint_U \frac{\mu(1-t|\mu|^2)\phi}{(1-|\mu|^2)(1-t^2|\mu|^2)}dxdy$$
$$= \frac{1-t}{1+t}\left[\iint_U \frac{\mu\phi}{1-|\mu|^2}dxdy + t\iint_U \frac{\mu\phi}{1-t^2|\mu|^2}dxdy\right]$$
$$\le \frac{1-t}{1+t}\left[\frac{k^2}{1-k^2} + \frac{tk}{1-t^2k^2}\right]$$
$$= \frac{(1-t)k(1-tk^2)}{(1-k^2)(1-t^2k^2)} - \frac{(1-t)k}{(1+t)(1+k)}.$$

In the above inequality we have made use of Lemma 1.1. On the other hand,

$$\iint_U \frac{|\mu_1|^2|\phi|}{1-|\mu_1|^2}dxdy = \iint_U \frac{(1-t)^2|\mu|^2|\phi|}{(1-|\mu|^2)(1-t^2|\mu|^2)}dxdy$$
$$\le \frac{(1-t)^2k^2}{(1-k^2)(1-t^2k^2)}.$$

We obtain by Lemma 1.2

$$\frac{k(g^*)}{1-k(g^*)} \le \frac{(1-t)k(1-tk^2)}{(1-k^2)(1-t^2k^2)} - \frac{(1-t)k}{(1+t)(1+k)} + \frac{(1-t)^2k^2}{(1-k^2)(1-t^2k^2)}$$
$$= \frac{\sqrt{K}-1}{2} - \frac{K-1}{2\sqrt{K}(K+\sqrt{K}+1)}.$$

Hence

$$K(f_i) \le \sqrt{K}K(g^*) = \sqrt{K}(1+\frac{2k(g^*)}{1-k(g^*)}) = K - \frac{K-1}{K+\sqrt{K}+1}, \qquad i = 1, 2. \tag{2.1}$$

The above proof is simpler than that of Theorem 1 in [2], and the result (2.1) is better than that of Theorem 1 in [2] except $1 \le K < K_0 = 3.38297\cdots$[1].

§3. Feasibility of the Reich Procedure

Our main result is

Theorem 3.1. *The Reich procedure is feasible for any* $K(f)$ *and* $0 < t \le (K(f)+1)/(\sqrt{K(f)}+1)^2$.

[1] K_0 is the root of the equation $K^2 - 2K\sqrt{K} + 1 = 0$.

The proof of Theorem 3.1 is much more difficult than that of the result (2.1) and I benefit by the graduation thesis of my student Wu Zhemin which completely improves Theorem 1 in [2] and the result (2.1) by introducing a parameter.

Proof. For $f \in Q_I$ and its complex dilatation μ, write $K = K(f)$, $0 < t \le (K+1)/(\sqrt{K}+1)^2$ and $\mu_1(z) = t\mu(z)$. For f_i in 2), $i = 1, 2$, we have

$$K(f_1) = K(g^{*-1} \circ g_1) \le K(g^*)K(g_1), \tag{3.1}$$

$$K(f_2) = K(g_2 \circ g^*) \le K(g^*)K(g_2). \tag{3.2}$$

Since the complex dilatation of g_1 is μ_1, $k(g_1) = tk$,

$$K(g_1) = \frac{1+tk}{1-tk} \le \sqrt{K}, \qquad 0 < t \le \frac{K+1}{(\sqrt{K}+1)^2}. \tag{3.3}$$

If we set $\mu_2(z) = g_{2\bar{z}}/g_{2z}$, then

$$\mu_2(g_1(z)) = \frac{(1-t)\mu(z)}{1-t|\mu(z)|^2}\frac{g_{1z}}{\overline{g_{1z}}}, \qquad K(g_2) = \frac{1+\frac{(1-t)k}{1-tk^2}}{1-\frac{(1-t)k}{1-tk^2}} = \frac{(1+k)(1-tk)}{(1-k)(1+tk)}. \tag{3.4}$$

Combining (3.1), (3.3) and (3.2), (3.4), we need only to prove

$$\sup K(g^*) < \frac{1+tk}{1-tk} \quad \text{or} \quad \sup \frac{k(g^*)}{1-k(g^*)} < \frac{tk}{1-tk},$$

where the supremum is taken, for fixed t, over every g^* corresponding to μ with $\|\mu\|_\infty \le k$. Therefore, we need only to prove the following

Fundamental Lemma. *Suppose $\mu \in F$, $\|\mu\|_\infty = k > 0$, $0 < t < 1$, $g = f^{t\mu}$. If g^* is an extremal mapping in Q_g, $K(g^*) = \frac{1+k^*}{1-k^*}$, then*

$$\begin{aligned}
\frac{k^*}{1-k^*} &\le \frac{tk^2}{1+tk}\min\left\{\begin{array}{l} \frac{(1+t)[1+2k-k^2-2(1-k)tk-(1+k^2)t^2k^2]}{(1+k^2)(1-tk)(1-t^2k^2)}, \\ \frac{2(1+t)-(1-t)tk-2t^2k^2}{(1-k)(2-t^2k^2)} \end{array}\right\} \\
&\le \frac{tk^2}{1+tk}\left\{\begin{array}{ll} \frac{2(1+t)-(1-t)tk-2t^2k^2}{(1-k)(2-t^2k^2)}, & 0 < k \le 2-\sqrt{3}, 0 < t \le \frac{1}{2} \\ \frac{(1+t)[1+2k-k^2-2(1-k)tk-(1+k^2)t^2k^2]}{(1+k^2)(1-tk)(1-t^2k^2)}, & \textit{otherwise} \end{array}\right\} \\
&\equiv \frac{\phi(t,k)}{1-\phi(t,k)} < \frac{tk}{1-tk}
\end{aligned}$$

or

$$k^* \le \phi(t,k) < tk.$$

For the proof of Fundamental Lemma we first prove the following

Lemma 3.1. *Suppose that*

$$P(x) = \frac{\frac{k}{1-t^2k^2} - \frac{\sqrt{x}}{1-t^2x}}{\frac{k}{1-k^2} - \frac{\sqrt{x}}{1-x}}, \qquad 0 \le x \le k^2,$$

where $0 \le k < 1$, $0 \le t < 1$. Then

$$P(0) \ge P(x) \ge P(k^2).$$

Proof. We first show that $P(x)$ is left continuous at $x = k^2$, i.e., $\lim\limits_{x \nearrow k^2} P(x)$ exists. Because $P(x)$ can be rewritten as

$$P(x) = \frac{1-k^2}{1-t^2k^2} - \frac{\sqrt{x}}{1-x}\,\frac{\frac{1-x}{1-t^2x} - \frac{1-k^2}{1-t^2k^2}}{\frac{k}{1-k^2} - \frac{\sqrt{x}}{1-x}}, \qquad 0 \le x \le k^2,$$

we have

$$\lim_{x\nearrow k^2} P(x) = \frac{1-k^2}{1-t^2k^2} - \frac{k}{1-k^2} \lim_{x\nearrow k^2} \frac{\frac{1-t^2}{(1-t^2x)^2}}{\frac{1+x}{2\sqrt{x}(1-x)^2}}$$
$$= \frac{1-k^2}{1-t^2k^2}\left(1 - \frac{2k^2}{1+k^2}\frac{1-t^2}{1-t^2k^2}\right)$$
$$= \frac{(1-k^2)^2(1+t^2k^2)}{(1+k^2)(1-t^2k^2)^2} \equiv P(k^2). \tag{3.5}$$

For the rest we need only to prove $P'(x) \le 0$.

Write

$$\alpha = k/(1-t^2k^2), \qquad \beta(x) = \sqrt{x}/(1-t^2x),$$
$$\gamma = k/(1-k^2), \qquad \delta(x) = \sqrt{x}/(1-x).$$

Hence

$$P'(x) = \frac{-\beta'(x)(\gamma-\delta(x)) + (\alpha-\beta(x))\delta'(x)}{(\gamma-\delta(x))^2}$$
$$= \left\{\left(\frac{\sqrt{x}}{1-x} - \frac{k}{1-k^2}\right)\frac{1+t^2x}{2\sqrt{x}(1-t^2x)^2} + \left(\frac{k}{1-t^2k^2} - \frac{\sqrt{x}}{1-t^2x}\right)\frac{1+x}{2\sqrt{x}(1-x)^2}\right\}\frac{1}{(\gamma-\delta(x))^2}$$
$$= \frac{1}{2\sqrt{x}}\left\{\frac{\sqrt{x}}{1-x}\frac{1}{1-t^2x}\left(\frac{1+t^2x}{1-t^2x} - \frac{1+x}{1-x}\right) + k\left[\frac{1+x}{(1-t^2k^2)(1-x)^2} - \frac{1+t^2x}{(1-k^2)(1-t^2x)^2}\right]\right\}\frac{1}{(\gamma-\delta(x))^2}$$
$$= -\frac{1-t^2}{2\sqrt{x}}\{2(1-k^2)(1-t^2k^2)x^{3/2} + k[k^2-(3-(1+t^2)k^2)x + (1+t^2-3t^2k^2)x^2 + t^2x^3]\}/(1-k^2)(1-t^2k^2)(1-x)^2(1-t^2x)^2(r-\delta(x))^2.$$

We are to prove

$$f(y) = k^3 - [3-(1+t^2)k^2]ky^2 + 2(1-k^2)(1-t^2k^2)y^3 + (1+t^2-3t^2k^2)ky^4 + t^2ky^6 \ge 0, \qquad 0 \le y \le k.$$

By immediate computation we have

$$f''(y) = 2[(1+t^2)k^2-3]k + 12(1-k^2)(1-t^2k^2)y + 12(1+t^2-3t^2k^2)ky^2 + 30t^2ky^4$$

and

$$f'''(y) = 12(1-k^2)(1-t^2k^2) + 24(1+t^2-3t^2k^2)ky + 120t^2ky^3.$$

Because $f^{(5)}(y) = 720t^2ky \ge 0$, $0 \le y \le k$, we see that $f'''(y)$ is convex downward. In addition,

$$f'''(0) = 12(1-k^2)(1-t^2k^2) > 0,$$
$$f'''(k) = 12(1-k^2)(1-t^2k^2) + 24(1+t^2)k^2 + 48t^2k^4 > 0.$$

Therefore either $f'''(y) \ge 0$ or $f'''(y)$ has two real zeros on $0 \le y \le k$. But the latter is impossible. It is proved that $f''(y)$ is increasing on $0 \le y \le k$. From

$$f''(0) = 2[(1+t^2)k^2-3]k < 0$$

and

$$f''(k) = 2[3 + (1+t^2)k^2 + 3t^2k^4]k > 0$$

we know that $f''(y)$ has a unique zero on $0 \le y \le k$ and it is the minimum point of $f'(y)$. Because $f'(0) = 0$ and

$$f'(k) = 2[(1+t^2)k^2 - 3]k^2 + 6(1-k^2)(1-t^2k^2)k^2 + 4(1+t^2-3t^2k^2)k^4 + 6t^2k^6 = 0,$$

we have $f'(y) \le 0$, $0 \le y \le k$. Thus $f(y)$ is decreasing on $0 \le y \le k$. Moreover,

$$f(0) = k^3,$$

$$f(k) = k^3 + [(1+t^2)k^2 - 3]k + 2(1-k^2)(1-t^2k^2)k^3 + (1+t^2-3t^2k^2)k^5 + t^2k^7 = 0.$$

We have $f(y) \ge 0$, $0 \le y \le k$. Lemma 3.1 is proved.

Let us turn to the proof of Fundamental Lemma. On the one hand, we have by Lemma 1.2

$$\frac{k^*}{1-k^*} \le I(t\mu) + \Delta(t\mu), \tag{3.6}$$

where

$$I(t\mu) = \sup_{\substack{\phi\in B\\ \|\phi\|\le 1}} \left| \iint_U \frac{t\mu(z)\phi(z)}{1-|t\mu(z)|^2} dxdy \right|, \tag{3.7}$$

$$\Delta(t\mu) = \sup_{\substack{\phi\in B\\ \|\phi\|\le 1}} \iint_U \frac{|t\mu(z)|^2|\phi(z)|}{1-|t\mu(z)|^2} dxdy. \tag{3.8}$$

On the other hand, it holds for arbitrary real constant s depending only on k that

$$\begin{aligned}\iint_U \frac{t\mu\phi}{1-t^2|\mu|^2} dxdy =& s(1-t)\iint_U \frac{\mu\phi}{1-|\mu|^2} dxdy \\ &+ (1-t)\iint_U \frac{\mu[\frac{t}{1-t}(1-|\mu|^2) - s(1-t^2|\mu|^2)]\phi}{(1-|\mu|^2)(1-t^2|\mu|^2)} dxdy.\end{aligned}$$

Write

$$h(x) = \frac{\sqrt{x}|A(x)|}{(1-x)(1-t^2x)}, \qquad 0 \le x \le k^2,$$

where

$$A(x) = \frac{t}{1-t}(1-x) - s(1-t^2x).$$

Applying Lemma 1.1 we obtain

$$\left| \iint_U \frac{t\mu\phi}{1-t^2|\mu|^2} dxdy \right| \le |s|(1-t) \iint_U \frac{|\mu|^2|\phi|}{1-|\mu|^2} dxdy + (1-t)\iint_U h(|\mu|^2)|\phi| dxdy. \tag{3.9}$$

Obviously $A(x) \ge 0$ is equivalent to

$$s \le \frac{t}{1-t}\frac{1-x}{1-t^2x} \equiv E(x), \qquad 0 \le x \le k^2. \tag{3.10}$$

Under the condition (3.10), it is easy to see that $h(x) \le h(k^2)$ is equivalent to the inequality

$$s \le \frac{t}{1-t}\frac{\frac{k}{1-t^2k^2} - \frac{\sqrt{x}}{1-t^2x}}{\frac{k}{1-k^2} - \frac{\sqrt{x}}{1-x}} \equiv \frac{t}{1-t}P(x).$$

Because $E'(x) = -\frac{t}{1-t}\frac{1-t^2}{(1-t^2x)^2} < 0$, we have $E(0) \geq E(x) \geq E(k^2) = \frac{t}{1-t}\frac{1-k^2}{1-t^2k^2} = \frac{t}{1-t}P(0)$. In view of Lemma 3.1 we know $h(x) \leq h(k^2)$, if

$$0 \leq s \leq \frac{t}{1-t}P(k^2). \tag{3.11}$$

We get thereby from (3.9) that

$$\left|\iint\limits_U \frac{t\mu\phi}{1-t^2|\mu|^2}dxdy\right| \leq s(1-t)\frac{k^2}{1-k^2} + \frac{k[t(1-k^2)-s(1-t)(1-t^2k^2)]}{(1-k^2)(1-t^2k^2)}$$
$$= \frac{tk}{1-t^2k^2} - \frac{s(1-t)k}{1+k}. \tag{3.12}$$

In addition,

$$\iint\limits_U \frac{|t\mu|^2|\phi|}{1-|t\mu|^2}dxdy \leq \frac{t^2k^2}{1-t^2k^2}. \tag{3.13}$$

Hence we deduce by (3.6), (3.7), (3.8), (3.12) and (3.13) that

$$\frac{k^*}{1-k^*} \leq \frac{tk(1+tk)}{1-t^2k^2} - \frac{s(1-t)k}{1+k} = \frac{tk}{1-tk} - s(1-t)\frac{k}{1+k}.$$

Thus it can be seen that the larger the value of s, the better the estimate of $\frac{k^*}{1-k^*}$. According to (3.11) and applying (3.5), we take

$$s = \frac{t}{1-t}\frac{(1-k^2)^2(1+t^2k^2)}{(1+k^2)(1-t^2k^2)^2},$$

and then it follows that

$$\frac{k^*}{1-k^*} \leq \frac{tk}{1-tk} - tk\frac{(1-k)(1-k^2)(1+t^2k^2)}{(1+k^2)(1-t^2k^2)^2}$$
$$= \frac{t(1+t)k^2}{(1+k^2)(1-t^2k^2)^2}[1+2k-k^2-2(1-k)tk-(1+k^2)t^2k^2]. \tag{3.14}$$

If we take $s = \frac{t}{1-t}\frac{2-tk^2}{2-t^2k^2}$, then $|A(x)| \leq -A(k^2)$. Hence $h(x) \leq h(k^2)$ still holds. By (3.9) we have

$$\left|\iint\limits_U \frac{t\mu\phi}{1-t^2|\mu|^2}dxdy\right| \leq s(1-t)\frac{k^2}{1-k^2} + \frac{k[s(1-t)(1-t^2k^2)-t(1-k^2)]}{(1-k^2)(1-t^2k^2)}$$
$$= s(1-t)\frac{k}{1-k} - \frac{tk}{1-t^2k^2} = t\frac{2-tk^2}{2-t^2k^2}\frac{k}{1-k} - \frac{tk}{1-t^2k^2}. \tag{3.15}$$

Combining (3.6), (3.7), (3.8), (3.15) and (3.13), we get

$$\frac{k^*}{1-k^*} \leq t\frac{2-tk^2}{2-t^2k^2}\frac{k}{1-k} - \frac{tk}{1-t^2k^2} + \frac{t^2k^2}{1-t^2k^2}$$
$$= tk\left(\frac{2-tk^2}{2-t^2k^2}\frac{1}{1-k} - \frac{1}{1+tk}\right) = \frac{tk^2[2(1+t)-(1-t)tk-2t^2k^2]}{(1-k)(1+tk)(2-t^2k^2)}. \tag{3.16}$$

Merging (3.14) and (3.16), we find that

$$\frac{k^*}{1-k^*} \leq \frac{tk^2}{1+tk}\min\left\{\begin{array}{c}\frac{(1+t)[1+2k-k^2-2(1-k)tk-(1+k^2)t^2k^2]}{(1+k^2)(1-tk)(1-t^2k^2)}, \\ \frac{2(1+t)-(1-t)tk-2t^2k^2}{(1-k)(2-t^2k^2)}\end{array}\right\}.$$

In what follows we prove that if $0 \le k \le 2-\sqrt{3}$, $0 \le t \le \frac{1}{2}$, then the inequality

$$\frac{2(1+t)-(1-t)tk-2t^2k^2}{(1-k)(2-t^2k^2)} \le \frac{(1+t)[1+2k-k^2-2(1-k)tk-(1+k^2)t^2k^2]}{(1+k^2)(1-tk)(1-t^2k^2)}$$

holds. Multiply the two sides by k and write $x = tk$. The inequality (3.17) becomes

$$\frac{2k+(2-k)x+(1-2k)x^2}{(1-k)(2-x^2)} \le \frac{(k+x)[1+2k-k^2-2(1-k)x-(1+k^2)x^2]}{(1+k^2)(1-x)(1-x^2)}.$$

A simple computation tells us that

$$\begin{aligned}&(1-k)(2-x^2)(k+x)[1+2k-k^2-2(1-k)x-(1+k^2)x^2]\\&-(1+k^2)(1-x)(1-x^2)[2k+(2-k)x+(1-2k)x^2]\\=&2k^2(1-4k+k^2)+k(1+k^2)x-(3-8k+2k^2-4k^3-k^4)x^2-2k(1+k^2)x^3\\&+(1-4k-k^4)x^4+k(1+k^2)x^5 \equiv F(x).\end{aligned}$$

Obviously $F^{(5)}(x) = 120k(1+k^2) > 0$. Therefore

$$F^{(3)}(x) = -12k(1+k^2)+24(1-4k-k^4)x+60k(1+k^2)x^2$$

is a convex downward function. And further, from

$$F^{(3)}(0) = -12k(1+k^2) < 0$$

and

$$F^{(3)}\left(\frac{k}{2}\right) = -48k^2+3k^3+3k^5 < 0$$

we know $F^{(3)}(x) \le 0$, $0 \le x \le \frac{k}{2}$. Thus

$$\begin{aligned}F''(x) =& -2(3-8k+2k^2-4k^3-k^4)-12k(1+k^2)x\\&+12(1-4k-k^4)x^2+20k(1+k^2)x^3\end{aligned}$$

is decreasing on $0 \le x \le \frac{k}{2}$. Because $1-4k+k^2 \ge 0$ for $0 \le k \le 2-\sqrt{3}$, we have

$$\begin{aligned}F''(0) &= -2(3-8k+2k^2-4k^3-k^4)\\&= -2[3(1-4k+k^2)+k(1-4k+k^2)+3k+3k^2-5k^3-k^4] < 0.\end{aligned}$$

Hence $F''(x) \le 0$, $0 \le x \le \frac{k}{2}$. So $F(x)$ is convex upward on $0 \le x \le \frac{k}{2}$. But

$$\begin{aligned}&F(0) = 2k^2(1-4k+k^2) \ge 0,\\&F\left(\frac{k}{2}\right) = \frac{3}{2}k^2(1-4k+k^2)+\frac{1}{4}k^2+\frac{5}{16}k^4+\frac{3}{4}k^5+\frac{1}{32}k^6(1-k^2) > 0.\end{aligned}$$

Consequently $F(x) \ge 0$, $0 \le x \le \frac{k}{2}$. The inequality (3.17) is proved.

What remains is only to prove

$$\frac{\phi(t,k)}{1-\phi(t,k)} < \frac{tk}{1-tk}, \qquad tk \ne 0.$$

In fact,

$$\begin{aligned}\frac{tk}{1-tk}-\frac{\phi(t,k)}{1-\phi(t,k)} &\ge \frac{tk}{1-tk}-\frac{tk^2}{1+tk}\frac{(1+t)[1+2k-k^2-2(1-k)tk-(1+k^2)t^2k^2]}{(1+k^2)(1-tk)(1-t^2k^2)}\\&= tk\frac{(1-k)(1-k^2)(1+t^2k^2)}{(1+k^2)(1-t^2k^2)^2} > 0.\end{aligned}$$

Fundamental Lemma is completely proved.

The fact that the decomposition obtained by the Reich procedure can replace the first decomposition will be proved in another paper.

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