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ON THE UPPER ESTIMATES OF FUNDAMENTAL SOLUTIONS OF PARABOLIC EQUATIONS ON RIEMANNIAN MANIFOLDS***

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Abstract

The authors first derive the gradient estimates and Harnack inequalities for positive solutions of the equation

$$\Delta u(x,t)+b(x,t)\cdot\nabla u(x,t)+h(x,t)u(x,t)-\frac{\partial u(x,t)}{\partial t}=0$$

on complete Riemannian manifolds, and then derive the upper bounds of any positive L^2 fundamental solution of the equation when $h(x,t)$ and $b(x,t)$ are independent of t.

Keywords Parabolic equation, Gradient estimate, Harnack inequality, Fundamental solution, Riemannian manifolds.
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§1. Introduction

Let M be a complete Riemannian manifold. We consider the parabolic equation

$$\Delta u(x,t)+b(x,t)\nabla u(x,t)+h(x,t)u(x,t)-\frac{\partial u(x,t)}{\partial t}=0 \tag{1.1}$$

on M. The function $h(x,t)$ is assumed to be C^2 in the first variable and C^1 in the second variable, the tensor field $b(x,t)$ is assumed to be C^1 in both the variables.

Yau[6] first obtained the gradient estimates of positive harmonic functions on complete Riemannian manifolds. Later Li and Yau[4] derived the gradient estimates for positive solution of the parabolic equation

$$\Delta u(x,t)+h(x,t)u(x,t)-\frac{\partial u(x,t)}{\partial t}=0 \tag{1.2}$$

and applied these estimates to the estimations of a positive L^2 fundamental solution of (1.2) when $h(x,t)$ is independent of t. In his dissertation[2,3], the first author considered the nonlinear elliptic equation

$$\Delta u(x)+b(x)\nabla u(x)+h(x)u^\alpha(x)=0 \tag{1.3}$$

and the nonlinear parabolic equation

$$\Delta u(x,t)+h(x,t)u^\alpha(x,t)-\frac{\partial u(x,t)}{\partial t}=0. \tag{1.4}$$

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Recently we noted that Cranston[1] also considered the equation (1.3) when $h(x) \equiv 0$. The main purpose of this paper is to show that the methods developed in [2,3,4,6] can also be applied to the equation

$$\Delta u(x,t) + b(x,t)\nabla u(x,t) + h(x,t)u^\alpha(x,t) - \frac{\partial u(x,t)}{\partial t} = 0. \tag{1.5}$$

For simplicity we mainly consider equation (1.1). We also applied these estimates to the estimations of a positive L^2 fundamental solution of (1.1) when $h(x,t)$ and $b(x,t)$ are independent of t.

§2. Gradient Estimates

In this section we mainly derive gradient estimates for positive solutions $u(x,t)$ on $M \times [0,\infty)$ of the equation

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)u(x,t) + b(x,t)\cdot\nabla u(x,t) + h(x,t)u(x,t) = 0. \tag{2.1}$$

We introduce a new function $W(x,t) = u^{-\beta}(x,t)$, where $\beta > 0$. By a simple computation we have

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)W = \frac{\beta+1}{\beta}\frac{|\nabla W|^2}{W} + \beta h W - b\cdot\nabla W. \tag{2.2}$$

Set

$$\phi_0 = \frac{|\nabla W|^2}{W^2} + s\beta^2 h(x), \tag{2.3}$$

$$\phi_1 = \frac{W_t}{W}, \tag{2.4}$$

$$\phi = \phi_0 + s\beta\phi_1, \tag{2.5}$$

where $s > 1$.

Now, we estimate the Laplacian of ϕ, which is essential to the derivation of the gradient estimate.

Let $e_1, e_2, \cdots, e_n$ be a local orthonormal frame field of M. By adopting the notation of moving frames, the subscripts i, j, and k will denote the covariant differentiations in the e_i, e_j and e_k directions respectively, where $1 \le i,j,k \le n$. Suppose $b = \sum_i b_i e_i$. A straightforward computation gives

$$\nabla\phi_0(x,t) = \frac{2\sum_i W_iW_{ij}}{W^2} - \frac{2\sum_i W_i^2W_j}{W^3} + s\beta^2 h_j, \tag{2.6}$$

$$\begin{aligned}\Delta\phi_0(x,t) = {} & \frac{2\sum_{ij} W_{ij}^2}{W^2} + \frac{2\sum_{ij} W_iW_{ijj}}{W^2} - 8\frac{\sum_{ij} W_iW_{ij}W_j}{W^3} + 6\frac{(\sum_i W_i^2)^2}{W^4} \\ & - 2\frac{\sum_{ij} W_i^2W_{jj}}{W^3} + s\beta^2\sum_j h_{jj}.\end{aligned} \tag{2.7}$$

We denote the Ricci tensor of M by R_{ij}. Then

$$\frac{2\sum_{ij} W_iW_{ijj}}{W^2} = \frac{2\sum_{ij} W_iW_{jji}}{W^2} + \frac{2\sum_{ij} R_{ij}W_iW_j}{W^2}. \tag{2.8}$$

By the inequality $\sum\limits_{i,j} W_{ij}^2 \ge \frac{1}{n}(\sum\limits_i W_{ii})^2 = \frac{1}{n}(\Delta W)^2$, we obtain

$$\begin{aligned}
&\frac{2\sum\limits_{ij} W_{ij}^2}{W^2} - 8\frac{\sum\limits_{ij} W_i W_{ij} W_j}{W^3} + 6\frac{(\sum\limits_i W_i^2)^2}{W^4} \\
&\ge \frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2 - 4\left(\frac{\sum\limits_{ij} W_i W_{ij} W_j}{W^3} - \frac{(\sum\limits_i W_i^2)^2}{W^4}\right) - 2\Big(\frac{1}{\varepsilon} - 1\Big)\frac{(\sum\limits_i W_i^2)^2}{W^4},
\end{aligned} \tag{2.9}$$

where $0 < \varepsilon < 1$.

$$\frac{\partial \phi_0}{\partial t} = \frac{2\sum\limits_i W_i W_{it}}{W^2} - \frac{2\sum\limits_i W_i^2 W_t}{W^3} + s\beta^2 h_t. \tag{2.10}$$

By (2.8), (2.9), (2.10) and (2.2) we have

$$\begin{aligned}
\Big(\Delta - \frac{\partial}{\partial t}\Big)\phi_0 \ge{}& \frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2 - 2\Big(\frac{1}{\varepsilon} - 1\Big)\frac{(\sum\limits_i W_i^2)^2}{W^4} + \frac{2}{\beta}\nabla\phi_0 \cdot \nabla \log W \\
&+ 2\beta(1-s)\sum_i h_i \frac{W_i}{W} + 2\frac{\sum\limits_{ij} W_i(R_{ij} - b_{ij})W_j}{W^2} \\
&+ s\beta^2(\Delta h + \nabla h b - h_t) - \nabla\phi_0 b.
\end{aligned} \tag{2.11}$$

Computing directly one obtains

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)\phi_1 = \frac{2}{\beta}\nabla\phi_1 \cdot \nabla \log W + s\beta^2 h_t - \sum_i b_{it}\frac{W_i}{W} - b\nabla\phi_1. \tag{2.12}$$

(2.11) and (2.12) yield

$$\begin{aligned}
\Big(\Delta - \frac{\partial}{\partial t}\Big)\phi \ge{}& \frac{2(1-\varepsilon)}{n}\left(\frac{\Delta W}{W}\right)^2 - 2\Big(\frac{1}{\varepsilon} - 1\Big)\frac{(\sum\limits_i W_i^2)^2}{W^4} + \frac{2}{\beta}\nabla\phi \cdot \nabla \log W \\
&+ 2\beta(1-s)\sum_i h_i \cdot \frac{W_i}{W} + 2\frac{\sum\limits_{ij} W_i(R_{ij} - b_{ij})W_j}{W^2} + s\beta^2(\Delta h + \nabla h b) \\
&- \nabla\phi \cdot b - s\beta \cdot \sum_i b_{it}\frac{W_i}{W}.
\end{aligned} \tag{2.13}$$

By (2.2),

$$\begin{aligned}
\frac{\Delta W}{W} &= \frac{\beta + 1}{\beta}\frac{|\nabla W|^2}{W^2} + \beta h - \sum_i b_i \frac{W_i}{W} + \frac{W_t}{W} \\
&= \frac{1}{s\beta}\phi + \Big(\frac{\beta+1}{\beta} - \frac{1}{s\beta}\Big)\frac{|\nabla W|^2}{W^2} - \sum_i \frac{b_i W_i}{W}.
\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13) one obtains

$$\begin{aligned}\Big(\Delta-\frac{\partial}{\partial t}\Big)\phi \geq{}& \frac{2(1-\varepsilon)}{n}\frac{1}{s^2\beta^2}\phi^2+\left(\frac{2(1-\varepsilon)}{n}+\frac{(s+s\beta-1)^2}{s^2\beta^2}-2\Big(\frac{1}{\varepsilon}-1\Big)\right)\frac{|\nabla W|^4}{W^4}\\ &+\frac{4(1-\varepsilon)}{n}\frac{(s+s\beta-1)}{s^2\beta^2}\phi\frac{|\nabla W|^2}{W^2}+\frac{2}{\beta}\nabla\phi\cdot\nabla\log W+2\beta(1-s)\sum_i h_i\frac{W_i}{W}\\ &+\frac{2\sum_{ij}W_i(R_{ij}-b_{ij})W_j}{W^2}+s\beta^2(\Delta h+\nabla hb)-\nabla\phi\cdot b-s\beta\sum_i b_{ti}\frac{W_i}{W}\\ &-\frac{4(1-\varepsilon)}{n}\frac{1}{s\beta}\sum_i b_i\frac{W_i}{W}\phi-\frac{4(1-\varepsilon)}{n}\frac{(s\beta+s-1)}{s\beta}\Big(\sum_i b_i\frac{W_i}{W}\Big)\frac{|\nabla W|^2}{W^2}\\ &+\frac{2(1-\varepsilon)}{n}\Big(\sum_i b_i\frac{W_i}{W}\Big)^2. \end{aligned}\tag{2.15}$$

Lemma 2.1. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. Suppose that $h(x,t)$ and $b(x,t)$ satisfy the following conditions:*

(1) $\forall(x,t)\in M\times[0,\infty)$, $\Delta h+\nabla hb\geq -K_1$;

(2) $\forall x\in M$, $R_{ij}-b_{ij}\geq -K_2g_{ij}$.

If we choose β sufficiently small such that $\frac{1}{\beta}>\frac{n}{\varepsilon}\frac{s-1}{s}$, then

$$\begin{aligned}\Big(\Delta-\frac{\partial}{\partial t}\Big)\phi\geq{}&\frac{1-\varepsilon}{n}\frac{1}{s^2\beta^2}\phi^2+\frac{4(1-\varepsilon)}{n}\frac{(s+s\beta-1)}{s^2\beta^2}\phi\frac{|\nabla W|^2}{W^2}+\frac{2}{\beta}\nabla\phi\cdot\nabla\log W\\ &-\nabla\phi\cdot b+s\beta^2(\Delta h+\nabla h\cdot b)-B,\end{aligned}\tag{2.16}$$

where

$$\begin{aligned}B={}&\frac{3}{4\sqrt[3]{4}}\left(\frac{ns^2(2(s-1)|\nabla h|+s|b_t|)^4}{(1-\varepsilon)(s-1)^2}\right)^{1/3}\beta^2+\frac{64}{n}(1-\varepsilon)\frac{s^2\beta^2}{(s-1)^2}|b|^4\\ &+\frac{4n}{1-\varepsilon}\left(\frac{6(1-\varepsilon)}{n}|b|^2+2K_2\right)^2\frac{s^2\beta^2}{(s-1)^2},\end{aligned}$$

$s>1$, $0<\varepsilon<1$.

Proof. By Hölder's inequality and Young's inequality, we have

$$\frac{4(1-\varepsilon)}{n}\frac{1}{s\beta}\left(\sum_i b_i\frac{W_i}{W}\right)\phi\leq\frac{1-\varepsilon}{n}\frac{\phi^2}{s^2\beta^2}+\frac{4(1-\varepsilon)}{n}\left(\sum_i b_i\frac{W_i}{W}\right)^2,\tag{2.17}$$

$$\begin{aligned}&\frac{4(1-\varepsilon)}{n}\frac{(s\beta+s-1)}{s}\left(\sum_i b_i\frac{W_i}{W}\right)\frac{|\nabla W|^2}{W^2}\\ &\leq\frac{3}{4}\frac{1-\varepsilon}{n}\frac{(s\beta+s-1)^2}{s^2\beta^2}\frac{|\nabla W|^4}{W^4}+\frac{64}{n}(1-\varepsilon)\frac{s^2\beta^2|b|^4}{(s-1)^2},\end{aligned}\tag{2.18}$$

$$\begin{aligned}&\frac{2(1-\varepsilon)}{n}\left(\sum_i b_i\frac{W_i}{W}\right)^2+\frac{2\sum_{ij}W_i(R_{ij}-b_{ij})W_j}{W^2}\\ &\geq-\left[\frac{1-\varepsilon}{4n}\frac{(s\beta+s-1)^2}{s^2\beta^2}\frac{|\nabla W|^4}{W^4}+\frac{4n}{1-\varepsilon}\left(\frac{2(1-\varepsilon)}{n}|b|^2+2K_2\right)^2\frac{s^2\beta^2}{(s-1)^2}\right].\end{aligned}\tag{2.19}$$

We set

$$A=\frac{64}{n}(1-\varepsilon)\frac{s^2\beta^2}{(s-1)^2}|b|^4+\frac{4n}{1-\varepsilon}\left(\frac{2(1-\varepsilon)}{n}|b|^2+2K_2\right)^2\frac{s^2\beta^2}{(s-1)^2}. \tag{2.20}$$

Substituting (2.17)-(2.20) into (2.15) one has

$$\begin{aligned}\Big(\Delta-\frac{\partial}{\partial t}\Big)\phi\geq&\frac{1-\varepsilon}{n}\frac{1}{s^2\beta^2}\phi^2+\left[\frac{1-\varepsilon}{n}\frac{(s\beta+s-1)^2}{s^2\beta^2}-\frac{2(1-\varepsilon)}{\varepsilon}\right]\frac{|\nabla W|^4}{W^4}\\&+\frac{4(1-\varepsilon)}{n}\frac{(s\beta+s-1)}{s^2\beta^2}\phi\frac{|\nabla W|^2}{W^2}+\frac{2}{\beta}\nabla\phi\cdot\nabla\log W+2\beta(1-s)\sum_i h_i\frac{W_i}{W}\\&-s\beta\sum_i b_{ti}\frac{W_i}{W}-\nabla\phi\cdot b+s\beta^2(\Delta h+\nabla h\cdot b)-A.\end{aligned} \tag{2.21}$$

Since $\frac{1}{\beta}>\frac{n}{\varepsilon}\frac{s-1}{s}$, we have

$$\begin{aligned}&\left[\frac{1-\varepsilon}{n}\frac{(s\beta+s-1)^2}{s^2\beta^2}-\frac{2(1-\varepsilon)}{\varepsilon}\right]\frac{|\nabla W|^4}{W^4}-2(s-1)\beta\sum_i h_i\frac{W_i}{W}-s\beta\sum_i b_{ti}\frac{W_i}{W}\\&\geq-\frac{3}{4\sqrt[3]{4}}\left[\frac{ns^2(2(s-1)|\nabla h|+s|b_t|)^4\beta^6}{(1-\varepsilon)(s-1)^2}\right]^{1/3}.\end{aligned} \tag{2.22}$$

Clearly (2.16) follows from (2.20), (2.21) and (2.22).

Now, we first give interior estimates, and then extend our local estimates to global ones.

Theorem 2.1. *Let M be an n-dimensional complete Riemannian manifold with possible empty boundary ∂M. Assume $p\in M$, and $B_p(2R)=\{x\in M\mid dist(x,p)<2R\}$ does not intersect the boundary ∂M. Let $h(x,t)$ be a function which is C^2 in the x-variable and C^1 in the t-variable. Let $b(x,t)$ be a tensor field which is C^1 in both the variables. Suppose that $h(x,t)$, $b(x,t)$ satisfy the following conditions:*

(1) $\forall(x,t)\in B_p(2R)\times[0,\infty)$, $\Delta h(x,t)+b(x,t)\nabla h(x,t)\geq -K_1(2R)$,

(2) *with local coordinates* $R_{ij}-b_{ij}\geq -K_2(2R)g_{ij}$, $\forall x\in B_p(2R)$ *and* $R_{ij}\geq -K_3(2R)g_{ij}$, *where R_{ij} is the Ricci tensor of M,*

(3) $\forall(x,t)\in B_p(2R)\times[0,\infty)$, $|\nabla h|\leq K_4(2R)$, $|b|\leq K_5(2R)$, $|b_t|\leq K_6(2R)$.

If $u(x,t)$ is a positive solution of (2.1), then in $B_p(R)\times[0,\infty)$, $u(x,t)$ satisfies the estimate

$$\begin{aligned}\frac{|\nabla u|^2}{u^2}+s\cdot h-s\frac{u_t}{u}\leq& ns^2\frac{1}{t}+s\sqrt{n}\sqrt{sK_1(2R)+B_1}\\&+ns^2\left(\frac{C_n}{R^2}+\frac{C_n}{R}(\sqrt{K_3(2R)}+K_5(2R))\right),\end{aligned}$$

where

$$\begin{aligned}B_1=C_n\Bigg[&\left(\frac{s^2(2(s-1)K_4(2R)+sK_6(2R))^4}{(s-1)^2}\right)^{1/3}+\frac{s^2}{(s-1)^2}K_5^4(2R)+\\&+(\frac{6}{n}K_5^2(2R)+2K_2(2R))^2\frac{s^2}{(s-1)^2}\Bigg],\qquad s>1.\end{aligned}$$

Proof. We define the function $F(x,t)=t\phi(x,t)$. Let $\widetilde{g}(r)$ be a C^2 function defined on $[0,\infty)$ such that

$$\widetilde{g}(r)=\begin{cases}1 & \text{if } r\in[0,1],\\ 0 & \text{if } r\geq 2.\end{cases}$$

$0\leq\widetilde{g}(r)\leq 1$, $\widetilde{g}'(r)\leq 0$, $\widetilde{g}''(r)\geq -C$, where C is a positive constant.

If $r(x)$ denotes the geodesic distance between x and some fixed point P, we set $g(x) = \widetilde{g}(\frac{r(x)}{R})$.

Using the argument of Calabi, we may assume the function $g(x) \cdot F(x,t)$ with support in $B_p(2R) \times [0, \infty)$ to be smooth.

Let (x_0, t_0) be the point where $g \cdot F$ achieves its maximum in $B_p(2R) \times [0, T]$.

Clearly, we may assume $g(x_0)F(x_0, t_0) > 0$.

By the maximal principle in differential equations, at (x_0, t_0), we have

$$\nabla(g \cdot F) = 0, \tag{2.23}$$

$$\frac{\partial(gF)}{\partial t} \geq 0, \tag{2.24}$$

$$\Delta(g \cdot F) \leq 0. \tag{2.25}$$

Obviously,

$$\frac{|\nabla g|^2}{g} = \frac{1}{R^2}\frac{|\widetilde{g}'|^2}{\widetilde{g}} \leq \frac{C}{R^2}, \tag{2.26}$$

$$\Delta g = \widetilde{g}''\frac{1}{R^2} + \frac{\widetilde{g}'}{R}\Delta r.$$

Applying the Laplacian comparison theorem[5], we have

$$\Delta r \leq \frac{n-1}{r}(1 + \sqrt{K_3(2R)} \cdot r).$$

So,

$$\Delta g \geq -\frac{C_n}{R^2} - \frac{C_n}{R}\sqrt{K_3(2R)}. \tag{2.27}$$

By (2.23), we have

$$\nabla F = -\frac{\nabla g}{g}F. \tag{2.28}$$

By (2.25), we have

$$\Delta g \cdot F + 2\nabla g \cdot \nabla F + g \cdot \Delta F \leq 0. \tag{2.29}$$

Using (2.24) and (2.29), we obtain

$$\Delta g \cdot F + 2\nabla g \cdot \nabla F + g(\Delta - \frac{\partial}{\partial t})F \leq 0. \tag{2.30}$$

Using Lemma 2.1, we have

$$\begin{aligned}(\Delta - \frac{\partial}{\partial t})F \geq & \frac{1-\varepsilon}{n}\frac{1}{s^2\beta^2}\frac{1}{t}F^2 - \frac{F}{t} + \frac{4(1-\varepsilon)}{n}\frac{s + s\beta - 1}{s^2\beta^2}F\frac{|\nabla W|^2}{W^2} \\ & + \frac{2}{\beta}\nabla F \cdot \nabla \log W - \nabla F \cdot b - s\beta^2 K_1(2R)t - \beta^2 B_1^* t,\end{aligned} \tag{2.31}$$

where

$$\begin{aligned}B_1^* = C_n \Bigg[& \left(\frac{s^2(2(s-1)K_4(2R) + sK_6(2R))^4}{(1-\varepsilon)(s-1)^2}\right)^{1/3} + \frac{(1-\varepsilon)s^2}{(s-1)^2}K_5^4(2R) \\ & + \frac{1}{1-\varepsilon}\left[\frac{6(1-\varepsilon)}{n}K_5^2(2R) + 2K_2(2R)\right]^2 \frac{s^2}{(s-1)^2}\Bigg].\end{aligned}$$

Substituting (2.31) into (2.30) and using (2.27), (2.28), one has

$$\begin{aligned}&\frac{1-\varepsilon}{n}\frac{1}{s^2\beta^2}\frac{1}{t}g^2F^2+\frac{4(1-\varepsilon)}{n}\frac{s+s\beta-1}{s^2\beta^2}gF\frac{|\nabla W|^2}{W^2}-\frac{2}{\beta}Fg\cdot\nabla g\frac{\nabla W}{W}+Fg\cdot\nabla g\cdot b\\&-2|\nabla g|^2F-\frac{F}{t}-s\beta^2K_1(2R)t-\beta^2B_1^*t-\left(\frac{C_n}{R^2}+\frac{C_n}{R}\sqrt{K_3(2R)}\right)F\le 0.\end{aligned}\tag{2.32}$$

Clearly,

$$\frac{2}{\beta}Fg\cdot\nabla g\frac{\nabla W}{W}\le\frac{4(1-\varepsilon)}{n}\frac{s-1}{s^2\beta^2}gF\frac{|\nabla W|^2}{W^2}+\frac{n}{4(1-\varepsilon)}\frac{s^2}{(s-1)}|\nabla g|^2F.\tag{2.33}$$

By (2.26) and (2.33), we have

$$\begin{aligned}&\frac{1-\varepsilon}{n}\frac{1}{s^2\beta^2}\frac{1}{t}g^2F^2-\left(\frac{1}{t}+\frac{C_n}{R^2}+\frac{C_n}{R}(\sqrt{K_3(2R)}+K_5(2R))\right)gF\\&-s\beta^2K_1(2R)t-\beta^2B_1^*t\le 0.\end{aligned}\tag{2.34}$$

(2.34) implies

$$\begin{aligned}gF\le&\sqrt{\frac{n}{1-\varepsilon}}\sqrt{sK_1(2R)+B_1^*}\cdot s\beta^2t+\frac{n}{1-\varepsilon}s^2\beta^2t\left(\frac{C_n}{R^2}+\frac{C_n}{R}(\sqrt{K_3(2R)}+K_5(2R))\right)\\&+\frac{n}{1-\varepsilon}s^2\beta^2.\end{aligned}\tag{2.35}$$

The theorem follows from (2.35).

Using Theorem 2.1, we can obtain the following global estimates.

Theorem 2.2. *Let M be an n-dimensional complete Riemannian manifold with Ricci tensor R_{ij}. Let $h(x,t)$ be a function which is C^2 in the x-variable and C^1 in the t-variable. Let $b(x,t)$ be a tensor field which is C^1 in both the variables. Suppose that $h(x,t)$ and $b(x,t)$ satisfy the following conditions:*

(1) $\forall(x,t)\in M\times[0,\infty)$, $\Delta h(x,t)+b(x,t)\nabla h(x,t)\ge -K_1$,

(2) *with local coordinates* $R_{ij}-b_{ij}\ge -K_2g_{ij}$, $\forall x\in M$ *and* $R_{ij}\ge -K_3(1+r^2(x))$, *where $r(x)$ is the geodesic distance between x and some fixed point.*

(3) $\forall(x,t)\in M\times[0,\infty)$, $|\nabla h|\le K_4$, $|b|\le K_5$, $|b_t|\le K_6$.

If $u(x,t)$ is a positive solution of (2.1), then

$$\frac{|\nabla u|^2}{u^2}+s\cdot h-s\frac{u_t}{u}\le ns^2\frac{1}{t}+s\sqrt{n}\sqrt{sK_1+B_2}+C_ns^2\sqrt{K_3},\tag{2.36}$$

where

$$\begin{aligned}B_2=C_n&\left[\left(\frac{s^2(2(s-1)K_4+sK_6)^4}{(s-1)^2}\right)^{1/3}+\frac{s^2}{(s-1)^2}K_5^4\right.\\&\left.+(\frac{6}{n}K_5^2+2K_2)^2\frac{s^2}{(s-1)^2}\right],\qquad s>1.\end{aligned}$$

Proof. Letting $R\to\infty$, by Theorem 2.1 we get this theorem.

§3. Harnack Inequality

In this section, we will utilize the gradient estimates derived in section 2 to obtain the following Harnack inequality.

Theorem 3.1. *Suppose that M, h, b satisfy the hypotheses of Theorem 2.2. If $u(x,t)$ is a positive solution of (2.1), then the following Harnack inequality must hold on u:*

$$u(x_1,t_1) \le (\frac{t_2}{t_1})^{ns} \exp(C_n(s\sqrt{K_3}+\sqrt{sK_1+B_2})(t_2-t_1)+A(x_1,x_2,t_2-t_1))u(x_2,t_2), \quad (3.1)$$

where $x_1,x_2 \in M$, $0<t_1<t_2<\infty$, and

$$A(x_1,x_2,t_2-t_1) = \inf_{\gamma\in\Gamma}\left\{\frac{s}{4(t_2-t_1)}\int_0^1 |\dot\gamma|^2 dv - (t_2-t_1)\int_0^1 h(\gamma(v),(1-v)t_2+vt_1)dv\right\}$$

with inf taken over all paths in M parametrized by $[0,1]$ joining x_1 and x_2.

Proof. Using Theorem 2.2, we know that if $f=\log u$ then

$$|\nabla f|^2 - sf_t \le ns^2\frac{1}{t} + C_n(s^2\sqrt{K_3}+s\sqrt{sK_1+B_2}) - s\cdot h \quad (3.2)$$

for all $(x,t)\in M\times[0,\infty)$.

For any two points (x_1,t_1) and (x_2,t_2) in $M\times[0,\infty)$ with $t_1<t_2$, we let γ be any curve given by $\gamma:[0,1]\to M$, with $\gamma(0)=x_2$ and $\gamma(1)=x_1$.

Define the curve $\eta:[0,1]\to M\times[0,\infty)$ by $\eta(v)=(\gamma(v),(1-v)t_2+vt_1)$.

Clearly $\eta(0)=(x_2,t_2)$, $\eta(1)=(x_1,t_1)$,

$$\begin{aligned} f(x_1,t_1)-f(x_2,t_2) &= \int_0^1 \frac{df(\eta(v))}{dv}dv \\ &\le \int_0^1 (|\dot\gamma||\nabla f| - (t_2-t_1)f_t)dv. \end{aligned} \quad (3.3)$$

By (3.2), we have

$$-f_t \le \frac{ns}{t} + C_n(s\sqrt{K_3}+\sqrt{sK_1+B_2}) - h - \frac{|\nabla f|^2}{s}.$$

So,

$$\begin{aligned} f(x_1,t_1)-f(x_2,t_2) \le \int_0^1 & |\dot\gamma||\nabla f| - \frac{t_2-t_1}{s}|\nabla f|^2 + \frac{n(t_2-t_1)s}{t} \\ &+ C_n(s\sqrt{K_3}+\sqrt{sK_1+B_2})(t_2-t_1) - (t_2-t_1)h]dv, \end{aligned}$$

where $t=(1-v)t_2+vt_1$.

However, as a function of $|\nabla f|$, the quadratic

$$\begin{aligned} &|\dot\gamma||\nabla f| - \frac{(t_2-t_1)}{s}|\nabla f|^2 + \frac{ns(t_2-t_1)}{t} + C_n(s\sqrt{K_3}+\sqrt{sK_1+B_2})(t_2-t_1) - h(t_2-t_1) \\ \le& \frac{s|\dot\gamma|^2}{4(t_2-t_1)} + \frac{ns(t_2-t_1)}{t} + C_n(s\sqrt{K_3}+\sqrt{sK_1+B_2})(t_2-t_1) - h(t_2-t_1). \end{aligned}$$

So

$$\begin{aligned} f(x_1,t_1)-f(x_2,t_2) \le& ns\log(\frac{t_2}{t_1}) + C_n(s\sqrt{K_3}+\sqrt{sK_1+B_2})(t_2-t_1) \\ &+ \frac{s}{4(t_2-t_1)}\int_0^1 |\dot\gamma|^2 dv - (t_2-t_1)\int_0^1 h(\gamma(v),(1-v)t_2+vt_1)dv. \end{aligned}$$

The theorem follows by taking exponentials of the above inequality.

§4. L^2 Estimates

In this section, we will derive L^2 estimates for solutions of the equation

$$\Delta u(x,t)-\frac{\partial u(x,t)}{\partial t}+b(x)\cdot\nabla u(x,t)+h(x)u(x,t)=0, \tag{4.1}$$

where $h(x)\in C^2(M)$ and $b(x)\in\mathcal{X}(M)$.

Theorem 4.1. *Let M be a complete Riemannian manifold. Suppose that $u(x,t)$ is an L^2 solution of* (4.1). *Assume that $A_1(x,t)$ is a bounded above Lipschitz function which satisfies*

$$|\nabla_x A_1|^2+\frac{\partial A_1}{\partial t}+2h(x)+2|b(x)|^2=0. \tag{4.2}$$

Then

$$\int_M e^{A_1(x,T)}u^2(x,T)dx\le\int_M e^{A_1(x,0)}u^2(x,0)dx,$$

where $T>0$.

Proof. We choose $\phi(x)\in C_0^\infty(M)$,

$$\phi(x)=\begin{cases}1, & x\in B_y(R),\\ 0, & x\notin B_y(2R),\end{cases}$$

$0\le\phi\le1$, $|\nabla\phi|\le\frac{C}{R}$, where C is an absolute constant. Since $u(x,t)$ satisfies (4.1), we have

$$\begin{aligned}0&=2\int_0^T\int_M\phi^2e^{A_1}u((\Delta-\frac{\partial}{\partial t}+h)u+b\cdot\nabla u)dxdt\\&=-2\int_0^T\int_M\nabla(\phi^2e^{A_1}u)\nabla udxdt-2\int_M\int_0^T\phi^2e^{A_1}u\frac{\partial u}{\partial t}dtdx+\\&\quad+2\int_0^T\int_M\phi^2e^{A_1}hu^2dxdt+2\int_0^T\int_M\phi^2e^{A_1}u\cdot\nabla u\cdot bdxdt\\&=-\int_0^T\int_M[4\phi e^{A_1}u\cdot\nabla\phi\cdot\nabla u+2\phi^2e^{A_1}u\nabla A_1\cdot\nabla u+2\phi^2e^{A_1}|\nabla u|^2]dxdt\\&\quad-\int_M[\phi^2e^{A_1}u^2]|_0^T dx+\int_0^T\int_M\phi^2e^{A_1}u^2\frac{\partial A_1}{\partial t}dxdt\\&\quad+2\int_0^T\int_M\phi^2e^{A_1}u^2hdxdt+2\int_0^T\int_M\phi^2e^{A_1}u\cdot\nabla u\cdot bdxdt.\end{aligned} \tag{4.3}$$

By Hölder's inequality,

$$\begin{aligned}&-4\int_0^T\int_M\phi e^{A_1}u\cdot\nabla\phi\cdot\nabla udxdt\\&\le\frac12\int_0^T\int_M\phi^2e^{A_1}|\nabla u|^2dxdt+8\int_0^T\int_M e^{A_1}|\nabla\phi|^2u^2dxdt,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&2\int_0^T\int_M\phi^2e^{A_1}u\nabla A_1\cdot\nabla udxdt\\&\le\int_0^T\int_M\phi^2e^{A_1}|\nabla u|^2dxdt+\int_0^T\int_M\phi^2e^{A_1}u^2|\nabla A_1|^2dxdt,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&2\int_0^T\int_M\phi^2e^{A_1}u\nabla u\cdot bdxdt\\&\le\frac12\int_0^T\int_M\phi^2e^{A_1}|\nabla u|^2dxdt+2\int_0^T\phi^2e^{A_1}u^2|b|^2dxdt.\end{aligned} \tag{4.6}$$

Substituting (4.4), (4.5) and (4.6) into (4.3) we have

$$\int_M \phi^2 e^{A_1} u^2\big|_{t=T}\, dx \le \int_M \phi^2 e^{A_1} u^2\big|_{t=0}\, dx + 8\int_0^T \int_M e^{A_1}|\nabla\phi|^2 u^2 dxdt. \tag{4.7}$$

Letting $R \to \infty$, from (4.7) we get this theorem.

§5. Upper Bounds of Fundamental Solutions

In this section, we will derive upper estimates of any positive L^2 fundamental solution of (4.1).

We set

$$A_2(x,y,t) = \inf_{\gamma\in\Gamma}\left\{\frac{1}{4t}\int_0^1 |\dot\gamma|^2 dv - t\int_0^1 (2h(\gamma(v)) + 2|b(\gamma(v))|^2)dv\right\},$$

where $\Gamma = \{\gamma : [0,1] \to M \mid \gamma(0) = y, \gamma(1) = x\}$.

Clearly one has[4]

$$|\nabla_x A_2|^2(x,y,t) + \frac{\partial A_2(x,y,t)}{\partial t} + 2h(x) + 2|b(x)|^2 = 0, \tag{5.1}$$

$$|\nabla_y A_2|^2(x,y,t) + \frac{\partial A_2(x,y,t)}{\partial t} + 2h(y) + 2|b(y)|^2 = 0. \tag{5.2}$$

Lemma 5.1. *Let M be a complete Riemannian manifold. $h(y) \in C^2(M)$, $b \in \mathcal{X}(M)$ satisfy the condition that $|\nabla h| \le K$ and $|b| \le K$. Suppose that $H(x,y,t)$ is a positive L^2 fundamental solution of* (4.1). *Let*

$$F(y,t) = \int_{s_1} H(x,z,t)\cdot H(y,z,T)dz$$

for $x \in M$, $s_1 \subset M$ and $0 \le t \le \tau < (1+2\delta)T$ $(\delta > 0)$. Then for any subset $s_2 \subset M$, we have

$$\int_{s_2} F^2(z,\tau)dz \le \int_{s_1} H^2(x,z,T)dz \cdot \sup_{z\in s_1}\exp(-A_2(x,z,(1+2\delta)T))$$
$$\sup_{z\in s_2}\exp(A_2(x,z,(1+2\delta)T-\tau)). \tag{5.3}$$

Proof. For $x \in M$ we assume that $r(y)$ is the geodesic distance between y and x. Since $|\nabla h| \le K$, we have $|h(y)| \le Kr(y)$.

Let us define the function $A_1(y,t) = -A_2(x,y,(1+2\delta)T-t)$, where $x \in M$, $0 \le t < (1+2\delta)T$.

Obviously $A_1(y,t)$ satisfies (4.2) and is bounded above. So,

$$\int_M e^{A_1(y,\tau)}F^2(y,\tau)dy \le \int_M e^{A_1(y,0)}F^2(y,0)dy. \tag{5.4}$$

Observing that

$$F(y,0) = \begin{cases} H(x,y,T) & \text{if } y \in s_1, \\ 0 & \text{if } y \notin s_1, \end{cases}$$

we have

$$\int_M \exp(-A_2(x,y,(1+2\delta)T))F^2(y,0)dy$$
$$\le \sup_{z\in s_1}\exp(-A_2(x,z,(1+2\delta)T))\int_{s_1} H^2(x,z,T)dz.$$

On the other hand,

$$\int_M \exp(-A_2(x,y,(1+2\delta)T-\tau))F^2(y,\tau)dy$$
$$\geq \int_{s_2} F^2(z,\tau)dz \inf_{z\in s_2} \exp(-A_2(x,z,(1+2\delta)T-\tau)).$$

This proves the lemma.

Theorem 5.1. *Suppose that* $M, h \in C^2(M)$, $b \in \mathcal{X}(M)$ *satisfy the hypotheses of Theorem 2.2.* $H(x,y,t)$ *is a positive* L^2 *fundamental solution of* (4.1). *Then*

$$\begin{aligned} H(x,y,t) \leq{} & (1+\delta)^{2ns} \exp(C_n(s\sqrt{K_3}+\sqrt{sK_1+B_3})\delta(2+\delta)T) \\ & \cdot \sup_{z\in s_1} \exp(A(y,z,\delta t)) \cdot \sup_{z\in s_1} \exp(-\frac{1}{2}A_2(x,z,(1+2\delta)(1+\delta)t)) \\ & \cdot \sup_{z\in s_2} \exp(A(x,z,\delta(1+\delta)t)) \cdot \sup_{z\in s_2} \exp(\frac{1}{2}A_2(x,z,\delta(1+\delta)t)) \\ & \cdot V^{-1}(s_1)\cdot V^{-1}(s_2) \end{aligned} \tag{5.5}$$

for any $\delta > 0$ *and any subsets* $s_1, s_2 \subset M$ *whose volumes* $V(s_1)$ *and* $V(s_2)$ *are finite, where*

$$B_3 = C_n\left[\left(\frac{4s(s-1)^2K_4^2}{s-1}\right)^{2/3} + \frac{s_2^2}{(s-1)^2}K_5^4 + (\frac{6}{n}K_5^2+2K_2)^2\frac{s^2}{(s-1)^2}\right], \qquad s>1.$$

Proof. We apply Theorem 3.1 to the function $F(y,t)$ of Lemma 5.1. This yields

$$\begin{aligned} \left(\int_{s_1} H^2(x,z,T)dz\right)^2 &= F^2(x,T) \\ &\leq (1+\delta)^{2ns}\exp(C_n(s\sqrt{K_3}+\sqrt{sK_1+B_3})\delta T)\exp(2A(x,y,\delta T))F^2(y,(1+\delta)T) \end{aligned}$$

by setting $t_1 = T$, $t_2 = (1+\delta)T$.

So,

$$\begin{aligned} \left(\int_{s_1} H^2(x,z,T)dz\right)^2 \leq{} & (1+\delta)^{2ns}\exp(C_n(s\sqrt{K_3}+\sqrt{sK_1+B_3})\delta T) \\ & \cdot \sup_{z\in s_2}\exp(2A(x,z,\delta T)) \cdot \int_{s_2} F^2(y,(1+\delta)T)dyV^{-1}(s_2). \end{aligned}$$

By Lemma 5.1 we have

$$\begin{aligned} \left(\int_{s_1} H^2(x,z,T)dz\right)^2 \leq{} & (1+\delta)^{2ns}\exp(C_n(s\sqrt{K_3}+\sqrt{sK_1+B_3})\delta T) \\ & \cdot \sup_{z\in s_2}\exp(2A(x,z,\delta T)) \cdot \sup_{z\in s_2}\exp(A_2(x,z,\delta T)) \\ & \cdot \sup_{z\in s_1}\exp(-A_2(x,z,(1+2\delta)T))V^{-1}(s_2). \end{aligned} \tag{5.6}$$

Applying Theorem 3.1 to the function $H(x,z,T)$ and setting $T = (1+\delta)t$, we obtain

$$\begin{aligned} H^2(x,y,t) &\leq (1+\delta)^{2ns}\exp(C_n(s\sqrt{K_3}+\sqrt{sK_1+B_3})\delta t)\exp(2A(y,z,\delta t))H^2(x,z,(1+\delta)t) \\ &\leq (1+\delta)^{2ns}\exp(C_n(s\sqrt{K_3}+\sqrt{sK_1+B_3})\delta t)\sup_{z\in s_1}\exp(2A(y,z,\delta t)) \\ &\quad \int_{s_1} H^2(x,z,T)dz\cdot V^{-1}(s_1). \end{aligned} \tag{5.7}$$

Clearly the theorem follows from (5.6) and (5.7).

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