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ON THE RANGE OF RANDOM WALKS IN RANDOM ENVIRONMENT**

ZHOU XIANYIN*

Abstract

The range of random walk on Z^d in symmetric random environment is investigated. As results, it is proved that the strong law of large numbers for the range of random walk on Z^d in some random environments holds if $d \geq 3$, and a weak law of large numbers holds for $d = 1$.

Keywords Law of large numbers, Ergodic theorem, Range of random walk, Random environment, Effective resistance.
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§1. Introduction

There are a lot of works on the random walk in random environment. Among them, two kinds of random environments, which include symmetric case and nonsymmetric case, have been widely investigated. However, most of them are concentrated on studying the recurrence or transience of random walk on Z^d in a nonsymmetric random environment. But, the recurrence or transience of random walk on Z^d in a symmetric random environment can be easily judged by comparing the effective resistance of an electrical network corresponding to this random walk. In fact, most of the works are concentrated on studying the invariance principle for random walk on Z^d in a symmetric random environment. For more details on this research direction, the reader is referred to see [5] or the references there in.

We also know that there are a lot of works on the range of the simple random walk on Z^d. To be precise, we let $X^d = \{X_n^d\}_{n\geq 0}$ be the simple random walk on Z^d, and

$$R_n^d = \#\{X_0^d, X_1^d, \cdots, X_n^d\}.$$

Then R_n^d is the range of X^d up to time n. As early as in 1951, Dvoretzky and Erdös[2] obtained the strong law of large numbers for R_n^d if $d \geq 2$. After that, Jain, Pruitt, Le Gall and Rosen made systematically investigations for the range of more general random walks. Both the law of large numbers and the central limit theorem for the range are obtained by them. For more details, the reader is referred to [3, 4, 8] or the references therein.

Since a symmetric random walk is closely linked with an electrical network, some techniques in electrical network can be used to study some problems related to the symmetric random walk. The main purpose of this paper is to study the asymptotic behaviour of the range of random walk in symmetric random environment. Let $\widetilde{X}^d = \{\widetilde{X}_n^d\}_{n\geq 0}$ be a random

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*Department of Mathematics, Beijing Normal University, Beijing 100875, China.
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walk on Z^d in symmetric random environment, and $\widetilde{R}_n^d = \#\{\widetilde{X}_0^d, \cdots, \widetilde{X}_n^d\}$. Assume that the environment process $\{\xi_k^d\}_{k\geq 0}$ with starting measure μ^* is reversible and ergodic. Then the strong law of large numbers for $\widetilde{R}_n^d$ with $d \geq 3$ holds almost surely with respect to the measure μ^* under suitable assumptions (see Corollary 4.1 below), and a weak law of large numbers for $\widetilde{R}_n^1$ also holds (see Theorem 4.1 below). To get Corollary 4.1, in Section 3 we will use the Birkhoff's Pointwise Ergodic Theorem (see [6]) to prove a general limit theorem for the range $\widetilde{R}_n^d$ (see Theorem 3.1). Unfortunately, we are so far unable to get a law of large numbers for $\widetilde{R}_n^2$, and also unable to get the central limit theorem for $\widetilde{R}_n^d$ with $d \geq 2$.

§2. A Preliminary Result

In this section, we introduce some notation and give a preliminary result for the invariance principle of random walk in symmetric random environment.

Consider the d dimensional lattice Z^d and assign a conductor $a_e(x)$ to the bond $(x, x+e)$ for any $x, e \in Z^d$ with $|e| = 1$. It is clear that $a_e(x) = a_{-e}(x+e)$. Denote by $\mathbb{S}$ the space of the environment

$$\mathbb{S} = S_\theta(Z^d) = \{a_e : \ Z^d \to [\theta, \theta^{-1}], \ |e| = 1\}$$

where $\theta \in [0, 1]$. Let μ be the environment measure on $\mathbb{S}$ which is translation invariant and ergodic with respect to space translations. As in [5], we let

$$\alpha(\xi) = \sum_{|e|=1} \xi_e(0), \quad \text{if } \xi \in \mathbb{S},$$

and S_{-e} be defined by $S_{-e}\xi = \xi^*$, where $\xi^*_{e'}(x) = \xi_{e'}(x+e)$, $\forall x, e' \in Z^d$ with $|e'| = 1$. The transition probability $(p(\xi, \xi'), \xi, \xi' \in \mathbb{S})$ of the environment process $\{\xi_k\}_{k\geq 0}$ with state space $\mathbb{S}$ is given by

$$p(\xi, \xi') = \begin{cases} \alpha(\xi)^{-1}\xi_e(0), \ \xi' = S_{-e}\xi, & \text{for some } e \in Z^d \text{ with } |e| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$Tf(\xi) = \sum_{|e|=1} p(\xi, S_{-e}\xi) f(S_{-e}\xi), \ \forall \xi \in \mathbb{S}$$

and θ_τ denote the time - translation operator which is defined by $\theta_\tau \xi(t) = \xi(t - \tau)$. For convenience, we denote by P_μ (resp. P_ξ) the law of the process $\{\xi_k\}_{k\geq 0}$ with starting measure μ (resp. with $\xi_0 = \xi$), and by E_μ (resp. E_ξ) the expectation with respect to P_μ (resp. P_ξ).

The random walk $\{X(k) = X(\xi, k)\}_{k\geq 0}$ starting at 0 with the given environment $\xi_0 = \xi \in \mathbb{S}$ is defined by

$$\begin{cases} X(0) = 0, \\ X(1) = e, \quad \text{iff } \xi_1 = S_{-e}\xi_0, \\ X(k) = \sum_{i=1}^{k} X(1) \circ \xi_{i-1}. \end{cases} \tag{2.1}$$

Obviously, if $a_e(x) = 1$ for any $x, e \in Z^d$ with $|e| = 1$, then $\{X(k)\}_{k=0}$ is the simple random walk on Z^d.

Remark 2.1. If the rates a_e are periodic with period less than or equal to two in a coordinate direction, then (2.1) is meaningless. However, the following arguments also work for this case, provided the state space $\mathbb{S}$ is suitably enlarged (see [5]).

Let μ^* be the measure on $\mathbb{S}$ defined by $\mu^*(d\xi) = \langle\alpha(\xi)\rangle^{-1}\alpha(\xi)\mu(d\xi)$, where $\langle\alpha(\xi)\rangle = \int \alpha(\xi)\mu(d\xi)$. Then, it is easy to check that the process $(\xi_k)_{k\geq 0}$ with starting measure μ^* is reversible and ergodic (see [5, Lemma 4.3]). Let $X_t^{(n)} = n^{-1/2}X([nt])$. We say that the process $X_t^{(n)} = (X_t^{(n)})_{t\geq 0}$ converges weakly in μ^*-measure to an R^d-value process Y, if for every bounded continuous function F on $D = D([0,\infty), R^d)$ (equipped with the Skorohod topology).

$$E_{\mu^*}(F(X^{(n)})|\xi_0 = \xi) \longrightarrow E(F(Y))$$

as $n \longrightarrow \infty$ in μ^*-probability. Let $D^* = (D^*_{ij}, i, j = 1, \cdots, d)$ be given by

$$\frac{1}{2}D^*_{ij} = \left\langle \frac{a_{e_i}(0)}{\alpha(a)} \right\rangle_{\mu^*} \delta_{ij} - (\phi_i^*(a), (I-T)^{-1}\phi_j^*(a))_{\mu^*},$$

where $\phi_i^*(a) = \alpha(a)^{-1}[a_{e_i}(0) - a_{e_i}(-e_i)]$ and I is the identity. The next theorem is actual ([5, Theorem 4.5 (i)]).

Theorem A. *Suppose that the following is satisfied:*

$$\mu(\{a_e(0) > 0\}) = 1; \quad \langle a_e(0)\rangle_\mu < \infty. \tag{2.2}$$

Then, the process $X^{(n)}$ converges weakly in μ^*-measure to a Brownian motion $\{w_{D^*}(t)\}_{t\geq 0}$ in R^d with the finite diffusion matrix D^*.

§3. A General Result

Let $R_n = \#\{X(0), \cdots, X(n)\}$, where $\{X(n)\}_{n\geq 0}$ is the random walk on Z^d in a random environment defined by (2.1). In this section, we use the ergodic theory to prove a limit theorem for the range R_n. In fact, the ergodic theory has been used to prove that the strong law of large numbers for the range of simple random walk on Z^d holds if $d \geq 3$ (see [7]). Fortunately, this argument also works for the random walk on Z^d in random environment if $d \geq 3$. The main result in this section is as follows.

Theorem 3.1. *Assume that μ is translation invariant and ergodic with respect to space translations, and (2.2) is satisfied, and let*

$$F = F(\xi) = P_\xi(X(\nu) = 0 \ for\ some\ \nu \geq 1), \ \forall \xi \in S_0(Z^d).$$

Then there is a subset $\Omega_0 \subset S_\theta(Z^d)$ with $\mu^*(\Omega_0) = 1$ such that

$$P_\xi\Big(\lim_{n\to\infty} \frac{R_n}{n} = 1 - E_{\mu^*}F\Big) = 1, \ \forall \xi \in \Omega_0.$$

To prove this theorem, we begin with several lemmas.

Lemma 3.1. *The following holds*

$$\lim_{n\to\infty} \frac{E_{\mu^*}R_n}{n} = 1 - E_{\mu^*}F.$$

Proof. Let

$$\phi_k = \begin{cases} 1, & k = n, \\ I_{\{X(\nu)\neq X(k),\ \nu=k+1,\cdots,n\}}, & k < n. \end{cases}$$

Then, we have

$$R_n = \sum_{k=0}^{n} \phi_k. \tag{3.1}$$

By definition, $X(1)$ can be represented as $f(\xi_0, \xi_1)$ (see [5]). Then $X(1) \circ \theta_{i-1} = f(\xi_{i-1}, \xi_i)$. Since the reversible process $\{\xi_k\}_{k\geq 0}$ is stationary with respect to P_{μ^*}, we have

$$\begin{aligned} E_{\mu^*}\phi_k &= P_{\mu^*}(X(\nu) \neq X(k),\ \nu = k+1, \cdots, n) \\ &= P_{\mu^*}\Big(\sum_{i=k}^{\nu} X(1)\circ\theta_i \neq 0,\ \nu = k, \cdots, n-1\Big) \\ &= P_{\mu^*}\Big(\sum_{i=0}^{\nu-k} X(1)\circ\theta_i \neq 0,\ \nu = k, \cdots, n-1\Big) \\ &= P_{\mu^*}X(\nu) \neq 0,\ \nu = 1, \cdots, n-(k+1)\}. \end{aligned}$$

Thus, for any fixed $k \geq 0$, $\lim_{n\to\infty} E_{\mu^*}\phi_{n-k} = 1 - E_{\mu^*}F$. By (3.1), we get the desired result immediately.

Lemma 3.2. *There is a subset $\Omega_0 \subset S_\theta(Z^d)$ with $\mu^*(\Omega_0) = 1$, such that*

$$P_\xi\Big(\overline{\lim_{n\to\infty}} \frac{R_n}{n} \leq 1 - E_{\mu^*}F\Big) = 1, \quad \forall \xi \in \Omega_0.$$

Proof. For any $M \geq 1$, let $R_{n,M} = \sum_{k=0}^{[n/M]+1} Z_k(M)$, where

$$Z_k(M) = \#\{X(kM), X(kM+1), \cdots, X((k+1)M-1)\}.$$

It is clear that $R_n \leq R_{n,M}$. Therefore

$$\overline{\lim_{n\to\infty}} \frac{R_n}{n} \leq \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{[n/M]+1} Z_k(M), \quad \forall M \geq 1.$$

If there is a subset $\Omega_0 \subset S_\theta(Z^d)$ with $\mu^*(\Omega_0) = 1$, such that

$$P_\xi\Big(\underline{\lim_{M\to\infty}}\ \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{[n/M]+1} Z_k(M) = 1 - E_{\mu^*}F\Big) = 1, \quad \forall \xi \in \Omega_0, \tag{3.2}$$

then the desirable result follows immediately.

We now use the ergodic theorem to show (3.2). Let $\widetilde{T} = \theta_M$. Then $\widetilde{T} : \mathbb{S} \to \mathbb{S}$ is a one-to-one map. By the stationary property of $\{\xi_i\}_{i\geq 0}$, we also know that $\tilde{T}$ is a measure-preserving transformation, i.e., $P_{\mu^*}(\tilde{T}^{-1}E) = P_{\mu^*}(E),\ \ \forall E \subset \mathbb{S}$. Let $f(\xi) = \#\{X(0), X(1), \cdots, X(M-1)\}$. Then we get the following immediately from the Birkhoff's Pointwise Ergodic Theorem (see [6, Chapter 2])

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{[n/M]+1} f(\tilde{T}^k\xi) = \frac{1}{M} E_{\mu^*} f, \quad a.e. - P_{\mu^*}.$$

In other words,

$$P_{\mu^*}\Big(\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{[n/M]+1} Z_k(M) = \frac{1}{M} E_{\mu^*} R_M\Big) = 1.$$

Note that $P_{\mu^*}(A) = \int_{\mathbb{S}} P_\xi(A)\mu^*(d\xi)$, $\forall A \subset \mathbb{S}$ and $P_\xi(A) \le 1$, $\forall A \subset \mathbb{S}$, $\forall \xi \in \mathbb{S}$. Therefore, there is a subset $\Omega_1 \subset \mathbb{S}$ with $\mu^*(\Omega_1) = 1$ such that

$$P_\xi\Big(\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{[n/M]+1} Z_k(M) = \frac{1}{M} E_{\mu^*} R_M\Big) = 1, \quad \forall M \in \mathbb{N},\ \forall \xi \in \Omega_1.$$

Thus, we get (3.2) immediately from Lemma 3.1.

Lemma 3.3. *There is a subset $\Omega_0 \subset \mathbb{S}$ with $\mu^*(\Omega_0) = 1$, such that*

$$P_\xi\Big(\varliminf_{n\to\infty} \frac{R_n}{n} \ge 1 - E_{\mu^*}F\Big) = 1, \quad \forall \xi \in \Omega_0.$$

Proof. Let $F_n = \sum_{k=0}^{n} \psi_k$, where

$$\psi_k = \begin{cases} 1, & \text{if } X(k+\nu) - X(k) \ne 0,\ \forall \nu = 1, 2, \cdots, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have $R_n \ge F_n$ and $E_{\mu^*}\psi_k = E_{\mu^*}\psi_0 = 1 - E_{\mu^*}F$, $\forall k = 0, 1, 2, \cdots$. As in the proof of Lemma 3.2, we can use the Birkhoff's Pointwise Ergodic Theorem to get the following

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \psi_k = 1 - E_{\mu^*}F, \quad a.e. - P_{\mu^*}.$$

By this, we get the desired result immediately.

Having Lemma 3.2 and Lemma 3.3, we have actually finished the proof of Theorem 3.1.

Remark 3.1. In fact, it is easy to check that $1 - E_{\mu^*}F = 0$ for the simple random walk on Z^d with $d \le 2$. To get a law of large numbers for R_n, we should study when $1 - E_{\mu^*}F > 0$. In the next section, we will study when $1 - E_{\mu^*}F > 0$ for transient random walks on Z^d in random environment.

§4. Law of Large Numbers

In this section, we first prove that a weak law of large numbers for the range of random walk on Z^1 in random environment holds, and then prove that the strong law of large numbers for the range of transient random walk holds in some random environments.

Let $\tilde{P}$ be the probability law of the Brownian motion $\{w_{D^*}(t)\}_{t\ge 0}$ starting at 0 which is determined in Theorem A, and $\widetilde{E}$ the expectation with respect to $\widetilde{P}$.

Theorem 4.1. *Suppose that $d = 1$, (2.2) is satisfied, and the probability measure μ is translation invariant and ergodic. Then for all bounded continuous function f on R^1*

$$E_{\mu^*}(f(n^{-1/2}R_n)|\xi_0 = \xi) \longrightarrow \widetilde{E}f(\sup\{w_{D^*}(s) : s \le 1\} - \inf\{w_{D^*}(s) : s \le 1\})$$

as $n \longrightarrow \infty$ in μ^*-probability.

Proof. Recall that if $d = 1$,

$$R_n = \#\{X(0), \cdots, X(n)\} = \sup\{X(s) : s \le n\} - \inf\{X(s) : s \le n\}.$$

Thus, we get the desired result immediately from Theorem A.

Now we consider the transient case. Remember the conclusion of Theorem 3.1. In fact, we need only to study when $1 - E_{\mu^*}F > 0$. Let $Y = (Y_k(\xi))_{k\ge 0}$ (or $(p_{ij}(\xi))_{i,j\in Z^d}$) be the random walk corresponding to the given environment $\xi = (a_e(x)) \in S_\theta(Z^d)$, i.e.,

$$p_{ij}(\xi) = \begin{cases} \frac{a_{j-i}(i)}{\sum\limits_{|e|=1} a_e(i)}, & \text{if } |i-j|=1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Q_{x,\xi}$ be the probability law of Y starting at x. Then, we can get the next lemma from (2.1) and the definition of Y.

Lemma 4.1. *For any $i_1, \cdots, i_n \in Z^d$, $\forall n \geq 1$, the following holds:*

$$Q_{0,\xi}(Y_1(\xi) = i_1, \cdots, Y_n(\xi) = i_n) = P_\xi(X(1) = i_1, \cdots, X(n) = i_n).$$

Let $p_{\text{escape}}(\xi) = P_\xi(X(n) \neq 0, \forall n \geq 1)$. Then $p_{\text{escape}}(\xi) = 1 - F(\xi)$. Let $B_a(n) = \prod\limits_{i=1}^{d} [a_i + n, a_i - n]$ and $S_a(n) = \partial B_a(n)$, i.e.,

$$S_a(n) = \{(x-1, \cdots, x_d) \in B_a(n) : \exists i = 1, \cdots, d, \text{ s.t. } x_i = a_i + n, \text{ or }, x_i = a_i - n\},$$

where $a = (a_1, \cdots, a_d)$. Given an environment $\xi = (a_e(x)) \in S_\theta(Z^d)$, we consider the electrical network $B_a(n)$ in which a conductor $a_e(x)$ is assigned to the bond $(x, x+e)$ for any $x, e \in Z^d$ with $|e| = 1$. Let $R_{eff}^{(n)}(\xi)$ be the effective resistance of $B_0(n)$ between 0 and $S_0(n)$, and $R_{eff}(\xi) = \lim\limits_{n\to\infty} R_{eff}^{(n)}(\xi)$. By a well known result (see [1]) and Lemma 4.1, we know

$$p_{\text{escape}}(\xi) = \frac{1}{\sum\limits_{|e|=1} a_e(0) R_{eff}(\xi)}.$$

Thus, if $\sum\limits_{|e|=1} a_e(0) \in (0, \infty)$ for almost all $\xi \in S_\theta(Z^d)$ with respect to μ^*, then one needs only to study when the following holds for studying when $1 - E_{\mu^*} F > 0$

$$\mu^*\{\xi : R_{eff}(\xi) < \infty\} > 0.$$

Remember the relation between μ and μ^*. Then, it suffices to show

$$\mu\{\xi : R_{eff}(\xi) < \infty\} > 0. \tag{4.1}$$

For this purpose, we begin with a lemma.

Lemma 4.2. *Let $\xi_0, \xi_1, \cdots$, be independent random variables with identical distribution in a probability space $(\Omega, \mathcal{F}, P)$. If there is a constant $H \in (0, \infty)$ such that*

$$E[\exp(t\xi_0)] < \infty, \quad \forall t \leq H,$$

then there is a constant $C \in (0, \infty)$ such that

$$P\Big(\sum_{i=0}^{n-1} \xi_i \geq (\log n)^2 n\Big) \leq C \exp(-(\log n)^2).$$

Proof. By [7, Lemma III.5], there are constants g and $T \in (0, \infty)$, such that

$$E[\exp(t\xi_0)] \leq e^{gt^2}, \quad \forall t \in (-T, T).$$

Then, by [7, Theorem III.15] we know that there is a constant $n_0 \geq 1$ such that

$$P\Big(\sum_{i=0}^{n-1} \xi_i \geq n(\log n)^2\Big) \leq \exp(-(\log n)^4), \quad \forall n \geq n_0.$$

This implies the desired result.

Theorem 4.2. *Suppose that $a_e(x) \in (0,\infty)$ for any $e, x \in Z^d$ with $|e| = 1$ are independent random variables with identical distribution relative to a probability measure μ. If $d \geq 3$ and there is a constant $C \in (0,\infty)$ such that $\int \exp(ea_e^{-1}(0))d\mu < \infty, \quad \forall t \leq H$, then $1 - E_{\mu^*}F > 0$.*

Proof. Without loss of the generality, we may assume that μ is translation invariant and ergodic with respect to space translations. By the arguments as before, we know that it suffices to show (4.1). By Cutting Law on resistance (see [1]), we also know that it is enough to show (4.1) for $d = 3$. To this end, we first construct a tree in Z^3 in the following way. We start three rays off from the origin going north, east and up. Whenever a ray intersects the plane $x + y + z = 2^n - 1$ for some n, it splits into three rays, going north, east and up. This process is actually illustrated in [1, Figure 6.17]. One easily sees that the Figure 6.17 in [1] can be thought of as the tree which is also shown in [1, Figure 6.18]. For convenience, we denote the Figure 6.17 in [1] by G. Let A_n be the set of those points at which the plane $x + y + z = 2^n - 1$ and those rays described as above intersect. Then, one easily sees that $\#(A_n) = 3^n$. Let $A_n = \{x_1, x_2, \cdots, x_{3^n}\}$, and $x_k(i)$ for $i = 1,2,3$ be the intersection points at which the plane $x+y+z = 2^{n+1} - 1$ and the three rays starting from x_k intersect for any $k = 1,2,\cdots,3^n$. Then $A_{n+1} = \{x_1(i),\cdots,x_{3^n}(i),\ i = 1,2,3\}$. Let $R_j^{(n)}(i)$ be the effective resistance of the line segment which connects x_j and $x_j(i)$ for $i = 1,2,3$, and $j = 1,\cdots,3^n$. Then $R_j^{(n)}(i)$ can be written as $R_j^{(n)}(i) = \sum_{i=1}^{2^n} \zeta_k$, where $\zeta_1,\cdots,\zeta_{2^n}$ are independent random variables with the same distribution as $a_e^{-1}(0)$. Let $B_j^{(n)}(i) = \{R_j^{(n)}(i) \geq n^2 2^n\}$. By Lemma 4.2 and our hypothesis, we know that there is a constant $C_1 \in (0,\infty)$ such that

$$\mu(B_j^{(n)}(i)) \leq C_1 \exp(-n^2) \tag{4.2}$$

for any $j = 1,2,\cdots,3^n$, $i = 1,2,3$, $n \geq 1$.

We now use (4.2) to show (4.1). Let

$$\Omega_N = \bigcap_{n=N}^{\infty} \bigcap_{j=1}^{3^n} \bigcap_{i=1}^{3} (B_j^{(n)}(i))^c$$

Then, (4.2) tells us that

$$\begin{aligned}\mu(\Omega_N^c) &\leq \sum_{n=N}^{\infty} \sum_{j=1}^{3^n} \sum_{i=1}^{3} \mu(B_j^{(n)}(i)) \leq 3C_1 \sum_{n=N}^{\infty} 3^n \exp(-n^2) \\ &\leq C_2 \exp(-N^2/2), \qquad \forall N \geq 1\end{aligned}$$

for some constant $C_2 \in (0,\infty)$. On the other hand, there is a constant $C_3 \in (0,\infty)$ such that if $n \leq N$,

$$\mu(R_j^{(n)}(i) \geq 2^{2N}) \leq \sum_{k=1}^{2^n} \mu(\zeta_k \geq 2^N) \leq C_3 \exp(-N^2).$$

Hence, there is a constant $C_4 \in (0,\infty)$ such that

$$\mu(\bigcup_{n=1}^{N} \bigcup_{j=1}^{3^n} \bigcup_{i=1}^{3} \{R_j^{(n)}(i) \geq 2^{2N}\}) \leq C_4 \exp(-N^2/2).$$

Thus, one can choose a suitable $N_0 \geq 1$ such that

$$\mu(\Omega_0) > 0, \tag{4.3}$$

where

$$\Omega_0 = \Omega_{N_0} \cap \bigcap_{n=1}^{N_0} \bigcap_{j=1}^{3^n} \bigcap_{i=1}^{3} \{R_j^{(n)}(i) \leq 2^{2N}\}.$$

Now we assign a resistor with resistance 2^{2N_0} to the line segment $\overline{x_n x_n(i)}$ for $i = 1, 2, 3$ and any $n \leq N_0$, and a resistor with resistance $n^2 2^n$ to the line segment $\overline{x_n x_n(i)}$ for $i = 1, 2, 3$ and any $n > N_0$, where $\overline{xy}$ is the line segment connecting x and y. By Thomson's principle and Cutting Law (see [1]), it is easy to check that if $\xi \in \Omega_0$,

$$R_{eff}(\xi) \leq \sum_{n=0}^{N_0} 2^{2N_0} \left(\frac{2}{3}\right)^n + \sum_{n=N_0+1}^{\infty} n^2 \left(\frac{2}{3}\right)^n < \infty.$$

Thus from (4.3) we get $\mu\{\xi : R_{eff}(\xi) < \infty\} \geq \mu(\Omega_0) > 0$.

So far, the proof is complete.

From Theorem 3.1 and Theorem 4.2, we get the next result.

Corollary 4.1. *Assume that all hypotheses of Theorem 4.2 are satisfied. Then there are a constant $\delta_0 \in (0, 1]$ and a subset $\Omega_0 \subset \Omega$ with $\mu^*(\Omega_0) = 1$ such that*

$$P_\xi\left(\lim_{n\to\infty} \frac{R_n}{n} = \delta_0\right) = 1, \quad \forall \xi \in \Omega_0.$$

In other words, the strong law of large numbers for R_n holds in this case.

Remark 4.1. In Theorem 4.2, we only assume $a_e(x) \in (0, \infty)$ for any $e, x \in Z^d$ with $|e| = 1$, so we need a further assumption on $a_e(x)$. Actually, if $d \geq 3$ and there is a constant $\theta \in (0, 1]$ such that $\theta \leq a_e(x) \leq \theta^{-1}$, $\forall x, e \in Z^d$ *with* $|e| = 1$, then by the Thomson's principle and Cutting Law on resistance (see [1]) or Theorem 4.2 above one can check that $R_{eff}(\xi) < \infty$, $\forall \xi \in S_\theta(Z^d)$, $d \geq 3$. Thus we get immediately $1 - E_{\mu^*} F > 0$. In other words, the strong law of large numbers for R_n holds in this case if μ is also translation invariant and ergodic with respect to space translations.

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References

[1] Doyle, P. G. & Snell, J. L., Random walks and electric networks, Washington DC, Math. Assoc. Amer., (1984).

[2] Dvoretzky, A. & Erdös, P., Some problems on random walk in space, Proc. Second Berkeley Symp. Math. Stati. Prob., Univ. California Press, Berkeley, 1951, 352-367.

[3] Jain, N.C. & Pruitt, W. E., Further limit theorem for the range of random walk, *J. Analyse Math.*, **27**(1974), 94-117.

[4] Le Gall, J. F. & Rosen, J., The range of stable random walks, *Ann. Prob.*, **19**(1991), 650-705.

[5] de Masi, A., Ferrari, P. A., Goldstein, S. & Wick, W. D., An invariance principle for reversible Markov process: Applications to random motions in random environment, *J. Stati. Phys.*, **55**(1989), 787-855.

[6] Petersen, K., Ergodic theory, *Cambridge Studies in Adv. Math.*, **2**(1983), Cambridge Univ. Press.

[7] Petrov, V. V., Sums of independent random variables, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, **82**(1975), Springer-Verlag.

[8] Spritzer, F., Principles of random walks, Van Nostrand, Princeton, N.J. Springer-Verlag, 1964.