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EXPLICIT CONSTRUCTION OF HARMONIC MAPS FROM R^2 TO $U(N)$**

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Abstract

Darboux transformation method is used for constructing harmonic maps from R^2 to $U(N)$. The explicit expressions for Darboux matrices are used to obtain new harmonic maps from a known one. The algorithm is purely algebraic and can be repeated successively to obtain an infinite sequence of harmonic maps. Single and multiple solitons are obtained with geometric characterizations and it is proved that the interaction between solitons is elastic. By introducing the singular Darboux transformations, an explicit method to construct new unitons is presented.

Keywords Harmonic map, Explicit construction, Darboux transformation method.
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§1. Introduction

In the present paper, we consider the explicit construction of harmonic maps from R^2 to $U(N)$. This is one of the most interesting problems in geometry and mathematical physics. It is well-known that the equations of harmonic maps from R^{1+1} and R^2 to a Lie group admit a Lax pair with a spectral parameter. Hence the technique for solving integrable soliton equations is a powerful tool to study the harmonic maps form R^{1+1} and R^2 to Lie groups, especially, to $U(N)$. Among various methods the Darboux transformation method has the advantage that new solutions can be obtained explicitly by using purely algebraic algorithm and the same algorithm can be used successively to obtain an infinite sequence of explicit solutions. For the R^{1+1} case, the general method was introduced in [1]. By using this method, single and multiple soliton solutions were constructed with explicit formulas and the interaction of solitons has been proved to be elastic[2]. Harmonic maps from R^{1+1} to $U(N)$ have been studied also by Beggs[3], but the soliton solutions have not been expressed explicitly and the interaction of solitons has not been considered.

For the harmonic map from R^2 or S^2 to $U(N)$, there is a famous paper of K. Uhlenbeck[4]. She uses the loop action and Birkohoff factorization to construct new harmonic maps. Besides, the Bäcklund transformation and singular Bäcklund transformations are mentioned as tools for obtaining new harmonic maps and unitons. We find that Darboux transformation method, which we have used to study the harmonic maps from R^{1+1} to $U(N)$ in [1], is also valid for the case of R^2 to $U(N)$. Explicit solutions can be obtained too. The object of

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the present paper is to present this method and give the explicit formulas for new harmonic maps and unitons.

In §2, we recall the generic formulation of harmonic maps from R^2 to $U(N)$ and the Lax pair. §3 is devoted to the Darboux transformations. In particular, explicit formulas for new harmonic maps are established. In §4, we construct single and multiple soliton-like solutions, they are the Darboux transformation of a kind of trivial solutions. Global behavior of these solitons is sketched. §5 is devoted to obtain the explicit formula for obtaining unitons from a known one by introducing the singular Darboux transformation.

§2. Harmonic Maps from R^2 to $U(N)$

A harmonic map $\phi(x,y)$ from $R^2 = \{(x,y)\}$ to the group $U(N)$ is a critical point of the energy integral

$$S[\phi] = \int \mathrm{tr}(\phi_x\phi^{-1}\phi_x\phi^{-1} + \phi_y\phi^{-1}\phi_y\phi^{-1})dxdy \quad (\phi \in U(N)\,). \tag{2.1}$$

Let

$$U = \phi_x\phi^{-1},\ V = \phi_y\phi^{-1}. \tag{2.2}$$

We have

$$U_y - V_x + [U,V] = 0, \tag{2.3}$$

$$U_x + V_y = 0, \tag{2.4}$$

$$U + U^* = 0,\ V + V^* = 0. \tag{2.5}$$

Here (2.3) follows from (2.2), (2.5) is the condition for unitary group and (2.4) is the Euler equation for the variational problem. Thus, harmonic maps from R^2 to $U(N)$ are defined by (2.3), (2.4) and (2.5). If (2.3), (2.4) and (2.5) are satisfied, ϕ can be constructed by the integration of (2.2) provided that the initial data, say $\phi(0,0)$, satisfy the condition $\phi(0,0) \in U(N)$.

Introducing the complex coordinates $(\zeta, \bar\zeta)$ for R^2

$$\zeta = x + iy,\ \bar\zeta = x - iy; \tag{2.6}$$

$$\frac{\partial}{\partial\bar\zeta} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial\zeta} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big). \tag{2.7}$$

Denoting

$$A = \frac{1}{2}(U + iV) = \phi_{\bar\zeta}\phi^{-1},\ B = \frac{1}{2}(U - iV) = \phi_\zeta\phi^{-1}, \tag{2.8}$$

the equations (2.3)-(2.5) become

$$A_\zeta - B_{\bar\zeta} + [A,B] = 0, \tag{2.9}$$

$$A_\zeta + B_{\bar\zeta} = 0, \tag{2.10}$$

$$A^* = -B, \tag{2.11}$$

respectively. Here $*$ denotes the conjugate and transpose of matrices. The Lax pair of the equations (2.9) and (2.10) is

$$\Phi_{\bar\zeta} = \lambda A\Phi,\ \Phi_\zeta = \frac{\lambda}{2\lambda - 1}B\Phi, \tag{2.12}$$

i.e. the integrability condition (or zero-curvature condition) of (2.12)

$$\Phi_{\bar{\zeta}\zeta} = \Phi_{\zeta\bar{\zeta}}$$

for all λ ($\lambda \neq 1/2$) is equivalent to (2.9) and (2.10).

If $\Phi(\lambda)$ is a fundamental solution to the Lax pair (2.12) and valued in $U(N)$ at some point in R^2 for all λ with $|2\lambda - 1| = 1$, then $\Phi(\lambda)$ is valued in $U(N)$ for all $(x, y) \in R^2$ and all λ with $|2\lambda - 1| = 1$. Moreover, $\Phi(1)$ is a harmonic map. $\Phi(\lambda)$ is essentially the extended harmonic map in [4].

§3. Darboux Transformation

In our previous work [1,2], we use Darboux transformation to obtain explicit formulas for the harmonic maps from $R^{1,1}$ to $U(N)$ and elucidate their behavior. The method is applicable to the case $R^2 \to U(N)$ with some modification.

Let $\Phi(\lambda)$ be an extended harmonic maps of $R^2 \to U(N)$, and A, B its potentials. We want to construct an $N \times N$ matrix $\alpha(\zeta, \bar{\zeta})$, which is independent of λ, such that

$$\Phi_1(\lambda) = S\Phi = (I + \lambda\alpha)\Phi \tag{3.1}$$

satisfies

$$\Phi_{1\bar{\zeta}} = \lambda A_1 \Phi_1, \ \Phi_{1\zeta} = \frac{\lambda}{2\lambda - 1} B_1 \Phi_1 \tag{3.2}$$

with some A_1, B_1 satisfying the $U(N)$ condition (2.11), then $S = I + \lambda\alpha$ is called a Darboux matrix and (3.1) the Darboux transformation. Substituting (3.1) to (3.2), it is seen that

$$A_1 = A + \alpha_{\bar{\zeta}}, \ B_1 = B - \alpha_\zeta; \tag{3.3}$$

$$\alpha_{\bar{\zeta}}\alpha = \alpha A - A\alpha, \ \alpha_\zeta \alpha + 2\alpha_\zeta = B\alpha - \alpha B. \tag{3.4}$$

(3.4) is a system of nonlinear equations of the matrix α. Explicit solutions α of (3.4) can be constructed by using $\Phi(\lambda)$ in the following way.

Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be N numbers such that at least two of them are unequal and $\lambda_\alpha \neq 0, \frac{1}{2}, 1$ ($\alpha = 1, 2, \cdots, N$). Choose N constant columns l_ρ ($\rho = 1, 2, \cdots, N$) and let

$$h_\rho = \Phi(\lambda_\rho) l_\rho \ \ (\rho = 1, 2, \cdots, N) \tag{3.5}$$

such that

$$H = [h_1, h_2, \cdots, h_N] \tag{3.6}$$

is a non-degenerate matrix. Note that h_ρ is a column solution to the Lax pair (2.12) with $\lambda = \lambda_\rho$. We have

Theorem 3.1. *The matrix*

$$\alpha = -H\Lambda^{-1}H^{-1} \ \ \textit{with} \ \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \tag{3.7}$$

is a solution to (3.4).

Proof. From the definition, h_ρ satisfies

$$h_{\rho\bar{\zeta}} = \lambda_\rho A h_\rho, \ h_{\rho\zeta} = \frac{\lambda_\rho}{2\lambda_\rho - 1} B h_\rho. \tag{3.8}$$

Hence

$$H_{\bar{\zeta}} = AH\Lambda,\ H_\zeta = BH\Lambda'.$$

Here

$$\Lambda' = \begin{bmatrix} \dfrac{\lambda_1}{2\lambda_1 - 1} & & \\ & \ddots & \\ & & \dfrac{\lambda_N}{2\lambda_N - 1} \end{bmatrix}. \tag{3.9}$$

Thus

$$\begin{aligned} \alpha_{\bar{\zeta}} &= -H_{\bar{\zeta}}\Lambda^{-1}H^{-1} + H\Lambda^{-1}H^{-1}H_{\bar{\zeta}}H^{-1} \\ &= -A + H\Lambda^{-1}H^{-1}AH\Lambda H^{-1} \end{aligned} \tag{3.10}$$

and further we have

$$\alpha_{\bar{\zeta}}\alpha = -A\alpha + \alpha A. \tag{3.11}$$

The first equation of (3.4) is satisfied. The second one of (3.4) can be verified similarly.

Remark 3.1. It is easily seen that (3.4) is completely integrable and each solution α can be determined by the value of α at a given point. Consequently, (3.7) gives all solutions of (3.4) which are similar to $-\Lambda^{-1}$ at the given point, and then at every point the property holds true.

From the expressions (3.3) of A_1, B_1 and the $U(N)$ condition (2.11), we should have

$$(\alpha_{\bar{\zeta}})^* = \alpha_\zeta \text{ or } (\alpha^* - \alpha)_\zeta = 0. \tag{3.12}$$

Moreover, in order that the solution obtained can be defined on the whole R^2 we should have $\det H \neq 0$ in R^2. The following choice of λ_ρ's and l_ρ's gives the explicit formula of α, which satisfies the above requirement.

We choose a complex number $\underset{1}{\lambda}$, and let

$$\begin{aligned} \underset{2}{\lambda} &= \frac{\bar{\underset{1}{\lambda}}}{2\bar{\underset{1}{\lambda}} - 1}, \\ \lambda_\rho &= \begin{cases} \underset{1}{\lambda} & (\rho = 1, \cdots, k), \\ \underset{2}{\lambda} & (\rho = k+1, \cdots, N). \end{cases} \end{aligned} \tag{3.13}$$

Here $\underset{1}{\lambda}$ satisfies

(i) $|2\underset{1}{\lambda} - 1| \neq 1$,

We choose l_ρ's such that

(ii) $\Phi(\underset{1}{\lambda})L_1$ and $\Phi(\underset{2}{\lambda})L_2$ are of rank k and $N-k$ respectively at some point (say $(0,0)$), where

$$L_1 = [l_1, \cdots, l_k, 0, \cdots, 0],\ L_2 = [0, \cdots, 0, l_{k+1}, \cdots, l_N], \tag{3.14}$$

and

(iii) at a fixed point, say $(0,0)$,

$$h_p^* h_a = 0\ (a = 1, \cdots, k; p = k+1, \cdots, N). \tag{3.15}$$

We note that

$$(h_p^*)_{\bar{\zeta}} = (h_{p\zeta})^* = \Big(\frac{\underset{2}{\lambda}}{2\underset{2}{\lambda}-1} B h_p\Big)^* = -\underset{1}{\lambda} h_p^* A$$

and hence

$$(h_p^* h_a)_{\bar{\zeta}} = 0. \tag{3.16}$$

Similarly, we have

$$(h_p^* h_a)_{\zeta} = 0. \tag{3.17}$$

Therefore (iii) holds on R^2 if it holds at a fixed point. Thus we have

Lemma. *Let h_a and h_p be two column solutions of* (2.12) *corresponding to the parameter $\underset{1}{\lambda}$ and $\underset{2}{\lambda}$ respectively. If $h_p^* h_a = 0$ at a point in R^2, then $h_p^* h_a = 0$ everywhere.*

Besides, it is easy to see that if h_a's (resp. h_p's) are linearly independent at some point, they should be linearly independent everywhere. From this construction, we have $\det H \neq 0$ on R^2.

Theorem 3.2. *If the λ_ρ's* (*given by* (3.13)) *and the constant columns l_ρ's satisfy the requirements* (i) (ii) *and* (iii), *then the potential*

$$A_1 = A + \alpha_{\bar{\zeta}},\ B_1 = B - \alpha_{\zeta},$$

where α is given by the explicit expression (3.7), *satisfies the $U(N)$ condition $A_1^* = -B_1$* (*i.e. α satisfies* (3.12)), *and thus defines a new harmonic map from R^2 to $U(N)$.*

Proof. From (3.7), we see that $\alpha H = -H\Lambda^{-1}$, i.e.

$$\alpha h_a = -\frac{1}{\underset{1}{\lambda}} h_a,\ \alpha h_p = -\frac{1}{\underset{2}{\lambda}} h_p,\ (a = 1, \cdots, k;\ p = k+1, \cdots, N), \tag{3.18}$$

hence

$$h_a^* \alpha^* = -\frac{1}{\bar{\underset{1}{\lambda}}} h_a^*,\ h_p^* \alpha^* = -\frac{1}{\bar{\underset{2}{\lambda}}} h_p^*. \tag{3.19}$$

Consequently

$$\begin{aligned} h_a^*(\alpha^* - \alpha) h_b &= \Big(-\frac{1}{\bar{\underset{1}{\lambda}}} + \frac{1}{\underset{1}{\lambda}}\Big) h_a^* h_b, \\ h_p^*(\alpha^* - \alpha) h_q &= \Big(-\frac{1}{\bar{\underset{2}{\lambda}}} + \frac{1}{\underset{2}{\lambda}}\Big) h_p^* h_q, \\ h_p^*(\alpha^* - \alpha) h_a &= \Big(-\frac{1}{\bar{\underset{2}{\lambda}}} + \frac{1}{\underset{1}{\lambda}}\Big) h_p^* h_a = 0, \\ h_a^*(\alpha^* - \alpha) h_p &= \Big(-\frac{1}{\bar{\underset{1}{\lambda}}} + \frac{1}{\underset{2}{\lambda}}\Big) h_a^* h_p = 0, \end{aligned} \tag{3.20}$$

where $a, b = 1, \cdots, k$; $p, q = k+1, \cdots, N$. From the relation

$$2\underset{1}{\lambda}\bar{\underset{2}{\lambda}} = \underset{1}{\lambda} + \bar{\underset{2}{\lambda}}$$

for $\underset{1}{\lambda}, \bar{\underset{2}{\lambda}}$, we obtain

$$-\frac{1}{\bar{\underset{2}{\lambda}}} + \frac{1}{\underset{2}{\lambda}} = -\frac{2\underset{1}{\lambda} - 1}{\underset{1}{\lambda}} + \frac{2\bar{\underset{1}{\lambda}} - 1}{\bar{\underset{1}{\lambda}}} = -\frac{1}{\bar{\underset{1}{\lambda}}} + \frac{1}{\underset{1}{\lambda}}. \tag{3.21}$$

Thus, from (3.20) we have

$$h_\rho^*(\alpha^*-\alpha)h_\sigma = h_\rho^*\Big(\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{1}{\bar\lambda}}\Big)I\,h_\sigma\ \ (\rho,\sigma=1,\cdots,N). \tag{3.22}$$

It follows that

$$\alpha^*-\alpha=\Big(\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{1}{\bar\lambda}}\Big)I, \tag{3.23}$$

since h_ρ's are linearly independent. By differentiating (3.23) with respect to ζ, we see that the $U(N)$ condition (3.12) follows immediately. From Theorem 3.1, we know that A_1, B_1 satisfies (2.9), (2.10). The proof is completed.

In the following we first show that the Darboux matrix $S=I+\lambda\alpha$ can be expressed by an Hermitian projection π to $(N-k)$-dimensional subspaces of C^N.

In fact, from (3.23) we have

$$\alpha^*+\frac{1}{\underset{1}{\bar\lambda}}I=\alpha+\frac{1}{\underset{1}{\lambda}}I.$$

So the matrix $\alpha+\frac{1}{\underset{1}{\lambda}}I$ is Hermitian. Since $\alpha=-H\Lambda^{-1}H^{-1}$, and the eigenvalues of α are $-\frac{1}{\underset{1}{\lambda}}$ and $-\frac{1}{\underset{2}{\lambda}}$, we have

$$\alpha+\frac{1}{\underset{1}{\lambda}}I=\Big(\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{2}{\lambda}}\Big)\beta\begin{bmatrix}0 & 0\\ 0 & I_{N-k}\end{bmatrix}\beta^*=\Big(\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{2}{\lambda}}\Big)\pi, \tag{3.24}$$

where $\beta\in U(N)$, I_{N-k} is the $(N-k)\times(N-k)$ unit matrix and π is an Hermitian projection.

Thus

$$\alpha=-\frac{1}{\underset{1}{\lambda}}I+\Big(\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{2}{\lambda}}\Big)\pi=-\frac{1}{\underset{2}{\lambda}}\pi-\frac{1}{\underset{1}{\lambda}}\pi^\perp, \tag{3.25}$$

$$S=I+\lambda\alpha=\Big(1-\frac{\lambda}{\underset{2}{\lambda}}\Big)\pi+\Big(1-\frac{\lambda}{\underset{1}{\lambda}}\Big)\pi^\perp. \tag{3.26}$$

Moreover, by calculation we can prove that, for $|2\lambda-1|=1$,

$$\Big|1-\frac{\lambda}{\underset{1}{\lambda}}\Big|^2=\Big|1-\frac{\lambda}{\underset{2}{\lambda}}\Big|^2. \tag{3.27}$$

Hence

$$S^*S=\Big|1-\frac{\lambda}{\underset{2}{\lambda}}\Big|^2\pi+\Big|1-\frac{\lambda}{\underset{1}{\lambda}}\Big|^2\pi^\perp=\Big|1-\frac{\lambda}{\underset{1}{\lambda}}\Big|^2 I. \tag{3.28}$$

Thus we have

Theorem 3.3. *A new extended solution $\Phi^1(\lambda)$ is obtained from the original extended solution $\Phi(\lambda)$ by the transformation*

$$\Phi^1(\lambda)=S\Big(1-\frac{\lambda}{\underset{2}{\lambda}}\Big)^{-1}\Phi(\lambda) \tag{3.29}$$

and the corresponding new harmonic map is

$$\Phi^1(1)=S\Big(1-\frac{1}{\underset{2}{\lambda}}\Big)^{-1}\Phi(1). \tag{3.30}$$

We turn to deduce the differential equations satisfied by π. From (3.25), we have

$$\alpha = -\frac{1}{\lambda_1} + \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\Big)\pi.$$

Substitute it into (3.4), we obtain

$$\begin{aligned} \pi_{\bar\zeta} &= -\lambda_1 \pi A + \lambda_2 A\pi + (\lambda_1 - \lambda_2)\pi A \pi, \\ \pi_{\zeta} &= \bar\lambda_1 B\pi - \bar\lambda_2 \pi B + (\bar\lambda_2 - \bar\lambda_1)\pi B\pi. \end{aligned} \tag{3.31}$$

Thus we have

Theorem 3.4. *The projective operator*

$$\pi = \Big(\alpha + \frac{1}{\lambda_1}\Big)\Big/\Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\Big) \tag{3.32}$$

is a solution to (3.31).

Remark 3.2. Equations in (3.31) are just the pair of equations (25) of Bäcklund transformations for harmonic maps in Uhlenbeck's paper [4] with different notations. The differences of the notations between [4] and the present paper are as follows: (i) The order of multiplication of matrices in the present paper is different from that of [4]. (ii) Our A, B correspond to $2A_{\bar\zeta}$ and $2A_{\zeta}$ in [4], respectively. (iii) Our λ corresponds to $\frac{1-\lambda}{2}$ in [4].

Remark 3.3. From (3.30) and (3.26), it is seen that the Darboux transformation gives new harmonic maps

$$\Phi^1(1)K = (\pi + \gamma\pi^{\perp})\Phi(1)\cdot K. \tag{3.33}$$

Here K is an arbitrary constant matrix in $U(N)$ and

$$\gamma = \Big(1 - \frac{1}{\lambda_1}\Big)\Big/\Big(1 - \frac{1}{\lambda_2}\Big). \tag{3.34}$$

From (3.27), we have $|\gamma| = 1$.

(3.33) is actually the formula for new solution of Theorem 6.3 in [4]. But in our case, π can be constructed explicitly in terms of the extended solutions of the harmonic map $\Phi(1)$.

Remark 3.4. The system (3.31) is completely integrable. Hence each solution π of (3.31) is completely determined by the initial data $\pi(0,0)$. From our construction, for any fixed λ_1, we can choose l_α's such that

$$\alpha(0,0) = -\frac{1}{\lambda_1} + \Big(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\Big)\pi(0,0).$$

Thus our construction exhausts all solutions to the system (3.31).

§4. Soliton Solutions

In this section, we construct the single soliton solutions as applications of the Darboux matrix method. For simplifying the calculation, we take $N = 2$ in the following. The results for general N are similar.

The elements in $SU(2)$ are matrices

$$\begin{bmatrix} \gamma & \beta \\ -\bar\beta & \bar\gamma \end{bmatrix} \tag{4.1}$$

with $|\beta|^2+|\gamma|^2=1$. Take the seed solution in the following form

$$g_0=\begin{bmatrix} e^{\tau\bar{\zeta}-\bar{\tau}\zeta} & 0 \\ 0 & e^{-(\tau\bar{\zeta}-\bar{\tau}\zeta)} \end{bmatrix}, \tag{4.2}$$

where τ is a constant and

$$A=\begin{bmatrix} \tau & 0 \\ 0 & -\tau \end{bmatrix},\ B=\begin{bmatrix} -\bar{\tau} & 0 \\ 0 & \bar{\tau} \end{bmatrix}. \tag{4.3}$$

Substituting (4.3) into the Lax pair (2.12) and integrating, we obtain

$$\begin{aligned}\Phi_0&=\begin{bmatrix} \exp(\lambda\tau\bar{\zeta}-\frac{\lambda}{2\lambda-1}\bar{\tau}\zeta) & 0 \\ 0 & \exp(-\lambda\tau\bar{\zeta}+\frac{\lambda}{2\lambda-1}\bar{\tau}\zeta) \end{bmatrix}\\ &=\begin{bmatrix} l(\lambda) & 0 \\ 0 & l^{-1}(\lambda) \end{bmatrix}\end{aligned} \tag{4.4}$$

with

$$l(\lambda)=\exp(\lambda\tau\bar{\zeta}-\frac{\lambda}{2\lambda-1}\bar{\tau}\zeta). \tag{4.5}$$

Let λ_1,λ_2 be two distinct constants related by

$$2\lambda_1\bar{\lambda}_2=\lambda_1+\bar{\lambda}_2. \tag{4.6}$$

As in §3, we take

$$H=\begin{bmatrix} l(\lambda_1) & -\bar{a}l(\lambda_2) \\ al^{-1}(\lambda_1) & l^{-1}(\lambda_2) \end{bmatrix}. \tag{4.7}$$

It is seen that

$$l^{-1}(\lambda_1)=\overline{l(\lambda_2)},\ l^{-1}(\lambda_2)=\overline{l(\lambda_1)} \tag{4.8}$$

and

$$h_2^*h_1=0. \tag{4.9}$$

Moreover,

$$\det H=|l(\lambda_1)|^2+|a|^2|l(\lambda_1)|^{-2}. \tag{4.10}$$

From (3.7), we obtain

$$\alpha=\frac{-1}{e^p+|a|^2e^{-p}}\begin{bmatrix} \dfrac{e^p}{\lambda_1}+|a|^2\dfrac{e^{-p}}{\lambda_2} & \Big(\dfrac{1}{\lambda_1}-\dfrac{1}{\lambda_2}\Big)\bar{a}e^{iq} \\ \Big(\dfrac{1}{\lambda_1}-\dfrac{1}{\lambda_2}\Big)ae^{-iq} & \dfrac{e^p}{\lambda_2}+|a|^2\dfrac{e^{-p}}{\lambda_1} \end{bmatrix}, \tag{4.11}$$

where

$$p=(\lambda_1-\lambda_2)\tau\bar{\zeta}+(\bar{\lambda}_1-\bar{\lambda}_2)\bar{\tau}\zeta, \tag{4.12}$$

$$iq=(\lambda_1+\lambda_2)\tau\bar{\zeta}-(\bar{\lambda}_1+\bar{\lambda}_2)\bar{\tau}\zeta. \tag{4.13}$$

Note that p,q are real, and linear with respect to x and y. From (3.30), the corresponding new harmonic map is

$$\Phi^1(1) = (I+\alpha)\Big(1-\frac{1}{\underset{2}{\lambda}}\Big)^{-1}\Phi_0(1) = \frac{-1}{e^p+|a|^2e^{-p}}\Big(1-\frac{1}{\underset{2}{\lambda}}\Big)^{-1}\times$$

$$\begin{bmatrix} e^p\Big(\frac{1}{\underset{1}{\lambda}}-1\Big)+|a|^2e^{-p}\Big(\frac{1}{\underset{2}{\lambda}}-1\Big)e^{\tau\bar{\zeta}-\bar{\tau}\zeta} & \Big(\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{2}{\lambda}}\Big)\bar{a}e^{-iq}e^{-(\tau\bar{\zeta}-\bar{\tau}\zeta)} \\ \Big(\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{2}{\lambda}}\Big)ae^{-iq}e^{\tau\bar{\zeta}-\bar{\tau}\zeta} & e^p\Big(\frac{1}{\underset{2}{\lambda}}-1\Big)+|a|^2e^{-p}\Big(\frac{1}{\underset{1}{\lambda}}-1\Big)e^{-(\tau\bar{\zeta}-\bar{\tau}\zeta)} \end{bmatrix}. \quad (4.14)$$

We write

$$\Phi^1(1) = \begin{bmatrix} \gamma & \beta \\ -\bar{\beta} & \bar{\gamma} \end{bmatrix}$$

and

$$\beta = \rho_1 e^{i\theta_1},\ \gamma = \rho_2 e^{i\theta_2}, \quad (4.15)$$

then

$$\rho_1 = \frac{k|\bar{a}|}{e^p+|a|^2e^{-p}} = \frac{k}{2}\,\mathrm{sech}(p-\ln|a|)\ \Big(k = \Big|1-\frac{1}{\underset{2}{\lambda}}\Big|^{-1}\Big|\frac{1}{\underset{1}{\lambda}}-\frac{1}{\underset{2}{\lambda}}\Big|\Big), \quad (4.16)$$

$$\rho_2 = (1-\rho_1^2)^{\frac{1}{2}}. \quad (4.17)$$

We see that $\rho_1 \to 0$ and $\rho_2 \to 1$ when $p \to \pm\infty$. Hence we call $\Phi^1(1)$ a single soliton solution.

Let $\gamma = x_1+ix_2, \beta = x_3+ix_4$, $\Phi^1(1)$ can be considered as a harmonic map from R^2 to S^3.

We describe the geometric character of the harmonic map $\Phi^1(1)$. Let l be a straight line in R^2, which is not parallel to the line $p =$const. It is easy to see that if (x, y) approaches to infinity along the line l, then p approaches to ∞ and $\rho_1 \to 0$. Hence the image of l approaches to the equator $x_3 = x_4 = 0$ of S^3. Such kind of straight lines are called generic lines. The straight lines $p =$const., are called special lines and their images do not approach to the equator. Thus, for the single soliton solution $\Phi^1(1)$, there is one family of special lines $p =$const. whose images are some curves with $\rho_1 =$const.

We define the kth Darboux transformation of the trivial solution (4.2) to be k-soliton solution and describe their asymptotic behavior as we have done in [2]. Their explicit expressions for the extended solutions can be obtained recursively, i.e.

$$\begin{aligned} \Phi^k(\lambda) &= (I+\lambda\alpha_{k-1})\cdots(I+\lambda\alpha_0)\Phi_0(\lambda)\cdot \\ &\cdot\Big(1-\frac{\lambda}{\underset{2}{\lambda}^{(0)}}\Big)^{-1}\Big(1-\frac{\lambda}{\underset{2}{\lambda}^{(1)}}\Big)^{-1}\cdots\Big(1-\frac{\lambda}{\underset{2}{\lambda}^{(k-1)}}\Big)^{-1} \end{aligned} \quad (4.18)$$

and the harmonic maps are

$$\begin{aligned} \Phi^k(1) &= (I+\alpha_{k-1})\cdots(I+\alpha_0)\Phi_0(1)\cdot \\ &\cdot\Big(1-\frac{1}{\underset{2}{\lambda}^{(0)}}\Big)^{-1}\Big(1-\frac{1}{\underset{2}{\lambda}^{(1)}}\Big)^{-1}\cdots\Big(1-\frac{1}{\underset{2}{\lambda}^{(k-1)}}\Big)^{-1}. \end{aligned} \quad (4.19)$$

Here $(\underset{1}{\lambda}^{(i)}, \underset{2}{\lambda}^{(i)})$ for $i = 0, 1, \cdots, k-1$ are the parameters which are used in the successive Darboux transformations and satisfy the condition: all the $|2\underset{2}{\lambda}^{(i)}-1|$ $(i = 0, 1, \cdots, k-1)$ are distinct.

Moreover, in (4.18) and (4.19), α_i $(i=0,1,\cdots,k-1)$ are constructed from $\Phi^i(\lambda)$ by using Theorems 3.1-3.3. Define

$$p_i=(\underset{1}{\lambda}^{(i)}-\underset{2}{\lambda}^{(i)})\tau\bar{\zeta}+(\underset{1}{\bar{\lambda}}^{(i)}-\underset{2}{\bar{\lambda}}^{(i)})\bar{\tau}\zeta.$$

A straight line l in R^2 is called generic if it is not parallel to the lines $p_i=$const. $(i=0,1,\cdots,k-1)$, and the k families of lines $p_i=$const. are called special lines. We have

Theorem 4.1. *The k-soliton solution* $\Phi^k(1)=\begin{pmatrix}\gamma^{(k)} & \beta^{(k)}\\ -\bar{\beta}^{(k)} & \bar{\gamma}^{(k)}\end{pmatrix}\in SU(2)$ *which can be considered as a harmonic map from R^2 to S^3 has the following properties:*

(i) $\beta^{(k)}$ *approaches to* 0 *when* (x,y) *approaches to infinity along a generic line l, i.e. the image of the line l approaches to the equator $x_3=x_4=0$ asymptotically.*

(ii) *There are k families of special lines $p_i=const.$ $(i=0,1,\cdots,k-1)$, and $\Phi^k(1)$ behaves asymptotically as a single soliton when (x,y) approaches to infinity along each special line.*

The proof is similar to the proof of the main theorem for the R^{1+1} case[2].

If we consider y as the time coordinate and $y=$const. are generic lines, Theorem 4.1 implies that when $y\to\pm\infty$, a k-soliton is splitting up into k single solitons asymptotically, and the interaction of solitons is elastic if we consider the magnitudes ρ_1,ρ_2 only.

§5. Transformation of Unitons

From now on, we write the parameter $\lambda=\frac{1-\mu}{2}$, and the Lax pair (2.12) can be written as

$$\frac{\partial\Psi(\mu)}{\partial\bar{\zeta}}=\frac{1-\mu}{2}A\Psi,\ \frac{\partial\Psi(\mu)}{\partial\zeta}=\frac{1-\mu^{-1}}{2}B\Psi. \tag{5.1}$$

Here

$$\Psi(\mu)=\Phi(\frac{1-\mu}{2}). \tag{5.2}$$

The parameter μ is used in [4] where it is denoted by λ.

The concept of uniton was introduced by Uhlenbeck in [4]. Let g be a harmonic map. If there is an extended solution $\Psi(\mu)$ which satisfies the following conditions

(a) $\Psi(\mu)=\sum\limits_{a=0}^{n}T_a\mu^a$ (a polynomial of μ),

(b) $\Psi(1)=I$,

(c) $\Psi(-1)=g^{-1}Q$ $(Q\in U(N),$ a constant matrix),

(d) $\big(\Psi(\bar{\mu})\big)^*=\big(\Psi(\mu^{-1})\big)^{-1}$ $(\mu\neq 0)$.

g is called a uniton, and $\Psi(\mu)$ is called extended solution of a uniton or simply extended uniton.

From the above definition, we see that the soliton solutions which we obtained in the above section are not unitons.

By using the parameter μ

$$\underset{1}{\lambda}=\frac{1-\mu_1}{2},\ \underset{2}{\lambda}=\frac{1-\mu_2}{2}=\frac{1-\bar{\mu}_1^{-1}}{2}. \tag{5.3}$$

From (3.29), the Darboux transformation of an extended uniton is

$$\Psi^1(\mu)=\big(I+\frac{1-\mu}{2}\alpha\big)\big(1-\frac{1-\mu}{1-\mu_2}\big)^{-1}\Psi(\mu). \tag{5.4}$$

In general, it cannot be an extended uniton, since $\Psi^1(\mu)$ is not a polynomial of μ. Uhlenbeck introduced the singular Bäcklund transformation to obtain a uniton from a known uniton. Here we introduce the singular Darboux transformation as the limit of a sequence of Darboux transformations. We treat this problem as follows.

Let

$$\begin{aligned} L_1 &= [l_1, l_2, \cdots, l_k, 0, \cdots, 0], \\ L_2 &= [0, \cdots, 0, l_{k+1}, \cdots, l_N] \end{aligned} \tag{5.5}$$

be two constant matrices of rank k and $N-k$ $(0 < k < N)$ respectively. Here l_α $(\alpha = 1, \cdots, N)$ are N constant columns satisfying

$$l_p^* l_a = 0 \ (p = k+1, \cdots, N;\ a = 1, 2, \cdots, k). \tag{5.6}$$

We take $\mu_1 = \epsilon$, $\mu_2 = \bar{\epsilon}^{-1}$ $(\epsilon \neq 0)$ and apply Darboux transformation to the extended uniton $\Psi(\mu)$. Let

$$H_\epsilon = \big[\underset{1}{h}, \cdots, \underset{N}{h}\big] = \big[\Psi(\epsilon)l_1, \cdots, \Psi(\epsilon)l_k, \Psi(\bar{\epsilon}^{-1})l_{k+1}, \cdots, \Psi(\bar{\epsilon}^{-1})l_N\big]. \tag{5.7}$$

From the condition (d), we have

$$\begin{aligned} h_p^* h_a &= \big(\Psi(\bar{\epsilon}^{-1})l_p\big)^* \Psi(\epsilon) l_a \\ &= l_p^* \Psi^*(\bar{\epsilon}^{-1}) \Psi(\epsilon) l_a \\ &= l_p^* l_a = 0. \end{aligned} \tag{5.8}$$

We take ϵ such that $\det\Psi(\epsilon) \neq 0$, $\det\Psi(\bar{\epsilon}^{-1}) \neq 0$, hence

$$\det H_\epsilon \neq 0. \tag{5.9}$$

Let

$$\begin{aligned} \alpha_\epsilon &= -H_\epsilon \Lambda_\epsilon^{-1} H_\epsilon^{-1}, \\ \pi_\epsilon &= \frac{1}{2(1-\epsilon\bar{\epsilon})}\big[(1-\epsilon)(1-\bar{\epsilon})\alpha_\epsilon + 2(1-\bar{\epsilon})I\big]. \end{aligned} \tag{5.10}$$

The new extended solutions of the harmonic maps obtained by Darboux transformation are

$$\Psi_\epsilon^{(1)}(\mu) = (\pi_\epsilon + \mu\pi_\epsilon^\perp)\big(1 - \frac{1-\mu}{1-\bar{\epsilon}^{-1}}\big)^{-1}\Psi(\mu). \tag{5.11}$$

In order to elucidate the limiting process of $\epsilon \to 0$, we need the following lemma.

Lemma. *Let $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) be the subspace of C^N spanned by $l_1, l_2, \cdots, l_k$ (resp. $l_{k+1}, \cdots, l_N$). Then α_ϵ depends only on $\mathcal{L}_1$ and $\mathcal{L}_2$, and is independent of the choice of the basis of $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Proof. We note at first that $\mathcal{L}_2 = \mathcal{L}_1^\perp$ is determined by $\mathcal{L}_1$. The change of basis of $\mathcal{L}_1$ and $\mathcal{L}_2$ means that we use

$$\widetilde{L}_1 = L_1 \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{L}_2 = L_2 \begin{bmatrix} 0 & 0 \\ 0 & K_2 \end{bmatrix} \tag{5.12}$$

to replace L_1 and L_2. Here K_1 (resp. K_2) is a regular square matrix of order k (resp. $N-k$). We write H_ϵ and H_ϵ^{-1} by

$$H_\epsilon = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad H_\epsilon^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{5.13}$$

respectively, here the blocks of H and H^{-1} are $k\times k$, $k\times(N-k)$, $(N-k)\times k$, $(N-k)\times(N-k)$ matrices.

The matrices $\widetilde{H}_\epsilon$, $\widetilde{H}_\epsilon^{-1}$ which are constructed by using the column vectors of $\widetilde{L}_1$ and $\widetilde{L}_2$ are:

$$\widetilde{H}_\epsilon = \begin{bmatrix} A_{11}K_1 & A_{12}K_2 \\ A_{21}K_1 & A_{22}K_2 \end{bmatrix} \tag{5.14}$$

and

$$\widetilde{H}_\epsilon^{-1} = \begin{bmatrix} K_1^{-1}B_{11} & K_1^{-1}B_{12} \\ K_2^{-1}B_{21} & K_2^{-1}B_{22} \end{bmatrix}. \tag{5.15}$$

By calculation, it is easily seen that

$$\tilde{\alpha}_\epsilon = -\widetilde{H}_\epsilon\Lambda_\epsilon^{-1}\widetilde{H}_\epsilon^{-1} = -H_\epsilon\Lambda_\epsilon^{-1}H_\epsilon^{-1} = \alpha_\epsilon. \tag{5.16}$$

The lemma is proved.

We choose a special base of $\mathcal{L}_1$ such that

$$L_1 = L_1^{(0)} + L_1^{(1)} + \cdots + L_1^{(n)} \tag{5.17}$$

with

$$\begin{aligned} L_1^{(0)} &= [\ \tilde{l}_1,\cdots,\tilde{l}_{a_0},\quad 0,\cdots,0,\quad 0,\cdots,0,\ \cdots,\quad 0,\cdots,0;\quad 0,\cdots,0\ ],\\ L_1^{(1)} &= [\ 0,\cdots,0,\quad \tilde{l}_{a_0+1},\cdots,\tilde{l}_{a_1},\ 0,\cdots,0,\ \cdots,\quad 0,\cdots,0;\quad 0,\cdots,0\ ],\\ &\cdots\cdots\\ L_1^{(n)} &= [\ 0,\cdots,0,\quad 0,\cdots,0,\quad 0,\cdots,0,\ \cdots,\ \tilde{l}_{a_{n-1}+1},\cdots,\tilde{l}_{a_n};\ 0,\cdots,0\ ]; \end{aligned} \tag{5.18}$$

$$\begin{aligned} &T_iL_1^{(j)} = 0 \quad (j>i),\\ &\operatorname{rank}\{T_jL_1^{(j)}\} = \operatorname{rank}\{L_1^{(j)}\} = a_j - a_{j-1},\\ &\qquad (k = a_n \ge a_{n-1} \ge \cdots a_0 \ge 0, a_{-1} = 0). \end{aligned} \tag{5.19}$$

Define

$$\widetilde{L}_1 = L_1^{(0)} + L_1^{(1)}\epsilon^{-1} + \cdots + L_1^{(n)}\epsilon^{-n} \quad (\epsilon \ne 0). \tag{5.20}$$

Then

$$\Psi(\epsilon)\widetilde{L}_1 = T_0L_1^{(0)} + T_1L_1^{(1)} + \cdots + T_nL_1^{(n)} + \epsilon F_1, \tag{5.21}$$

where F_1 is a polynomial of ϵ. Similarly, we choose a special base of $\mathcal{L}_2$ and define

$$\widetilde{L}_2 = L_2^{(n)}\bar{\epsilon}^n + L_2^{(n-1)}\bar{\epsilon}^{n-1} + \cdots + L_2^{(0)}. \tag{5.22}$$

Here $L_2^{(i)}$ satisfy

$$\begin{aligned} &T_jL_2^{(i)} = 0 \quad (j>i),\\ &\operatorname{rank}\{T_jL_2^{(j)}\} = \operatorname{rank}\{L_2^{(j)}\}. \end{aligned} \tag{5.23}$$

Then

$$\Psi(\bar{\epsilon})\widetilde{L}_2 = T_0L_2^{(0)} + T_1L_2^{(1)} + \cdots + T_nL_2^{(n)} + \bar{\epsilon}F_2, \tag{5.24}$$

where F_2 is a polynomial of $\bar{\epsilon}$. Moreover, we take

$$\widetilde{H}_\epsilon = \Psi(\epsilon)\widetilde{L}_1 + \Psi(\bar{\epsilon}^{-1})\widetilde{L}_2. \tag{5.25}$$

From the lemma, we have

$$\begin{aligned}\alpha_\epsilon &= -\widetilde{H}_\epsilon \Lambda_\epsilon^{-1} \widetilde{H}_\epsilon^{-1} \\ &= -\frac{2}{1-\bar{\epsilon}^{-1}}\pi_\epsilon - \frac{2}{1-\epsilon}\pi_\epsilon^\perp. \end{aligned} \tag{5.26}$$

Here

$$\Lambda_\epsilon^{-1} = \begin{bmatrix} \frac{2}{1-\epsilon} I_k & 0 \\ 0 & \frac{2}{1-\bar{\epsilon}^{-1}} I_{N-k} \end{bmatrix} \tag{5.27}$$

and π_ϵ is a projection on $(N-k)$-dimensional subspaces of C^N.

Let $\epsilon \to 0$. We have

$$\lim_{\epsilon\to 0} H_\epsilon = \sum_{a=0}^{n} T_a L_1^{(a)} + \sum_{a=0}^{n} T_a L_2^{(a)} = H. \tag{5.28}$$

Evidently, H is a regular matrix. Let

$$\lim_{\epsilon\to 0} \alpha_\epsilon = -H \begin{bmatrix} 2I_k & 0 \\ 0 & 0 \end{bmatrix} H^{-1} = \alpha, \tag{5.29}$$

$$\lim_{\epsilon\to 0} \pi_\epsilon^\perp = H \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} H^{-1} = \pi^\perp, \tag{5.30}$$

$$\lim_{\epsilon\to 0} \pi_\epsilon = H \begin{bmatrix} 0 & 0 \\ 0 & I_{N-k} \end{bmatrix} H^{-1} = \pi. \tag{5.31}$$

Using (3.29), we have

Theorem 5.1. *From an extended uniton $\Phi(\mu)$, we can construct the extended uniton*

$$\Phi^1(\mu) = \Big(I + \frac{1-\mu}{2}\alpha\Big)\Phi(\mu). \tag{5.32}$$

Here

$$\alpha = -H \begin{bmatrix} 2I_k & 0 \\ 0 & 0 \end{bmatrix} H^{-1} \tag{5.33}$$

and H is defined by (5.28).

Remark 5.1. Since

$$\begin{aligned} I + \frac{1-\mu}{2}\alpha &= \pi + \pi^\perp + (\mu - 1)\pi^\perp \\ &= \pi + \mu\pi^\perp, \end{aligned} \tag{5.34}$$

(5.32) can be written in

$$\Phi^1(\mu) = (\pi + \mu\pi^\perp)\Phi(\mu). \tag{5.35}$$

This is just the main formula in Theorem 12.1 of [4]. In our case, the projection π has explicit expression (5.31). We will discuss the properties and concrete applications of the singular Darboux transformation in a forthcoming paper.

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References

[1] Gu, C. H. & Zhou, Z. X., Darboux transformations for principal chiral fields, in the Proc. of the International Conference "Nonlinear Evolution Equations, Como, 1988", Manchester Univ. Press, 1990.

[2] Gu, C. H. & Hu, H. S., The soliton behavior of the principal chiral fields, Lectured in one of the main courses of the School on Qualitative Aspects and Applications of Nonlinear Evolution Equations, Trieste, 1990; in *Int. J. Mod. Phys. A* (Proc. of International Conference on Differential Geometrical Methods in Theoretical Physics, Tianjin, 1992), **3** (1993), 501–510.

[3] Beggs, E. J., Solitons in the chiral equations, *Commun. Math. Phys.,* **128** (1990), 131–139.

[4] Uhlenbeck, K., Harmonic maps into Lie groups (classical solutions of the chiral model), *J. Diff. Geom.,* **38** (1990), 1-50.