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The HAUSDORFF DIMENSION AND MEASURE OF THE GENERALIZED MORAN FRACTALS AND FOURIER SERIES**

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Abstract

This paper studies the Hausdorff dimensions, the Hausdorff measures of generalized Moran fractals and the convergence of the Fourier series of functions defined on some generalized Moran fractals. A general formula is given for the calculation of the Hausdorff dimensions of generalized Moran fractals and it is proved that their Hausdorff measures are finite positive numbers under some conditions. In addition, the authors define an orthonormal system Φ of functions defined on generalized Moran s-sets (gMs) and discuss the convergence of the Fourier series, with respect to Φ, of each function $f(x) \in L^1(gMs, H^s)$.

Keywords Haudorff dimension, Hausdorff measure, s-set, Differentiation base, Fourier series.

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§1. Introduction

Let F be a subset of the q-dimensional Euclidean space $\mathbf{R}^q$ and let s be a non-negative number. For any $\delta > 0$, define

$$H_\delta^s(F) = \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : F \subset \bigcup_{i=1}^{\infty} U_i, 0 < |U_i| \leq \delta\Big\},$$

where $|U|$ denotes the diameter of the set U.

We write

$$H^s(F) = \lim_{\delta \to 0} H_\delta^s(F)$$

and call $H^s(F)$ the s-dimensional Hausdorff measure of F.

For given $F \subset \mathbf{R}^q$, there is a unique number, $\dim F$, called the Hausdorff dimension of F, such that

$$H^t(F) = \infty \quad \text{if} \quad 0 \leq t < \dim F,$$

$$H^t(F) = 0 \quad \text{if} \quad \dim F < t < \infty.$$

We know that it is often difficult to determine the Hausdorff dimension of a fractal and harder still to find or even to estimate its Hausdorff measure. So far, we may only find the Hausdorff dimensions of some specific fractals, such as self-similar fractals, classical

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Moran fractals, generalized Cantor fractals (see [1,2]). But the Hausdorff dimensions of the generalized Moran fractals, which are recently discussed by Su Feng[3] and encompass a very wide class of geometric objects such as self-similar fractals and classical Moran fractals, and the information about their Hausdorff measures are still unsettled questions. In this paper we give the general formula for the calculation of the Hausdorff dimensions of these generalized Moran fractals and prove that their Hausdorff measures are positive numbers under some conditions. In addition, we also discuss the convergence of the Fourier series of functions defined on generalized Moran s-sets.

Let J be a nonempty compact subset of $\mathbf{R}^q$, let n be a positive integer, $n \geq 2$, and let t_{ki} be fixed numbers between 0 and $1, k = 1, 2, \ldots, n$. We also assume that J is regular: $J = cl(intJ)$. A generalized Moran fractal based on seed set J and similarity ratios $t_{11}, t_{12}, \cdots, t_{1n}, t_{21}, t_{22}, \cdots, t_{2n}, \cdots, t_{k1}, t_{k2}, \cdots, t_{kn}, \cdots$ is a set E which can be expressed as

$$E = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in S_k} J(\sigma),$$

where $S_k = \{1, 2, \cdots, n\}^k$ and the sets $J(\sigma)$ are given recursively by the conditions that $J = J(\phi)$ and if $J(\sigma)$, for $\sigma \in S_k$, has been determined, then the sets $J(\sigma * 1), J(\sigma * 2), \cdots, J(\sigma * n)$ on the $(k+1)$-th level are nonoverlapping subsets of $J(\sigma)$ such that for each $i, J(\sigma * i)$ is geometrically similar to $J(\sigma)$ via a similarity map with similarity ratios $t_{k+1,i}$. If $\sigma = (\sigma(1), \cdots, \sigma(k))$, then by $\sigma * i$, the concatenation of σ and i, we mean $\sigma * i = (\sigma(1), \cdots, \sigma(k), i)$.

So, if $\sigma = (\sigma(1), \cdots, \sigma(k)) \in S_k$, then

$$|J(\sigma)| = |J| t_k(\sigma) = |J| \prod_{i=1}^{k} t_{i,\sigma(i)}.$$

In section 2, we obtain the calculation formula of the Hausdorff dimension of E and prove that the Haudorff measure is a positive number under some conditions. In section 3, we define an orthonormal system Φ of functions on the generalized Moran s-sets (gMs) and discuss the convergence of the Fourier series, with respect to Φ, of any function $f \in L^1(gMs, H^s)$.

§2. The Hausdorff Dimension and Measure of E

Let the similarity ratios $\{t_{ki}\}, i = 1, \cdots, n, k = 1, 2, \cdots$, satisfy

(A) $c = \inf\limits_{k,i} t_{ki} > 0$;

(B) $\lim\limits_{k\to\infty} \sup\limits_{\sigma \in S_k} |J(\sigma)| = 0$.

For convenience, we assume that $|J| = 1$ and a set $J(\sigma)$, used in the construction of E, is a net set. If $J(\sigma)$ is a net set on the k-th level, then we write $J_k(\sigma)$. If $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) \in S_k$, for every $p, 1 \leq p \leq k$, we denote by $\sigma[p]$ the p-tuple formed by the first p coordinates of σ, that is, $\sigma[p] = (\sigma(1), \sigma(2), \cdots, \sigma(p)) \in S_p$.

For each $m = 1, 2, \cdots$, let

$$f_m(\beta) = \prod_{k=1}^{m} \sum_{i=1}^{n} t_{ki}^{\beta}.$$

Since

$$f_m^{'}(\beta)=\sum_{k=1}^{m}\left[\sum_{i=1}^{n}t_{ki}^{\beta}\log t_{ki}\right]\prod_{1\le j\ne k\le m}\sum_{i=1}^{n}t_{ji}^{\beta}<0,$$

we have

Lemma 2.1. *For given m, the function $f_m(\beta)$ is a strictly decreasing continous function on $[0,\infty)$ and $f_m(0)=n^m\ge 2$, $\lim\limits_{\beta\to\infty}f_m(\beta)=0$.*

Using Lemma 2.1, we immediately show that the equation

$$\prod_{k=1}^{m}\sum_{i=1}^{n}t_{ki}^{\beta}=1 \tag{2.1}$$

has a unique solution, which we write as β_m.

Lemma 2.2. *Let $\{\beta_m\}_{m\ge1}$ be solutions of equations* (2.1). *Then $s=\liminf\limits_{m\to\infty}\beta_m>0$.*

Proof. Since $\beta_m>0$, there exists the lower limit: $\liminf\limits_{m\to\infty}\beta_m=s$.

For all k and $i=1,2,\cdots,n$, if we let $t_{ki}=c$ $(=\inf_{k,i})$, then we can obtain a classical Moran fractal E_1 and $\dim E_1=s_1$, where s_1 is the unique solution of $nc^{s_1}=1$, i.e., $s_1=-\frac{\log n}{\log c}>0$.

On the other hand, since

$$\prod_{k=1}^{m}\sum_{i=1}^{n}t_{ki}^{s_1}\ge\prod_{k=1}^{m}\sum_{i=1}^{n}c^{s_1}=1,$$

we obtain $\beta_m\ge s_1>0$ by using Lemma 2.1. So $s=\liminf\limits_{m\to\infty}\beta_m\ge s_1>0$.

We shall prove that the Hausdorff dimension of E is $s=\liminf\limits_{m\to\infty}\beta_m$. For this, we introduce again a lemma.

Lemma 2.3. *For given $\delta>0$, let*

$$\mathcal{F}=\{J_k(\sigma):J_k(\sigma)\ \textit{is a net set},\quad k\ge k_1,|J_k(\sigma)|\le\delta\},$$

$$\mathcal{F}_1=\{U:U\ \textit{is the union of some elements of}\ \mathcal{F}\},$$

$$H_{\mathcal{F}}^{t,\delta}(E)=\inf\left\{\sum_i|V_i|^t:E\subset\cup_iV_i,0<|V_i|\le\delta,V_i\in\mathcal{F}\right\},$$

$$H_{\mathcal{F}_1}^{t,\delta}(E)=\inf\left\{\sum_i|U_i|^t:E\subset\cup_iU_i,0<|U_i|\le\delta,U_i\in\mathcal{F}_1\right\}.$$

Then $H_\delta^t(E)\ge H_{\mathcal{F}_1}^{t,\delta}(E)\ge bH_{\mathcal{F}}^{t,\delta}(E)$, where b is a positive number, k_1 is a fixed positive integer, $t\ge0$ and E is the above-mentioned generalized Moran fractal.

The proof of Lemma 2.3 is contained in the process of proving Theorem 2.1 in [3].

Theorem 2.1. $\dim E=\liminf\limits_{m\to\infty}\beta_m$.

Proof. Let $s=\liminf\limits_{m\to\infty}\beta_m$. Then exists a subsequence $\{\beta_{ni}\}_{i\ge1}$ of $\{\beta_m\}_{m\ge1}$ such that $\lim\limits_{i\to\infty}\beta_{ni}=s$. That is, for any given $\varepsilon>0$, there is a positive integer k_1 such that $\beta_{n_k}\le s+\varepsilon$ as $k\ge k_1$. Using Lemma 2.1, we have

$$\prod_{j=1}^{n_k}\sum_{i=1}^{n}t_{ji}^{s+\varepsilon}\le\prod_{j=1}^{n_k}\sum_{i=1}^{n}t_{ji}^{\beta_{n_k}}=1. \tag{2.2}$$

For any given $\delta>0$ and the above ε, if the net sets $\{J_{k_0}(\sigma)\}_{\sigma\in S_{k_0}}$ on the k_0-th level satisfy $|J_{k_0}(\sigma)|\le\delta$, then $\{J_{n_k}(\sigma)\}_{\sigma\in S_{n_k}}$ is a δ-cover of E (if $k_0\le n_{k_1}$, we let $n_k=n_{k_1}$; if $k_0>n_{k_1}$, there must be an n_{k_j} such that $n_{k_j} \geq k_0$, and then we let $n_k = n_{k_j}$), and

$$\sum_{\sigma\in S_{n_k}} |J_{n_k}(\sigma)|^{s+\varepsilon} = \prod_{j=1}^{n_k}\sum_{i=1}^{n} t_{ji}^{s+\varepsilon} \leq 1$$

by using (2.2) in the last step. So $\dim E \leq s+\varepsilon$. By the arbitrariness of ε, we have $\dim E \leq s$.

For any given $\varepsilon > 0$, by the definition of the lower limit, there exists a positive integer k_2 such that $\beta_k \geq s-\varepsilon$ as $k \geq k_2$. So we have

$$\prod_{j=1}^{k}\sum_{i=1}^{n} t_{ji}^{s-\varepsilon} \geq \prod_{j=1}^{k}\sum_{i=1}^{n} t_{ji}^{\beta_k} = 1 \tag{2.3}$$

by using Lemma 2.1.

Let $\mathcal{F} = \{J(\sigma) : \sigma \in S_k, k \geq k_2, |J(\sigma)| \leq \delta\}$ and let $\mathcal{V} = \{J_{k_i}(\sigma_i)\} \subset \mathcal{F}$ be a δ-cover of E. We shall prove that $H_{\mathcal{F}}^{s-\varepsilon,\delta}(E) \geq 1$.

By expanding each net set slightly and using the compactness of E, we may assume that the collection $\mathcal{V}$ is a finite collection of closed sets and also the sets in $\mathcal{V}$ are pairwise disjoint (we may remove those sets contained in any others by virtue of the net property).

We write $\mathcal{V} = \{J_{k_i}(\sigma_i)\}_{i=1}^{j}$, where $k_1 < k_j$.

If $J_{k_j}(\sigma_j) \in \mathcal{V}$, then $\{J_{k_j}(\sigma_j[k_j-1]*i)\}_{i=1}^{n} \subset \mathcal{V}$ (the reason is that $\mathcal{V}$ is a disjoint δ-cover of E), where $J_{k_j}(\sigma_j[k_j-1]*i)$, on the k_j-th level, are nonoverlapping subsets of $J_{k_j-1}(\sigma_j[k_j-1])$.

We may assume $k_1 \leq k_2 \leq \cdots \leq k_j$, and $k_{i_1-1} < k_{i_1} = k_{i_1+1} = \cdots = k_j (i_1 < j)$. Then

$$\sum_{i=i_1}^{j} |J_{k_i}(\sigma_i)|^{s-\varepsilon} = \sum_{i=i_1}^{j}[t_{k_i-1}(\sigma_i[k_i-1])]^{s-\varepsilon} t_{k_j,\sigma(k_i)}^{s-\varepsilon}.$$

If $\{J_{k_j-1}(\sigma_i[k_j-1])\}_{i=i_1}^{j}$ are pairwise disjoint only at $i = i_1, i_2, \cdots, j'$, then

$$\{J_{k_i}(\sigma_i)\}_{i=1}^{i_1-1} \cup \{J_{k_j-1}(\sigma_i[k_j-1])\}_{i=i_1}^{j'}$$

are pairwise disjoint since $\{J_{k_i}(\sigma_i)\}_{i=1}^{j}$ are pairwise disjoint and

$$\sum_{i=i_1}^{j} t_{k_i}(\sigma_i)^{s-\varepsilon} = \sum_{i=i_1}^{j'}[t_{k_j-1}(\sigma_i[k_j-1])]^{s-\varepsilon}\sum_{i=1}^{n} t_{k_j,i}^{s-\varepsilon}.$$

Now we estimate the size of $\sum\limits_{i=1}^{j} |J_{k_i}(\sigma_i)|^{s-\varepsilon}$.

(a) If $\sum\limits_{i=1}^{n} t_{k_j,i}^{s-\varepsilon} \geq 1$, then

$$\begin{aligned}\sum_{i=1}^{j}|J_{k_i}(\sigma_i)|^{s-\varepsilon} &= \sum_{i=1}^{i_1-1}|J_{k_i}(\sigma_i)|^{s-\varepsilon} + \sum_{i=i_1}^{j'}[t_{k_j-1}(\sigma_i[k_j-1])]^{s-\varepsilon}\sum_{i=1}^{n} t_{k_j,i}^{s-\varepsilon} \\ &\geq \sum_{i=1}^{i_1-1}|J_{k_i}(\sigma_i)|^{s-\varepsilon} + \sum_{i=i_1}^{j'}[t_{k_j-1}(\sigma_i[k_j-1])]^{s-\varepsilon}.\end{aligned}$$

The collection $\{J_{k_i}(\sigma_i)\}_{i=1}^{j}$ has been reduced to a lower level, that is, now it is highest level is $k_j - 1$. We may only consider the collection $\{J_{k_i}(\sigma_i)\}_{i=1}^{i_1-1} \cup \{J_{k_j-1}(\sigma_i[k_j-1])\}_{i=i_1}^{j'}$ in the preceding steps.

(b) If $\sum\limits_{i=1}^{n} t_{k_j,i}^{s-\varepsilon} < 1$, we first consider $\{J_{k_i}(\sigma_i)\}_{i=1}^{i_1-1} \cup \{J_{k_j-1}(\sigma_i[k_j-1])\}_{i=i_1}^{j'}$ using the preceding process and obtain

$$\begin{aligned}&\sum_{i=1}^{i_1-1} |J_{k_i}(\sigma_i)|^{s-\varepsilon} + \sum_{i=i_1}^{j'} [t_{k_j-1}(\sigma_i[k_j-1])]^{s-\varepsilon}\\ &= \sum_{i=1}^{i_2-1} |J_{k_i}(\sigma_i)|^{s-\varepsilon} + \sum_{i=i_2}^{j''} [t_{k_j-2}(\sigma_i[k_j-2])]^{s-\varepsilon} \sum_{i=1}^{n} t_{k_j-1,i}^{s-\varepsilon},\end{aligned}$$

where $k_{i_2-1} < k_{i_2} = k_{i_2+1} = \cdots = k_j - 1$ and the selection of j'' is similar to that of j'. Thus

$$\begin{aligned}\sum_{i=1}^{j} |J_{k_i}(\sigma_i)|^{s-\varepsilon} &= \sum_{i=1}^{i_1-1} |J_{k_i}(\sigma_i)|^{s-\varepsilon} + \sum_{i=i_1}^{j} [t_{k_j-1}(\sigma_i[k_j-1])]^{s-\varepsilon} \sum_{i=1}^{n} t_{k_j,i}^{s-\varepsilon}\\ &\geq \sum_{i=1}^{i_2-1} |J_{k_i}(\sigma_i)|^{s-\varepsilon} + \sum_{i=i_2}^{j''} [t_{k_j-2}(\sigma_i[k_j-2])]^{s-\varepsilon} \sum_{i=1}^{n} t_{k_j-1,i}^{s-\varepsilon} \sum_{i=1}^{n} t_{k_j,i}^{s-\varepsilon}.\end{aligned}$$

If $\sum\limits_{i=1}^{n} t_{k_j-1,i}^{s-\varepsilon} \cdot \sum\limits_{i=1}^{n} t_{k_j,i}^{s-\varepsilon} \geq 1$, then we go on in the step of (a). If not, we go on in the step of (b). By finite such steps, there must be a positive integer m such that $k_1 \leq m \leq k_j$ and

$$\sum_{i=1}^{j} |J_{k_i}(\sigma_i)|^{s-\varepsilon} \geq \prod_{j=1}^{m} \sum_{i=1}^{n} t_{ji}^{s-\varepsilon} \geq 1$$

by using (2.3) in the last inequality.

Thus, $H_{\mathcal{F}}^{s-\varepsilon,\delta}(E) \geq 1$. By using Lemma 2.3 we immediately obtain $H_\delta^{s-\varepsilon}(E) \geq b > 0$ and $\dim E \geq s - \varepsilon$, and so $\dim E \geq s$ since ε is arbitrary. The proof is finished.

Theorem 2.2. *Suppose that any two of the following three conditions are satisfied:*

(a) $\sum\limits_{i=1}^{n} t_{ki}^{s} \leq 1$,

(b) $\prod\limits_{j=1}^{m} \sum\limits_{i=1}^{n} t_{ji}^{s} \geq \alpha > 0$ *for* $m \geq k_1$,

(c) $|J_k(\sigma)|^s = \bigcirc(n^{-k})$.

Then $0 < H^s(E) < \infty$. *Here* k_1 *is a positive integer,* $s = \liminf\limits_{m\to\infty} \beta_m$.

Proof. (1) If (a) or (c) is satisfied, then $H^s(E) < \infty$.

For any $\delta > 0$, let the net sets $\{J_k(\sigma)\}$, on k-th level, is a δ-cover of E. Since

$$\sum_{\sigma\in S_k} |J_k(\sigma)|^s = \prod_{j=1}^{k} \sum_{i=1}^{n} t_{ji}^{s},$$

we have $\sum\limits_{\sigma\in S_k} |J_k(\sigma)|^s \leq 1$ when (a) is satisfied, or

$$\sum_{\sigma\in S_k} |J_k(\sigma)|^s \leq n^k \sup |J_k(\sigma)|^s = \bigcirc(1)$$

when (c) is satisfied. So $H^s(E) < \infty$.

(2) If (b) is satisfied, then $H^s(E) > 0$.

For any $\delta > 0$, let $\mathcal{F}' = \{J(\sigma) : \sigma \in S_k, k \geq k_1, |J(\sigma)| \leq \delta\}$. By using the similar techniques in the proof of Theorem 2.1 and noting (b), we have $H_{\mathcal{F}'}^{s,\delta}(E) \geq \alpha > 0$. Using

Lemma 2.3, we immediately obtain $H^s_\delta(E) \geq b\alpha > 0$. So we have $H^s(E) \geq b\alpha > 0$.

From the above analysis, we know that it is enough to prove the result of the theorem when (a) and (c) are satisfied, that is, it is enought to prove that $H^s(E) > 0$.

We will use the following lemma.

Lemma 2.4.[2] *Given a closed bounded set F in $\mathbf{R}^q$ of finite H^s-measure , then $H^s(F) > 0$ if and only if there exists an additive function $\phi(A)$ of half-open figures A such that*

(a) *For any figure A, $\phi(A) \geq 0$;*

(b) *If $A \supset F$, then $\phi(A) \geq b > 0$, where b is some fixed constant:*

(c) *There is a finite non-zero constant k such that if $|A| = \delta$,then $\phi(A) \leq k\delta^s$.*

Remark. A half-open figure is a set expressible as a finite union of half-open (e.g. open on the right) q-dimensional intervals.

Now we need only to define a suitable function $\phi(A)$ of half-open figure A so that ϕ satisfies the conditions of Lemma 2.4.

We first define a function f on the net $\{J_k(\sigma) : \sigma \in S_k, k \geq 1\}$:

$$f(J_k(\sigma)) = t^s_{1,\sigma(1)} \cdot t^s_{2,\sigma(2)} \cdot \cdots \cdot t^s_{k,\sigma(k)},$$

where $\sigma = (\sigma(1), \sigma(2), \cdots, \sigma(k)) \in S_k$.

Then for any half-open figure, we write

$$g(A) = \limsup_{k\to\infty} \sum_{J_k \subset A} f(J_k(\sigma)).$$

Let $A_\xi = \{(x_1, x_2, \cdots, x_q) : (x_1 + h_1, \cdots, x_q + h_q) \in A, 0 \leq h_i \leq \xi, 1 \leq i \leq q\}$. Then A_ξ is again half-open and $\phi(A) = \lim\limits_{\xi\to 0+} g(A_\xi)$ is an additive function of half-open figures (in fact, it can be checked by the inductive method).

It is clear that $\phi(A) \geq 0$.

If $A \supset E$, then

$$\begin{aligned}\phi(A) &= \lim_{\xi\to 0+} \limsup_{k\to\infty} \sum_{J_k(\sigma)\subset A_\xi} f(J_k(\sigma)) = \limsup_{k\to\infty} \prod_{j=1}^{k} \sum_{i=1}^{n} t^s_{ji} \\ &\geq \limsup_{k\to\infty} \inf\{t^s_{1,i_1} \cdot t^s_{2,i_2} \cdot \cdots \cdot t^s_{k,i_k}\} \cdot n^k = \bigcirc(1),\end{aligned}$$

that is, (b) in Lemma 2.4 is satisfied.

Let $|A| = \delta$. Since the net sets $J_1(\sigma(i)), J_2(\sigma(i), \sigma(j)), \cdots$ are similar to J but reduced in the ratios $t_{1,\sigma(i)}, t_{1,\sigma(i)}t_{2,\sigma(j)}, \cdots$, etc., where $\sigma(i) \in S_1, (\sigma(i), \sigma(j)) \in S_2, \cdots$. Then we may let these ratios be arranged in decreasing order and denoted by $d_1 \geq d_2 \geq d_3 \geq \cdots$.

Suppose that there is a positive integer k suck that $d_k \geq \delta \geq d_{k+1}$.

Let $C = \{y : |y - a| \leq 2\delta, a \in A\}$. We consider all the net sets $J(\sigma)$, in C, with reduction ratios lying between d_{k+1} and $cd_{k+1} (c = \inf t_{ki})$. If some of these are contained in others, we count only those with the largest reduction ratios. Let these net sets be $\{P_i\} (i = 1, 2, \cdots, N)$. Then we have $\phi(A) \leq \sum\limits_{i=1}^{N} f(P_i)$, using the definition of ϕ and the condition (a).

For every $P_i, f(P_i) \leq d^s_k$ and the volume of $P_i \geq (d_{k+1}c)^q V_J$, where V_J denotes the volume of J. So

$$N \le \frac{\frac{2^q[\Gamma(1/q)]^q}{q^q}(2\delta)^q}{(d_{k+1}c)^q V_J},$$

where Γ is the Γ-function.

Let

$$b = \frac{2^{2q}[\Gamma(1/q)]^q}{q^q V_J c^q}.$$

Then

$$\phi(A) \le b\frac{\delta^q d_k^s}{d_{k+1}^q} = b\frac{\delta^{q-s}}{d_k^{q-s}}\frac{d_k^q}{d_{k+1}^q}\delta^s \le \frac{b}{c^q}\delta^s.$$

The proof is finished.

Remark. When the generalized Moran fractals degenerate into the self-similar fractals or the Moran fractals with the similarity ratios $t_{ki} = t_i (i = 1, 2, \cdots, n)$, their Hausdorff dimension is $s : \sum\limits_{i=1}^{s} t_i^s = 1$ by Theorem 2.1. Moreover, in this case, (a) and (b) of Theorem 2.2 are satisfied, so $0 < H^s(E) < \infty$. These are consistent with the known results about the Hausdorff dimension and measure of the self-similar or the Moran fractal.

§3. The Fourier Series of Functions Defined on Some Generalized Moran Fractals

The study of the Fourier series of the functions defined on self-similar fractals or classical Moran fractals has been done by the authors of [4] and [5]. We shall say that the study of Fourier series of the functions defined on generalized Moran s-sets is also feasible. In this section, let $E \subset \mathbf{R}^q$ be a generalized Moran s-set, that is, E is a generalized Moran fractal which is H^s-measurable and $0 < H^s(E) < \infty$ (e.g. the generalized Moran fractals satisfying Theorem 2.2).

Definition 3.1. *For each $x \in E$, let $\mathcal{B}(x)$ be a collection of bounded measurable sets with positive measure containing x, such that there is at least a sequence $\{U_k\} \subset \mathcal{B}(x)$ with $|U_k| \to 0$. The whole collection $\mathcal{B} = \bigcup\limits_{x \in E} \mathcal{B}(x)$ will be called a differentiation basis.*

Definition 3.2. *Let $\mathcal{B}$ be a differentiation basis on (E, H^s), such that for each measurable set A and for almost every $x \in E$, if $\{U_k\}$ is an arbitrary sequence of $\mathcal{B}(x)$ contracting to x, then*

$$\lim_{k\to\infty} \frac{H^s(A \cap U_k)}{H^s(U_k)} = \mathcal{X}(x),$$

where $\mathcal{X}$ is the characteristic function. We call $\mathcal{B}$ a density basis.

Definition 3.3. *Given a differentiation basis $\mathcal{B}$ on (E, H^s), we define the maximal operator associated to the basis $\mathcal{B}$ by*

$$Mf(x) = \sup_{U \in \mathcal{B}(x)} \frac{1}{H^s(U)} \int\limits_U |f(y)|\, dH^s(y) \qquad \textit{for all } x \in E$$

for every function $f \in L^1(E, H^s)$.

Definition 3.4. *Let $\mathcal{B}$ be a differentiation basis on (E, H^s) and let $f \in L^p(E, H^s)$ $(1 \le p \le \infty)$. If*

$$\lim_{k\to\infty}\{\frac{1}{H^s(U_k)}\int_{U_k} f\, dH^s : \{U_k\} \subset \mathcal{B}(x), U_k \to 0\} = f(x)$$

for almost every $x \in E$, then we say that $\mathcal{B}$ differentiates $\int f$. We write $D(\int f, x) = f(x)$.

If for each $f \in L^p(E, H^s), \mathcal{B}$ differentiates $\int f$, then we say that $\mathcal{B}$ differentiates $L^p(E, H^s)$.

Lemma 3.1. *Let E be a generalized Moran s-set, and let $E_k(\sigma) = J_k(\sigma) \cap E$ for each $\sigma \in S_k$ and $k \ge 1$, where the meaning of $J_k(\sigma)$ is the same as above. Then $\mathcal{A} = \{E_k(\sigma) : \sigma \in S_k, k \ge 1\}$ is a density basis on (E, H^s).*

Proof. It is easy to check that $\mathcal{A}$ is a differentiation basis for (E, H^s) by Definition 3.1. Following the method of the proof of Theorem 3.3 in [6], we may prove that $\mathcal{A}$ is a density basis.

Lemma 3.2. *$\mathcal{A}$ differentiates $L^\infty(E, H^s)$.*

Proof. Following the similar steps used in the proof Theorem 1.4 in [7] and using Lemma 3.1 and the fact that Lusin's Theorem is still valid after the Hausdorff measure replaces the Lebsgue measure, we can obtain the proof of this lemma.

Lemma 3.3. *For every function $f \in L^1(E, H^s)$ and every number $\alpha > 0$, we have $H^s(\{x \in E : Mf(x) > \alpha\}) \le C\|f\|_1/\alpha$, where $C > 0$ is a constant independent of α and f.*

The verification method is similar to Theorem 3.5 in [6].

Theorem 3.1. *$\mathcal{A}$ differentiates $L^1(E, H^s)$, that is,*

$$D\Big(\int f, x\Big) = \lim_{k\to\infty}\Big\{\frac{1}{H^s(U_k)}\int_{U_k} f\, dH^s : \{U_k\} \subset \mathcal{A}(x), U_k \to x\Big\} = f(x)$$

for H^s-a.e.$x \in E$.

Proof. For each $f \in L^1(E, H^s)$, let

$$f_k(x) = \begin{cases} f(x), & \text{if } |f(x)| < k, \\ 0, & \text{if } |f(x)| \ge k, \end{cases}$$

for any $x \in E$ and let f^k be a function such that $f = f_k + f^k$.

For H^s-a.e. $x \in E$, by Lemma 3.2, we have $D(\int f_k, x) = f_k(x)$. So, for any $\varepsilon > 0$, we have

$$\begin{aligned} &H^s(\{x : |D(\int f, x) - f(x)| > \varepsilon\}) \\ &= H^s(\{x : |D(\int f^k, x) - f^k(x)| > \varepsilon\}) \\ &\le H^s(\{x : D(\int f^k, x) > \varepsilon/2\}) + H^s(\{x : f^k(x) > \varepsilon/2\}) \\ &\le H^s(\{x : Mf^k(x) > \varepsilon/2\}) + H^s(\{x : f^k(x) > \varepsilon/2\}), \end{aligned}$$

where M is a maximal operator associated to the basis $\mathcal{A}$. Since $f \in L^1(E, H^s)$ and Lemma 3.3 is true, we may obtain $D(\int f, x) = f(x)$ for H^s-a.e. $x \in E$. So the proof is finished.

Now we begin to discuss the convergence of the Fourier series of the functions defined on E.

We first define a function with support on the set E. It is $g_{-1}(x) = H^s(E) \quad$ for all $\; x \in E$.

Secondly, we define $n-1$ functions $g_0^h, 1 \leq h \leq n-1$, with support on the set $\bigcup_{\sigma(i)=1}^{h+1} E_{\sigma(i)} \subset E$ (here $E_{\sigma(i)}$ denotes $J(\sigma(i)) \cap E, \sigma(i) \in S_1$). They are

$$g_0^h(x) = \begin{cases} C_h^{-\frac{1}{2}}, & \text{if } x \in \bigcup_{\sigma(i)=1}^{h} E_{\sigma(i)}, \\ -C_h^{-\frac{1}{2}} H^s(E_{h+1})^{-1} \sum_{\sigma(i)=1}^{h} H^s(E_{\sigma(i)}), & \text{if } x \in E_{h+1}, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} C_h &= \Big[1 + H^s(E_{h+1})^{-1} \sum_{\sigma(i)=1}^{h} H^s(E_{\sigma(i)})\Big] \sum_{\sigma(i)=1}^{h} H^s(E_{\sigma(i)}) \\ &= H^s(E_{h+1})^{-1} \sum_{\sigma(i)=1}^{h} H^s(E_{\sigma(i)}) \sum_{\sigma(i)=1}^{h+1} H^s(E_{\sigma(i)}). \end{aligned}$$

Finally, for every $\sigma \in S_k$ and $k \geq 1$, we define $n-1$ functions $g_\sigma^h, 1 \leq h \leq n-1$, whose support is $\bigcup_{i=1}^{h+1} E_{\sigma i} \subset E_\sigma$ (here $E_{\sigma i}$ denotes $J(\sigma i) \cap E, i = 1, 2, \cdots, n$). They are defined as:

$$g_\sigma^h(x) = \begin{cases} C_{\sigma h}^{-\frac{1}{2}} H^s(E_\sigma)^{-1/2}, & \text{if } x \in \bigcup_{i=1}^{h} E_{\sigma i}, \\ -C_{\sigma h}^{-1/2} H^s(E_\sigma)^{-1/2} H^s(E_{\sigma,h+1})^{-1} \cdot & \\ \cdot \sum_{i=1}^{h} H^s(E_{\sigma i}), & \text{if } x \in E_{\sigma,h+1}, \\ 0, & \text{otherwise ,} \end{cases}$$

where

$$\begin{aligned} C_{\sigma h} &= H^s(E_\sigma)^{-1} \Big[1 + H^s(E_{\sigma,h+1})^{-1} \sum_{i=1}^{h} H^s(E_{\sigma i})\Big] \sum_{i=1}^{h} H^s(E_{\sigma i}) \\ &= H^s(E_\sigma)^{-1} H^s(E_{\sigma,h+1})^{-1} \sum_{i=1}^{h+1} H^s(E_{\sigma i}) \sum_{i=1}^{h} H^s(E_{\sigma i}). \end{aligned}$$

Let the system Φ be

$$\Phi = \{g_{-1}\} \cup \{g_0^h : 1 \leq h \leq n-1\} \cup \{g_\sigma^h : \sigma \in S_k, k \geq 1, 1 \leq h \leq n-1\}.$$

It is easy to show that $\Phi \subset L^\infty(E, H^s) \subset L^p(E, H^s)$, for $p \geq 1$.

By a method similar to that used in [5] we may show

Theorem 3.2. *The system Φ is orthonormal in the Hilbert space $L^2(E, H^s)$.*

For any $f(x) \in L^1(E, H^s)$, we define its Fourier series, with respect to Φ, as

$$f(x) \sim a_{-1} g_{-1}(x) + \sum_{h=1}^{n-1} a_0^h g_0^h(x) + \sum_{k=1}^{\infty} \sum_{\sigma \in S_k} \sum_{h=1}^{n-1} a_\sigma^h g_\sigma^h(x)$$

where

$$a_{-1} = (f, g_{-1}) = \int_E f(y) g_{-1}(y)\, dH^s(y),$$

$$a_0^h = (f, g_0^h) = \int_E f(y) g_0^h(y)\, dH^s(y),$$

$$a_\sigma^h = (f, g_\sigma^h) = \int_E f(y) g_\sigma^h(y)\, dH^s(y),$$

for $1 \le h \le n-1, \sigma \in S_k$ and $k \ge 1$, are the Fourier coefficients of f with the orthonormal system Φ.

Following the techniques of the proof of Theorem 3.2 in [5] and using Theorem 3.1, we can obtain the convergence theorems of the Fourier series, with respect to Φ, of every function $f \in L^1(E, H^s)$:

Theorem 3.3. *For each function $f \in L^1(E, H^s)$, the partial sums of its Fourier series with respect to Φ converge to f at H^s-a.e.$x \in E$.*

Throrem 3.4. *Let $p, 1 \le p \le \infty$. If $\{c_{-1}, c_0^h, c_\sigma^h\}$ is a sequence of real numbers which satisfies*

$$|c_{-1}| + \sum_{h=1}^{n-1} |c_0^h| \|g_0^h\|_p + \sum_{k=1}^{\infty} \sum_{\sigma \in S_k} \sum_{h=1}^{n-1} |c_\sigma^h| \|g_\sigma^h\|_p < \infty, \tag{3.1}$$

then there is a unique function $f \in L^p(E, H^s)$ such that $\{c_{-1}, c_0^h, c_\sigma^h\}$ are their Fourier coefficients and we have $\|\mathcal{S}_{m+1} f - f\|_p \to 0$. Moreover, if we have a function $f \in L^p$ and its Fourier coefficients $\{a_{-1}, a_0^h, a_\sigma^h\}$ satisfy (3.1), *then the Fourier series of f converges to f in L^p-norm.*

More details about the techniques of proving these theorems may be consulted in [5].

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