Chin. Ann. of Math.
16B: 2(1995),171-176.

THE GAUSS MAP OF SUBMANIFOLDS IN SPACES OF CONSTANT CURVATURE

Zhou Jianwei*

Abstract

This paper studies the Gauss map of submanifolds in space forms defined by Willmore and Saleemi. By using Morse functions, it is proved that the degree of Gauss map is the Euler number of the submanifold. The tight immersions are also studied.

Keywords Space form, Parallel displacement, Gauss map, Morse function, Tight immersion.

1991 MR Subject Classification 53A35, 53C42.

§1. Preliminaries

Let $N(\varepsilon)$ be the space form of constant curvature ε with dimension n. If $\varepsilon > 0$, $N(\varepsilon)$ is isometric to a sphere $S^n(1/\varepsilon)$ of radius $1/\sqrt{\varepsilon}$ in Euclidean space R^{n+1}. If $\varepsilon < 0$, then $N(\varepsilon)$ is isometric to a hyperbolic space $H^n(1/\varepsilon)$ imbedded as a hypersurface in Lorentz space R^{n+1}_{-1} with inner product $\langle\ ,\ \rangle_{-1}$ (see [5], p.101). In what follows we identify space $N(\varepsilon)$ with $S^n(1/\varepsilon)$ or $H^n(1/\varepsilon)$ according to $\varepsilon > 0$ or $\varepsilon < 0$. We also call $-q$ the antipodal point of $q \in H^n(1/\varepsilon)$. Note that $-q \notin H^n(1/\varepsilon)$. Denote $\langle\ ,\ \rangle >_{sgn\varepsilon}$ by $\langle\ ,\ \rangle_\varepsilon$ and $R^{n+1}_{sgn\varepsilon}$ by R^{n+1}_ε, where $\langle\ ,\ \rangle_{+1}$ is the Euclidean inner product on $R^{n+1}_{+1} = R^{n+1}$.

Lemma 1.1. *Let p, q be two points of $N(\varepsilon) \subset R^{n+1}_\varepsilon$ which are not antipodal points. Then the parallel displacement of $X_p \in T_pN(\varepsilon)$ along the geodesic γ from p to q is given by*

$$X_p - \frac{\varepsilon\langle X_p, q\rangle_\varepsilon}{1+\varepsilon\langle p, q\rangle_\varepsilon}(p+q).$$

Proof. Set

$$a = \frac{q - \varepsilon\langle p, q\rangle_\varepsilon p}{\sqrt{\frac{1}{\varepsilon} - \varepsilon\langle p, q\rangle_\varepsilon^2}}, \quad b = \frac{-p + \varepsilon\langle p, q\rangle_\varepsilon q}{\sqrt{\frac{1}{\varepsilon} - \varepsilon\langle p, q\rangle_\varepsilon^2}}.$$

The vectors a and b are parallel to the 2-plane of R^{n+1}_ε determined by o, p and q and the parallel displacement of a along γ to the point q is b. It is easy to verify that the vector $X_p - \langle X_p, a\rangle_\varepsilon a$ is tangent to $N(\varepsilon)$ at every point of γ and normal to γ, hence parallel along γ. Then the parallel displacement of X_p along γ to q is

$$X_p - \langle X_p, a\rangle_\varepsilon a + \langle X_p, a\rangle_\varepsilon b.$$

Manuscript received May 13, 1993.
*Department of Mathematics, Suzhou University, Suzhou 215006, China.

Let $q \in N(\varepsilon)$. We define coordinates on $N(\varepsilon) - \{-q\}$ by stereographic projection $\varphi : S^n(1/\varepsilon) - \{-q\} \longrightarrow R^n$, or $\varphi : H^n(1/\varepsilon) \longrightarrow D^n$, where

$$D^n = \Big\{x \in R^n | \langle x, x\rangle < \frac{1}{|\varepsilon|}\Big\}.$$

We can assume $q = \big(o, \frac{1}{\sqrt{|\varepsilon|}}\big)$ in some Euclidean (or Lorentz) coordinates of R^{n+1}_ε. Then the metric of $N(\varepsilon)$ is

$$ds^2 = d\varphi^{-1} \cdot d\varphi^{-1} = \frac{4\sum(dy^i)^2}{(1+\varepsilon|y|^2)^2}.$$

Lemma 1.2. *Let Y_p be a tangent vector at $x = \varphi(p) \in R^n(D^n)$. Then parallel displacement of Y_p along the geodesic from x to $o = \varphi(q)$ is*

$$\frac{1}{1+\varepsilon|x|^2} Y_p.$$

Proof. Let $\omega^i = \frac{2dy^i}{1+\varepsilon|y|^2}$, $e_i = \frac{1}{2}(1+\varepsilon|y|^2)\frac{\partial}{\partial y^i}$, $i = 1, 2, \cdots, n$. Then $\{e_i\}$ is an orthogonal frame field and parellel along any geodesic which is through the point $o = \varphi(q)$.

§2. Volume of Gauss Map g_q

Let $f : M \longrightarrow N(\varepsilon)$ be an immersion of a compact manifold M in the space form $N(\varepsilon)$, $\dim M = m$. Denote by $B(M)$ the unit normal bundle of M and S^{n-1}_q the unit sphere in $T_qN(\varepsilon)$, $q \in N(\varepsilon)$. We assume that M is oriented for convenience.

Definition 2.1. *Let q be a point of $N(\varepsilon)$ and $-q \notin f(M)$. The Gauss map of $B(M)$ based at point q is a map $g_q : B(M) \longrightarrow S^{n-1}_q$, for any $(p, v) \in B(M)$, $g_q(p, v)$ is the parallel displacement of v along the shortest geodesic joining $f(p)$ to q* (see [9]).

Unless otherwise stated, we agree on the following arranges of the indices:

$$1 \le A, B, \cdots \le n; \quad 1 \le i, j, \cdots \le m; \quad m+1 \le \alpha, \beta, \cdots \le n.$$

Let $e_1, \cdots, e_m, e_{m+1}, \cdots, e_n$ be local field of orthogonal frames on $N(\varepsilon)$ such that, restricted to M, $e_1, \cdots, e_m$ are tangent to M and the orientations determined by $e_1, \cdots, e_m$ and $e_1, \cdots, e_n, f$ are consistent with those of M and R^{n+1}_ε respectively. Associated with frames there are 1-forms ω^i, ω^B_A such that

$$df = \sum_i \omega^i e_i,$$
$$de_i = \sum_j \omega^j_i e_j + \sum_\alpha \omega^\alpha_i e_\alpha - \varepsilon\omega^i f,$$
$$de_\alpha = \sum_i \omega^i_\alpha e_i + \sum_\beta \omega^\beta_\alpha e_\beta.$$

Let $dv = dV \wedge d\sigma$ be the volume element of $B(M)$, where dV and $d\sigma$ are the volume elements of M and fibres of $B(M)$ respectively. Let $\omega^\alpha_i = \sum A^\alpha_{ij}\omega^j$, $A^\alpha_{ij} = A^\alpha_{ji}$. Then $\Pi = \sum A^\alpha_{ij}\omega^i\omega^j \otimes e_\alpha$ is the second fundamental form of f.

Let $d\mu$ be the volume element of unit sphere S^{n-1}_q. Now we compute $g^*_q d\mu$. Fixed an element $(p, v) \in B(M)$, $v = \sum v^\alpha e_\alpha$, $\sum {v^\alpha}^2 = 1$, we choose local field of orthogonal frames $e'_1, \cdots, e'_n$ such that $e'_i = e_i, e'_n = v$ at (p, v). The parallel displacement of them to q form an oriented orthogonal basis of $T_qN(\varepsilon)$; the image of e'_A are also denoted by $g_q(e'_A)$. Then we have $g_q^*d\mu = \bar\omega^1\wedge\cdots\wedge\bar\omega^{n-1}$, where $\bar\omega^A = \langle dg_q(p,v), g_q(e'_A)\rangle_\varepsilon$. Since the restriction of g_q in a fibre of $B(M)$ is an isometry, $d\sigma = \bar\omega^{m+1}\wedge\cdots\wedge\bar\omega^{n-1}$. As $\langle g_q(e_\alpha), g_q(e_i)\rangle = 0$,

$$\begin{aligned}\bar\omega^i = \sum_\alpha v^\alpha\Big[&\omega_\alpha^i - d\Big(\frac{\varepsilon\langle e_\alpha,q\rangle_\varepsilon}{1+\varepsilon\langle f,q\rangle_\varepsilon}\Big)\langle q,e_i\rangle_\varepsilon\\ &-\frac{\varepsilon\langle e_\alpha,q\rangle_\varepsilon}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}\langle df,e_i\rangle_\varepsilon\\ &-\frac{\varepsilon\langle e_i,q\rangle_\varepsilon}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}\langle de_\alpha,f(p)+q\rangle_\varepsilon\\ &+2d\Big(\frac{\varepsilon\langle e_\alpha,q\rangle_\varepsilon}{1+\varepsilon\langle f,q\rangle_\varepsilon}\Big)\langle e_i,q\rangle_\varepsilon\\ &+\frac{\varepsilon^2\langle e_\alpha,q\rangle_\varepsilon\langle e_i,q\rangle_\varepsilon}{(1+\varepsilon\langle f(p),q\rangle_\varepsilon)^2}\langle df,q\rangle_\varepsilon\Big]\\ =\sum_\alpha v^\alpha\Big[&\omega_\alpha^i-\frac{\varepsilon\langle e_\alpha,q\rangle_\varepsilon\omega^i}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}\Big].\end{aligned}$$

Thus we have proved the following proposition.

Proposition 2.1. $g_q^*d\mu = (-1)^m\det\Big(\sum_\alpha v^\alpha\Big(A_{ij}^\alpha + \delta_{ij}\frac{\varepsilon\langle e_\alpha,q\rangle_\varepsilon}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}\Big)\Big)dv.$

Similar to the case of immersed manifolds in Euclidean space, we call $G(p,v,q)$ defined by $g_q^*d\mu = G(p,v,q)dv$ the Lipschitz-Killing curvature.

Corollary 2.1. *The point (p,v) of $B(M)$ is a critical point of Gauss map g_q if and only if the Lipschitz-Killing curvature is zero at (p,v).*

Define function $h_v : M \longrightarrow R$ by

$$h_v(p) = \frac{\langle f(p),v\rangle_\varepsilon}{1+\varepsilon\langle f(p),q\rangle_\varepsilon},\quad p\in M,\quad v\in S_q^{n-1}.$$

Lemma 2.1. *If v is not in the image of the set of critical points of Gauss map g_q, h_v is a Morse function on M.*

Proof.

$$\begin{aligned}dh_v &= \frac{1}{1+\varepsilon\langle f,q\rangle_\varepsilon}\sum_i\omega^i\langle v, e_i - \frac{\varepsilon\langle e_i,q\rangle_\varepsilon}{1+\varepsilon\langle f,q\rangle_\varepsilon}(f+q)\rangle_\varepsilon\\ &= \frac{1}{1+\varepsilon\langle f,q\rangle_\varepsilon}\sum_i\omega^i\langle e_i, v - \frac{\varepsilon\langle v,f\rangle_\varepsilon}{1+\varepsilon\langle f,q\rangle_\varepsilon}(f+q)\rangle_\varepsilon.\end{aligned}$$

Hence p is a critical point of h_v if and only if the parallel displacement of v to $f(p)$ is normal to the tangent space T_pM.

In what follows we assume that p is a critical point of h_v. Then we can write

$$v - \frac{\varepsilon\langle f,v\rangle_\varepsilon}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}(f(p)+q) = \sum v^\alpha e_\alpha,$$
$$\sum(v^\alpha)^2 = 1.$$

From

$$(d^2h_v)_p = \frac{1}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}\left[\langle d(\sum\omega^i e_i),v\rangle_\varepsilon - \frac{\varepsilon\langle f(p),v\rangle_\varepsilon}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}\langle d(\sum\omega^i e_i),q\rangle_\varepsilon\right]$$
$$= \frac{\sum\omega^i\omega_i^\alpha}{1+\varepsilon\langle f(p),q\rangle_\varepsilon}\langle e_\alpha,\sum v^\alpha e_\alpha\rangle_\varepsilon - \frac{\varepsilon\sum\omega^i\omega^i\langle f,v\rangle_\varepsilon}{(1+\varepsilon<\langle f(p),q\rangle_\varepsilon)^2},$$

and

$$\langle\sum v^\alpha e_\alpha,q\rangle_\varepsilon = -\langle f(p),v\rangle_\varepsilon,$$

we know that the Hessian determinant of d^2h_v at p is

$$\left[\frac{-1}{(1+\varepsilon\langle f(p),q\rangle_\varepsilon}\right]^m G(p,\sum v^\alpha e_\alpha,q).$$

By Sard theorem, h_v is Morse function for almost every point of S_q^{n-1}.

We call

$$\tau(M,f,q) = \frac{1}{c(n-1)}\int_{B(M)}|G(p,v,q)|dv$$

the total absolute curvature of f. Let $\overline{\tau}(M,f)$ be the total absolute curvature of M in $R^n(D^n)$ with flat metric defined by Chern and Lashof. By Lemma 1.2 we have (see also [7])

Lemma 2.2. *Let $f: M \longrightarrow N(\varepsilon)$ be an immersion, the image of $q \in N(\varepsilon)$ under the stereographic projection be the origin of $R^n(D^n)$. Then we have $\tau(M,f,q) = \overline{\tau}(M,f)$.*

Theorem 2.1. *The degree of Gauss map is the Euler-Poincaré number $\chi(M)$.*

Proof. Let h_v be a Morse functions defined above and $p_1,\cdots,p_r,\cdots,p_k$ be its critical points, where the indices of $p_1,\cdots,p_r$ are odd and the indices of $p_{r+1},\cdots,p_k$ are even. Then

$$\chi(M) = k-2r$$

is independent of v. Denote by v_j the parallel displacement of v to $p_j, j=1,\cdots,k$. From $g_q^* d\mu = G(p,v,q)dv$, we know that the Gauss map preserves the orientation in a neighborhood of (p_j,v_j) in $B(M)$ if and only if $G(p_j,v_j,q)>0$. Hence we have proved that there is a neighborhood of v in S_q^{n-1} in which the degree of g_q is $\chi(M)$ ([4], §13,14).

From this theorem we have

$$\tau(M,f,q) = \frac{2}{c(n-1)}\int_{G>0}G(p,v,q)dv - \chi(M)$$
$$= \chi(M) - \frac{2}{c(n-1)}\int_{G<0}G(p,v,q)dv.$$

Since every Morse function h_v has at least two critical points with index 0 or m, we have $\tau(M,f,q)\geq 4-\chi(M)$ for M being even dimensional.

Theorem 2.2. *If $f: M\longrightarrow N(\varepsilon)$ is an embedding of a compact even dimensional manifold, $q\in N(\varepsilon)$, then $\tau(M,f,q) = 4-\chi(M)$ if and only if for every point (p,v) of $B(M)$ such that $G(p,v,q)>0$, $f(M)$ lies in one side of the hypersurface of $N(\varepsilon)$ defined by the hyperplane of R_ε^{n+1} :*

$$\frac{\langle v',x\rangle_\varepsilon}{1+\varepsilon\langle x,q\rangle_\varepsilon} = \frac{\langle v',f(p)\rangle_\varepsilon}{1+\varepsilon\langle f(p),q\rangle_\varepsilon},\quad x\in R_\varepsilon^{n+1},$$

where $v' = g_q(p,v)$.

Proof. This is a generalization of Theorem 3 of [3,I]. As

$$\tau(M,f,q)=4-\chi(M) \text{ if and only if } \frac{1}{c(n-1)}\int_{G>0}G(p,v,q)dv=2,$$

the number of critical points of even index of any Morse function $h_{v'}$ is exactly two. If this is the case, $f(M)$ lies between the two hypersurfaces of $N(\varepsilon)$ determined by $h_{v'}$ at the maximum and minimum. Note that the point $-q$ is on these hyperplanes.

§3. Tight Immersion

In this section we discuss the conditions for $\tau(M,f,q)$ to be minimum and independent of q. The proof of the following theorem is a combination of Lemma 2.2 of this paper and the Chern-Lashof Theorem[3].

Theorem 3.1. *Let $f: M\longrightarrow N(\varepsilon)$ be an immersion of a compact manifold M. If there are $m+4$ points q_i of $N(\varepsilon)$ which do not lie is any $(m+2)$-plane of R^{n+1}_ε, $\tau(M,f,q_i)=2$, $i=1,2,\cdots,m+4$, then f is an embedding and $f(M)$ is a geodesic sphere of $N(\varepsilon)$. Furthermore we have $\tau(M,f,q)=2$ for every point $q\in N(\varepsilon), -q\notin f(M)$.*

Denote

$$\gamma(M)=\min\Big\{\sum_i c_i(h)|h \text{ is a Morse function on } M\Big\},$$

where $c_i(h)$ is the number of critical points of h with index i.

Definition 3.1 *An immersion $f: M\longrightarrow N(\varepsilon)$ of a compact manifold M is called tight if $\tau(M,f,q)=\gamma(M)$ for every $q\in N(\varepsilon), -q\notin f(M)$.*

Let P be a subspace of constant curvature of $N(\varepsilon)$ of dimension $n-1$, the sign of the sectional curvature of P be the same as that of $N(\varepsilon)$. We know that P can be viewed as the intersection of $N(\varepsilon)$ with some affine plane of R^{n+1}_ε which is through the antipodal points of $N(\varepsilon)$. Let H be a closed half-space of $N(\varepsilon)$ defined by P.

Theorem 3.2. *Let M be a compact manifold such that*

$$\gamma(M)=\sum b_k(M,Z_2),$$

where $b_k(M,Z_2)$ is the k-th Betti number of M in field Z_2. The immersion f is tight if and only if for every closed halfspace H, the induced homomorphism

$$H_*(f^{-1}H)\longrightarrow H_*(M)$$

in Cech homology with Z_2 coefficients is injective.

Proof. Kuiper[6] proved this theorem for Euclidean cases. The general case follows from Lemma 2.2 and the construction of the stereographic projection of $N(\varepsilon)$.

By this theorem and the results of [2], [6] about the tightness and the tautness of immersion $f: M\longrightarrow S^n$, we have (see [2], p. 114-115)

Proposition 3.1. *The immersion $f: M\longrightarrow S^n(1/\varepsilon)$ is tight (in the sense here) if and only if f is a tight map of M in Euclidean space $R^{n+1}\supset S^n(1/\varepsilon)$.*

Hence we have many examples of tight immersions of $S^n(1/\varepsilon)$. In particular, if $f: S^m\longrightarrow S^n(1/\varepsilon)$ is tigh, then $f(S^m)$ is a metric sphere in $S^n(1/\varepsilon)$.

Proposition 3.2. *Let M be a compact manifold with $\gamma(M)=\sum b_k(M,Z_2)$. An immersion $f: M\longrightarrow H^n(1/\varepsilon)$ is tight if the immersion $f: M\longrightarrow N(1/\varepsilon)\subset R^{n+1}_{-1}$ of M in R^{n+1}_{-1} with the natural Euclidean metric is tight.*

By Theorem 3.1, we know that the conditions of this proposition are also necessary for $M = S^m$ and $n > m + 1$. But they are not necessary for $n = m + 1$. For example, let M be a metric circle in $H^2(-1)$, by a slight deformation of M such that the conditions of Theorem 3.2 still hold, one can get a curve M' in $H^2(-1)$ which is not a metric circle. M' is a tight submanifold of $H^2(-1)$ but not tight in R^3_{-1}. If we regard $H^2(-1)$ as a subspace of $H^n(-1)$ naturally $(n > 2)$, M' is not a tight submanifold of $H^n(-1)$ by Theorem 3.1.

References

[1] Cecil, T. E. & Ryan, P. J., Tight and taut immersions into hyperbolic space, *J. London Math. Soc.,* **19** (1979), 561-572.

[2] Cecil, T. E. & Ryan, P. J., Tight and taut immersions of manifolds, *Res. Notes Math.,* **107, Pitman, London, 1985**.

[3] Chern, S. S. & Lashof, R. K., On the total curvature of immersed manifolds, I, *Amer. J. Math.,* **79** (1957), 306-318; II, *Mich. Math.,* **5** (1958), 5-12.

[4] Dubrovin, B. A., Fomenko, A. T. & Novikov, S. P., Modern geometry —— methods and applications, Vol.**II**, Springer-Verlag, 1985.

[5] Klingenberg, W., Riemannian geometry, Walter de Gruyter, Berlin, New York, 1982.

[6] Kuiper, N. H., Tight embeddings and maps, submanifolds of geometrical class three in R^N, The Chern symposium, 1979.

[7] Winer, J. L., The Gauss map in spaces of constant curvature, *Proc. Amer. Math. Soc.,* **38** (1973), 157-160.

[8] Winer, J. L., Total curvature and total absolute curvature of immersed submanifolds of spheres, *J. Diff. Geom.,* **9** (1974), 391-400.

[9] Willmore, T. J. & Saleemi, B. A., The total absolute curvature of immersed manifolds, *J. London Math. Soc.,* **41** (1966), 153-160.