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STABILIZATION OF UNITARY GROUPS OVER POLYNOMIAL RINGS**

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Abstract

The author studies the stabilization for the unitary groups over polynomial rings and obtains for them some results analogous to the results of linear groups and symplectic groups. It is especially proved that $K_1U(A) = K_1U(R)$ where $A = R[X_1, \cdots, X_m]$, R is a ring of algebraic integers in a quadratic field $Q(\sqrt{d})$.

Keywords Stabilization theorem, Unitary group, Polynomial ring.
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§1. Introduction

In this article we study the stabilization for the unitary groups over polynomial rings and obtain for them some results analogous to the results of [6] on linear groups, and of [4] and [3] on symplectic groups. The patching method used here is different from [6], [7] and [4].

We assume that an involution $*$ is defined on a commutative ring R with 1. The involution $*$ also determines an involution on the ring M_nR of all n by n matrices (a_{ij}) by $(a_{ij})^* = (a^*_{ji})$. Set $\phi_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and

$$U_{2n}R = \{\theta \in GL_{2n}R : \theta\phi_n\theta^* = \phi_n\}$$
$$= \Big\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_{2n}R; \alpha\delta^* - \beta\gamma^* = I, \alpha\beta^* = \beta\alpha^*, \gamma\delta^* = \delta\gamma^*\Big\}.$$

If $\theta \in U_{2n}R$, then $\theta^{-1} = \phi_n\theta^*\phi_n^{-1}$.

Denote by C the set of elements in R such that $r^* = r$, and C_n the set of matrices in M_nR such that $(a_{ij}) = (a_{ij})^*$. In this paper we assume that the involution $*$ defined on R satisfies:

(Δ) for every maximal ideal M of R, the element u in $S^{-1}R/\mathrm{rad}(S^{-1}R)$ such that $u = u^*$ has an inverse image in $S^{-1}C$, where $S = R\backslash M$.

Examples (1) If 2 is invertible, any involution $*$ on R satisfies (Δ), since the subfield of $F = S^{-1}R/\mathrm{rad}(S^{-1}R)$ with $\mathrm{char}F \neq 2$ generated by ff^* $(f \in F)$ is equal to $\bar{C} = \{h \in F; h = h^*\}$.

(2) If $S^{-1}R/\mathrm{rad}(S^{-1}R)$ is a finite field, the involution $*$ on R satisfies (Δ).

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(3) If R is a ring of algebraic integers in a quadratic field $Q(\sqrt{d})$ where Q is the field of rational numbers, d is an integer not divisible by the square of any prime, the involution $*$ on R satisfies (Δ).

For a polynomial ring $A = R[X_1, \cdots, X_m]$, we define $X_i^* = X_i$ for all X_i.

Set $\sigma k = k + n$ if $k \le n$, $\sigma k = k - n$ if $k > n$. For $a \in A$, define

$$\rho_{ij}(a) = \begin{cases} I_{2n} + aE_{ij} - a^* E_{\sigma j, \sigma i}, & i \ne j \le n \text{ or } n+1 \le i \ne j; \\ I_{2n} + aE_{ij} + a^* E_{\sigma j, \sigma i}, & i \ne \sigma j; i \le n, n+1 \le j \text{ or } n+1 \le i, j \le n; \\ I_{2n} + aE_{i, \sigma i}, & a = a^*; \end{cases}$$

where E_{ij} denotes the matrix with 1 at the position (i, j) and zeros elsewhere.

Denote by $EU_{2n}A$ the subgroup generated by $\rho_{ij}(a)$ (elementary matrices) and $EU_{2n}q$ the subgroup of $EU_{2n}A$ generated by $\rho_{ij}(a) \in U_{2n}q$ where q is an involutory ideal ($q = q^*$) of A. The normal subgroup of $EU_{2n}A$ generated by $EU_{2n}q$ is denoted by $EU_{2n}(A, q)$. We set $U_{2n}q = U_{2n}A \cap GL_{2n}q = \ker(U_{2n}A \to U_{2n}A/q)$.

For any n we have a canonical imbedding $\psi : U_{2n}A \to U_{2n+2}A$ defined by

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \overset{\psi}{\to} \begin{pmatrix} \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha_3 & 0 & \alpha_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly, there are similar imbeddings for $EU_{2n}A$, $EU_{2n}q$, and $EU_{2n}(A, q)$. Put $U(A) = \varinjlim U_{2n}A$, $EU(A) = \varinjlim EU_{2n}A$. Then we have the following equations (see [2]):

$$EU(A) = [EU(A), EU(A)] = [U(A), U(A)], \quad EU(A, q) = [EU(A), EU(q)] = [U(A), U(q)].$$

Define $K_1U(A) = U(A)/EU(A)$ which is an abelian group. In this paper we use $Um_{2n}A$ to denote the set of first row (or column) of unitary matrices. For a $v \in A^{2n}$, we denote $v^*\phi_n$ by $\widetilde{v}$, and the ideal generated by the coordinates of v by $\circ(v)$. Set

$$TU_{2n} = \left\{ \begin{pmatrix} \alpha & \beta \\ & {\alpha^*}^{-1} \end{pmatrix} : \alpha\beta^* \in C_n, \ \begin{matrix} \alpha \text{ is an upper triangular matrix with 1 on the} \\ \text{diagonal} \end{matrix} \right\},$$

$$TL_{2n} = \left\{ \begin{pmatrix} \alpha & \\ \gamma & {\alpha^*}^{-1} \end{pmatrix} : \alpha^*\gamma \in C_n, \alpha \text{ is a lower triangular matrix with 1 on the diagonal} \right\}.$$

$$D = \{\mathrm{diag}(d_1, \cdots, d_{2n}) : d_i = {d_{\sigma i}^*}^{-1} \text{ for all } i\}.$$

Let W denote the subgroup consisting of the permutation matrices which are in $EU_{2n}A$, and $TW = WD = DW$. Let δ denote the matrix $\mathrm{diag}(X, \cdots, X, 1, \cdots, 1)$.

As usual, if $S \subset A$ is a multiplicative subset consisting of the powers of a fixed element s or being the complement of a maximal ideal M of A, we use A_S (resp. A_M) instead of $S^{-1}A$. If the element is not a zero divisor, we often identify the matrix $\beta \in M_{n,k}(A)$ with β_S. We put $\phi_S : A \to A_S$ for the canonical homomorphism. ϕ_S induces homomorphism between the various groups $U_{2n}A$ and $U_{2n}A_S$, $EU_{2n}A$ and $EU_{2n}A_S$.

We call a ring R locally principal if for every maximal ideal M of R the localization R_M is a principal ideal ring (see [3]). For example, every Dedekind ring is locally principal. Note that a locally principal ring has Krull dimension 1.

Our main results are:

Theorem 1.1. *Let R be a Noetherian ring, $A = R[X_1, \cdots, X_m]$. Assume that the involution defined on A satisfies (Δ). Then the canonical mapping $U_{2n}A/EU_{2n}A \to K_1U(A)$ is an isomorphism for $n \geq \max(3, \dim R + 2)$.*

Corollary 1.1. *With the conditions of Theorem* 1.1, *suppose that q is an involutory ideal of R, $B = q[X_1, \cdots, X_m]$. Then the canonical mapping $U_{2n}B/EU_{2n}(A, B) \to K_1U(A, B)$ is an isomorphism for $n \geq \max(3, \dim R + 2)$.*

The proof of Corollary 1.1 is just the same as Corollaries 7.9 and 2.13 in [7].

Theorem 1.2. *Let R be a locally principal ring and $A = R[X_1, \cdots, X_m]$. Assume that the involution defined on A satisfies (Δ). Then $U_{2n}A = U_{2n}R \cdot EU_{2n}A$ for all $n \geq 3$.*

Corollary 1.2. *With the conditions of Theorem* 1.2, *we have*

$$U_{2n}A/EU_{2n}A = U_{2n}R/EU_{2n}R$$

for all $n \geq 3$. So in limit we have $K_1U(A) = K_1U(R)$.

Corollary 1.3. *For a ring of algebraic integers in a quadratic field $Q(\sqrt{d})$, we have $U_{2n}A/EU_{2n}A = U_{2n}R/EU_{2n}R$ for all $n \geq 3$, and $K_1U(A) = K_1U(R)$.*

§2. Elementary Unitary Matrices

Lemma 2.1. *The following identities hold:*

(1) $\rho_{ij}(a + b) = \rho_{ij}(a)\rho_{ij}(b)$;

(2) $[\rho_{ij}(a), \rho_{jk}(b)] = \rho_{ik}(ab)$ *when $i, j, k, \sigma i, \sigma j, \sigma k$ are all distinct;*

(3) $[\rho_{ij}(a), \rho_{j,\sigma i}(b)] = \rho_{i,\sigma i}(ab + b^*a^*)$ *when $j \neq \sigma i$;*

(4) $[\rho_{ij}(a), \rho_{j,\sigma j}(b)] = \rho_{i,\sigma j}(ab)\rho_{i,\sigma i}(c)$ *when $j \neq \sigma i$, where $b \in C$ and $c = aba^*$ when $i, j \leq n$ or $n + 1 \leq i, j$; $c = -aba^*$ when $j \leq n < i$ or $i \leq n < j$.*

Lemma 2.2. *Assume that $n \geq 3$, $v, w \in A^n$ and $w^*v = 0$. Then $I_n + vw^* \in E_nA$* (see [6]).

Lemma 2.3. *Assume that $n \geq 3$, $v^* = (0, v_2^*) \in Um_{2n}A$. Then $I_{2n} + v\widetilde{w} - w\widetilde{v} \in EU_{2n}A$ where $\widetilde{w}v = \widetilde{w}w = 0$.*

Proof. $I_{2n} + v\widetilde{w} - w\widetilde{v}$ has the form

$$\begin{pmatrix} I - w_1v_2^* & 0 \\ * & I + v_2w_1^* \end{pmatrix} \text{ (write } w \text{ as } \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}).$$

It is easy to show that the above matrix lies in $EU_{2n}A$ by Lemma 2.2.

Lemma 2.4. *Assume that $n \geq 3$, $r \in A$, and g is an elementary matrix. Then*

(1) $g\rho_{ij}(Ar^2)g^{-1} \in EU_{2n}(Ar + Ar^*)$ *where $j \neq \sigma i$;*

(2) $g\rho_{i,\sigma i}(r^2Cr^{*^2})g^{-1} \in EU_{2n}(Ar + Ar^*)$.

Proof. Except for the following two cases, the conclusion is easily derived from Lemma 2.1:

Case 1. $\rho = \rho_{ij}(ar^2)$, $g = \rho_{ji}(b)$, where $j \neq \sigma i$;

Case 2. $\rho = \rho_{i,\sigma i}(r^2cr^{*^2})$, $g = \rho_{\sigma i,i}(b)$.

We will consider Case 2 only, the other case is similar. Suppose $1 \leq i \leq n$. Taking $j \neq \sigma i$

such that $n+1 \leq j$, we have

$$\begin{aligned} &\rho_{\sigma i,i}(b)\rho_{i,\sigma i}(r^2cr^{*^2})\rho_{\sigma i,i}(-b) \\ =&\rho_{\sigma i,i}(b)\rho_{ij}(-r^2cr^*)\rho_{\sigma i,i}(-b)\cdot[\rho_{\sigma i,i}(b)\rho_{i,\sigma j}(r^*)\rho_{\sigma i,i}(-b), \rho_{\sigma i,i}(b)\rho_{\sigma j,j}(rcr^*)\rho_{\sigma i,i}(-b)] \\ =&\rho_{ij}(-r^2cr^*)\rho_{\sigma j,i}(r^2c^*r^*b)\rho_{\sigma j,j}(-rc^*r^{*^2}br^2cr^*)\cdot \\ &[\rho_{i,\sigma j}(r)\rho_{\sigma i,\sigma j}(-br^*)\rho_{j,\sigma j}(rbr^*), \rho_{\sigma j,j}(rcr^*)] \in EU_{2n}(Ar+Ar^*). \end{aligned}$$

Remark 2.1. (1) In Lemma 2.4, if $r \in C$, then $g\rho_{ij}(Ar^2)g^{-1}$, $g\rho_{i,\sigma i}(r^2Cr^{*^2})g^{-1}$ lie in $EU_{2n}(Ar)$.

(2) If $r \in C$, the condition $n \geq 3$ can be weakened to $n \geq 2$.

Proposition 2.1. *If q is an involutory ideal of A and $n \geq 3$, then $EU_{2n}(A,q)$ is a normal subgroup of $U_{2n}A$*(see [9]).

Lemma 2.5.[8] *Assume that A is a commutative ring whose spectrum of maximal ideal is a Noetherian topological space of dimension $\leq d$ (i.e. A Noetherian and* $\dim A \leq d$*). Then*

(a) *if $n \geq d+2$, $EU_{2n}A$ acts transitively on the set of hyperbolic pair $\{v,w\}$ ($\widetilde{v}w = 1$, $\widetilde{v}v = \widetilde{w}w = 0$), and hence the canonical homomorphism $U_{2n-2}A \to K_1U(A)$ is surjective;*

(b) *if $n \geq d+2$, the canonical homomorphism $U_{2n}A/EU_{2n}A \to K_1U(A)$ is an isomorphism.*

Lemma 2.6. *Assume that $A \subset B \subset R$ is a tower of rings and A is a retract in R. Then $U_{2n}A \cap EU_{2n}R = EU_{2n}A$ and $(U_{2n}A \cdot EU_{2n}B) \cap EU_{2n}R = EU_{2n}B$.*

§3. Unitary Analogues of Quillen-Suslin Theorem

In this section, we want to prove

Theorem 3.1. *Suppose that n,k are positive integers, $n \geq 2$, and $\beta \in M_{2n,k}(R[X])$. For β there exists $\alpha \in EU_{2n}(R[X])$ such that $\alpha\beta \in M_{2n,k}(R)$ if and only if there exists for any $M \in \max(R)$ a matrix $\gamma \in EU_{2n}(R_M[X])$ such that $\gamma\beta_M \in M_{2n,k}(R_M)$.*

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.1. *Assume that $a \in R$, $\alpha, \beta \in M_{2n,k}(R[X,Y,Z])$, where $\alpha_a = \beta_a$ and $\alpha \equiv \beta$* mod *Z. Then there exists a natural number N such that $\alpha(X,Y,a^mZ) = \beta(X,Y,a^mZ)$ for all $m \geq N$* (see [6]).

Lemma 3.2. *Assume that $n \geq 3$, $a \in C$, $f \in R_a[Z]$, $\gamma \in EU_{2n}R_a$. Put $\sigma(Z) = \gamma\rho_{ij}(Zf)\gamma^{-1}$, when $\sigma j = i$, $f \in C$. Then there exists a natural number N and a matrix $\tau \in EU_{2n}(R[Z], ZR[Z])$ such that $\sigma(a^mZ) = \tau_a$ for all $m \geq N$.*

Proof. Suppose that $\gamma \in EU_{2n}R_a$ can be written as a product of k elementary matrices. Let us consider $\sigma(a^{4^k}Zf) = \gamma\rho_{ij}(a^{4^k}Zf)\gamma^{-1}$. By Proposition 2.1, $\sigma(a^{4^k}Zf) \in EU_{2n}(R_a[Z], ZR_a[Z])$. By Lemma 2.4, we have $\sigma(a^{4^k}Zf) \in EU_{2n}(aR_a[Z])$. So $\sigma(a^{4^k}Zf) \in EU_{2n}(R_a[Z], ZR_a[Z]) \cap EU_{2n}(aR_a[Z])$. Therefore $\sigma(a^{4^k}Zf)$ is a product of a finite number of elementary matrices which have the form $\rho_{ij}(ar_{ij})$ or $\rho_{ij}(aZfr_{ij})$. Writing r_{ij} as r'_{ij}/a^p and $f_i = f'_i/a^p$ (f_i is the coefficient of the terms of f). Then we have $\sigma(a^{4^k2p}Zf) \in \phi_S(EU_{2n}(R[Z], ZR[Z])$ where $S = \{a^i : i \geq 1\}$, i.e., there exists $\tau \in EU_{2n}(R[Z], ZR[Z])$ such that $\tau_a = \sigma(\alpha^mZf)$, where $m \geq 4^k2p = 2^{2k+1}p$.

Lemma 3.3. *Suppose that $a \in C$, n,k are natural numbers, $n \geq 2$, and $\beta \in M_{2n,k}(R[X])$. Assume that there exists $\alpha \in EU_{2n}(R_a[X])$ such that $\alpha\beta_a \in M_{2n,k}R_a$. Then there exists a* *natural number m such that when $b \equiv c \mod a^m$ there exists $\gamma \in EU_{2n}(R[X], XR[X])$ for which $\gamma\beta(bX) = \beta(cX)$* (see [7, p.2677] *or* [6, p.227]).

Remark 3.1. In Lemmas 3.2 and 3.3, if $a \notin C$ but $a \in R$, we can replace a by aa^* in $\sigma(a^m Z)$ and $b \equiv c \mod a^m$ respectively, i.e., for Lemma 3.2, there exists $\tau \in EU_{2n}(R[Z], ZR[Z])$ such that $\sigma((aa^*)^m Z) = \tau_a$, and for Lemma 3.3 when $b \equiv c \mod (aa^*)^m$ there exists $\gamma \in EU_{2n}(R[X], XR[X])$ for which $\gamma\beta(bX) = \beta(cX)$.

Lemma 3.4. *Suppose that n, k are natural numbers, $n \geq 2$, $\beta \in M_{2n,k}(R[X])$, $a_1, a_2 \in C$ or at least one of a_1, a_2 lies in C such that $a_1R + a_2R = R$. Assume that there exist $\alpha_i \in EU_{2n}(R_{a_i}[X])$ such that $\alpha_i\beta_{a_i} \in M_{2n,k}(R_{a_i})$. If now $\alpha = \alpha_1\alpha_2 \in EU_{2n}(R[X])$, then $\alpha\beta \in M_{2n,k}(R)$.*

Proof. We prove that for $a_1 \in C$, $a_2 \in R$. By Lemma 3.3 and Remark 3.1, we can find a natural number m satisfying the requirement for a_1 and $a_2a_2^*$. Since $a_1b + a_2d = 1$ for some $b, d \in R$, $a_1^*b^* + a_2^*d^* = 1$, so $a_1R + (a_1a_2^*)R = R$, $a_1^mR + (a_2a_2^*)^mR = R$. There exists $c \in R$ such that $c \equiv 1 \mod a_1^m$ and $c \equiv 0 \mod (a_2a_2^*)^m$. By construction, there exist α_1 and $\alpha_2 \in EU_{2n}(R[X])$ such that $\alpha_1\beta(X) = \beta(cX)$ and $\alpha_2\beta(cX) = \beta(0) \in M_{2n,k}(R)$. Let $\alpha = \alpha_2\alpha_1$. We are done.

Corollary 3.1. *If $n \geq 2$ and $\beta \in M_{2n,k}(R[X])$, the set $I(\beta) = \{a \in C$: there exists $\alpha \in EU_{2n}(R_a[X])$ such that $\alpha\beta_a \in M_{2n,k}R\}$ is an ideal of C.*

The proof depends on Lemma 3.4 (see [7, p.2678]).

Proof of Theorem 3.1. Necessity of the condition is obvious. If the condition holds, the set $S \cap C$ is nonempty where $S \subset R\backslash M$ for a maximal ideal of R. In fact, if $a \in S$, then either aa^* or $a + a^*$ lies in $S \cap C$. By Corollary 3.1, $I(\beta)$ is an ideal of C, but it cannot be contained in any maximal ideal of R, and cannot be contained in any maximal ideal of C too, so $1 \in I(\beta)$ and hence there exists $\alpha \in EU_{2n}(R[X])$ such that $\alpha\beta \in M_{2n,k}R$.

Corollary 3.2. *Assume that $n \geq 2$, $\beta \in U_{2n}(R[X])$. If for each $M \in \max(R)$ we have $\beta_M \in U_{2n}(R_M) \cdot EU_{2n}(R_M[X])$, then $\beta \in U_{2n}R \cdot EU_{2n}(R[X])$, and if we also have $\beta(0) \in EU_{2n}R$, then $\beta \in EU_{2n}(R[X])$* (see [7, p.2676]).

Corollary 3.3. *Assume that $n \geq 2$, $v^t \in Um_{2n}(R[X_1, \cdots, X_m])$. Then there is $\beta \in EU_{2n}(R[X_1, \cdots, X_m])$ such that $(\beta v)^t \in Um_{2n}R$ if and only if for each $M \in \max(R)$ there is $\beta_M \in EU_{2n}(R_M[X_1, \cdots, X_m])$ such that $(\beta_M v)^t \in Um_{2n}R_M$.*

Proof. The proof follows by induction on m, also follows the same line of argument as in Lemmas 3.5 and 3.6 in [3].

Corollary 3.4. *Assume that $n \geq 2$, $\beta \in U_{2n}(R[X_1, \cdots, X_m], (X_1, \cdots, X_m))$ where $(X_1, \cdots, X_m)$ denotes the ideal generated by $X_1, \cdots, X_m$. Then $\beta \in EU_{2n}(R[X_1, \cdots, X_m])$ if and only if the image of β in $U_{2n}(R_M[X_1, \cdots, X_m])$ lies in $EU_{2n}(R_M[X_1, \cdots, X_m])$ for every maximal ideal M of R.*

Lemma 3.5. *Suppose that $a \in C$, $b \in R$ such that $Ra + Rb = R$, and take $\alpha \in EU_{2n}R_{ab}$. Then there exist $\alpha_1 \in EU_{2n}R_b$ and $\alpha_2 \in EU_{2n}R_a$ such that $\alpha = (\alpha_1)_a \cdot (\alpha_2)_b$.*

Proof. By this condition, the lemma can be proved on the same line of argument as in [3, Lemma 3.7].

§4. Preliminary Results

Lemma 4.1. *Let $f \in R[X, X^{-1}]$. For $\rho_{ij}(f)$, we have*

$$\delta\rho_{ij}(f)\delta^{-1} = \begin{cases} \rho_{ij}(f) & \text{if } 1 \le i \ne j \le n, \\ \rho_{ij}(Xf) & \text{if } 1 \le i \le n, n+1 \le j \le 2n, \\ \rho_{ij}(X^{-1}f) & \text{if } n+1 \le i \le 2n, 1 \le j \le n. \end{cases}$$

Lemma 4.2. *Let $\alpha \in EU_{2n}(R[X]) \cap U_{2n}(R[X], XR[X])$. Then*

$$\delta\alpha\delta^{-1},\ \delta^{-1}\alpha\delta \in EU_{2n}(R[X]).$$

Proof. We prove only the first conclusion, as the second is similar. Let

$$\alpha = \prod_{k=1}^{m} \rho_{i(k),j(k)}(a_k + Xf_k),$$

where $a_k \in R$ and $f_k \in R[X]$, and $\gamma_p = \prod_{k=1}^{p} \rho_{i(k),j(k)}(a_k)$. Then $\gamma_m = I_{2n}$, and α can be expressed in the form $\prod_{k=1}^{m} \gamma_k \rho_{i(k),j(k)}(Xf_k)\gamma_k^{-1}$.

It is sufficient to show the conclusion in the case when $\alpha = \gamma\rho_{ij}(Xf)\gamma^{-1}$. Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ denote the ith and σjth columns of the matrix α respectively. Thus $\sigma = \delta(\gamma\rho_{i(k),j(k)}(Xf)\gamma^{-1})\delta^{-1}$ has the form

$$I_{2n} + \begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix} f \widetilde{\begin{pmatrix} Xw_1 \\ w_2 \end{pmatrix}} - \begin{pmatrix} Xw_1 \\ w_2 \end{pmatrix} f \widetilde{\begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix}} \quad \text{or} \quad I_{2n} + \begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix} f \widetilde{\begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix}}.$$

Since $\rho_{i,\sigma i}(b)(b \in C) = \rho_{i,\sigma j}(-b)[\rho_{ij}(1), \rho_{j,\sigma j}(b)]$, we can only consider this for $\rho_{ij}(Xf)$ where $j \ne \sigma i$. If $X = 0$, by Lemma 2.3 σ lies in $EU_{2n}R$. By Corollary 3.2, we can restrict ourselves to the case of local ring R. If $\circ(v_2) = R$ or $\circ(w_2) = R$, our assertion follows from Lemma 2.3 by conjugating σ by a series of elementary matrices $\rho_{ij}(r)$ for which $\delta\rho_{ij}(r)\delta^{-1} = \rho_{ij}(r)$ or $\rho_{ij}(Xr)$ to make Xv_1 or Xw_1 equal zero. If $\circ(v_2)$, $\circ(w_2) \ne R$, then obviously there exists a column $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ of the matrix whose column index differs from i and σj such that $\circ(u_2) = R$. By Lemma 2.1, we have

$$\sigma = \left(I_{2n} + \begin{pmatrix} Xu_1 \\ u_2 \end{pmatrix} f \widetilde{\begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix}} - \begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix} f \widetilde{\begin{pmatrix} Xu_1 \\ u_2 \end{pmatrix}} \right) \cdot$$

$$\left(I_{2n} + \begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix} f \begin{pmatrix} X\widetilde{(w_1 + u_1)} \\ w_2 + u_2 \end{pmatrix} - \begin{pmatrix} X(w_1 + u_1) \\ w_2 + u_2 \end{pmatrix} f \widetilde{\begin{pmatrix} Xv_1 \\ v_2 \end{pmatrix}} \right).$$

Since $\circ(w_2 + u_2) = R$, the above two factors both lie in $EU_{2n}(R[X])$.

Corollary 4.1. *If $n \ge 3$, $\beta \in EU_{2n}(R[X])$, and $\delta\beta\delta^{-1} \in EU_{2n}(R[X])$ (resp. $\delta^{-1}\beta\delta \in U_{2n}(R[X])$), then $\delta\beta\delta^{-1} \in EU_{2n}(R[X])$ (resp. $\delta^{-1}\beta\delta \in U_{2n}(R[X])$).*

Proof. It suffices to consider the matrix $\delta\beta\delta^{-1}$. Write β in the form $\beta_1\beta_2$ where $\beta_1 \in EU_{2n}R$ is the free term of β. By Lemma 4.2 $\delta\beta_2\delta^{-1} \in EU_{2n}(R[X])$, and the hypothesis $\delta\beta\delta^{-1} \in U_{2n}(R[X])$ means that β_1 has the form $\begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_1^{*-1} \end{pmatrix}$. Now we have (see Lemma

4.1)

$$\delta\beta_1\delta^{-1}=\begin{pmatrix}\alpha_1 & X\alpha_2\\ 0 & \alpha_1^{*^{-1}}\end{pmatrix}=\beta_1\begin{pmatrix}I_n & (x-1)\alpha_1^{-1}\alpha_2\\ 0 & I_n\end{pmatrix}\in EU_{2n}(R[X]).$$

From now on, we assume that R is a local ring (unless it is stated otherwise) with maximal ideal M and residue field k, and ϕ denotes the map: $R\to k$.

Lemma 4.3. *If $\gamma\in U_{2n}M$, then there exist $\alpha\in TU_{2n}M$ and $\alpha'\in TL_{2n}M$ such that $\alpha\gamma\alpha'\in D(M)$.*

Proof. Write γ in the form $\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix}$, where γ_{ij} are n by n matrices. By the hypothesis, $\gamma_{22}=I_n \mod M$, $\gamma_{22}\in GL_nM$. Since $\gamma_{21}\gamma_{22}^*\in C_n$, we have

$$\begin{pmatrix}I_n & -\gamma_{12}\gamma_{22}^{-1}\\ 0 & I_n\end{pmatrix}\gamma\begin{pmatrix}I_n & 0\\ -\gamma_{22}^{-1}\gamma_{21} & I_n\end{pmatrix}=\mathrm{diag}(u,u^{*^{-1}})$$

where $u\in GL_nM$. Then the conclusion follows from Lemma 6.1 of [7].

Corollary 4.2. $U_{2n}M\subset D(M)\cdot EU_{2n}(R,M)$.

Lemma 4.4. *$GL_nR=U\cdot TW\cdot U\cdot E_n(R,M)$ where U denotes the subgroup of GL_nR consisting of upper triangular matrices with 1 on the diagonal, $TW=WD$, W is the subgroup generated by permutation matrices, $D=\mathrm{diag}(d_1,\cdots,d_n)$* (see Lemma 4.3 in [6]).

Lemma 4.5. *$U_{2n}R=TU_{2n}R\cdot TW\cdot TU_{2n}R\cdot EU_{2n}(R,M)$.*

Proof. We will prove this by induction on n. Let $n=1$, $\alpha=\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}\in U_2R$. If $\alpha_3\in R^*$, then $-\alpha_1\alpha_3^{-1}\in C$, $-\alpha_3^{-1}\alpha_4\in C$. We have

$$\begin{pmatrix}1 & -\alpha_1\alpha_3^{-1}\\ & 1\end{pmatrix}\alpha\begin{pmatrix}1 & -\alpha_3^{-1}\alpha_4\\ & 1\end{pmatrix}=\begin{pmatrix}0 & -\alpha_3^{*^{-1}}\\ \alpha_3 & 0\end{pmatrix}$$

and $\alpha\in TU_2R\cdot TW\cdot TU_2R$. If $\alpha_3\notin R^*$, then $\alpha_4\in R^*$ and

$$\alpha=\begin{pmatrix}1 & \alpha_2\alpha_4^{-1}\\ & 1\end{pmatrix}\begin{pmatrix}\alpha_4^{*^{-1}} & \\ & \alpha_4\end{pmatrix}\begin{pmatrix}1 & \\ \alpha_4^{-1}\alpha_3 & 1\end{pmatrix}.$$

Now suppose $n>1$, $\alpha\in U_{2n}R$. Let $v=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ denote the first column of the matrix α. If $\circ(v_2)=R$, it is easy to see that there is $\gamma\in GL_nR$ such that γv_2 has only one nonzero coordinate which maybe 1 on be 1st place. Since $v_1^*v_2\in C$, there exists $\beta\in C_n$ such that $v_1=\beta v_2$. Set $\sigma=\begin{pmatrix}I_n & -\beta\\ & I_n\end{pmatrix}\begin{pmatrix}\gamma^{*^{-1}} & \\ & \gamma\end{pmatrix}$. Then σv has 1 on the $(n+1)$th place and other coordinates are zero. Multiplying $\omega_1=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}+I_{2(n-1)}$ from the left, we see that $\omega_1\sigma v$ has the form $(1,0,\cdots,0,\cdots,0)^t$. If $\circ(v_2)\in M$, then $\circ(v_1)=R$, and there exists $\gamma^{*^{-1}}\in GL_nR$ such that $\gamma^{*^{-1}}v_1$ has only one nonzero coordinate which is 1 on the 1st place. Set $\sigma=\prod_{i=n+1}^{2n}\rho_{i1}(*)\begin{pmatrix}\gamma^{*^{-1}} & \\ & \gamma\end{pmatrix}$, where $\prod_{i=n+1}^{2n}\rho_{i1}(*)$ lies in $EU_{2n}(R,M)$. Then σv has the form $(1,0,\cdots,0,0,\cdots,0)^t$. We can write α as

$$\begin{pmatrix}\gamma^{*^{-1}} & \\ & \gamma\end{pmatrix}\begin{pmatrix}I_n & \beta\\ & I_n\end{pmatrix}\begin{pmatrix} & & -1 & \\ & I_{n-1} & & \\ 1 & & & \\ & & & I_{n-1}\end{pmatrix}\begin{pmatrix}1 & & & \\ & \alpha_1 & & \alpha_2\\ & & 1 & \\ & \alpha_3 & & \alpha_4\end{pmatrix}T_1$$

or $\begin{pmatrix} \gamma^{*^{-1}} & \\ & \gamma \end{pmatrix} \prod_{i=n+1}^{2n} \rho_{i1}(*) \begin{pmatrix} 1 & & & \\ & \alpha_1 & & \alpha_2 \\ & & 1 & \\ & \alpha_3 & & \alpha_4 \end{pmatrix} T_1$, where $T_1 \in TU_{2n}R$. By the hypothesis, we can write $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in U_{2(n-1)}R$ as $T_2W'T_3E_1$ where $E_1 \in EU_{2(n-1)}(R,M), T_2, T_3 \in TU_{2(n-1)}R$, and $W' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + I_{2n-4}$. Then by Lemma 4.4 we can finish the proof.

Put $V_1 = U_{2n}R \cdot EU_{2n}(R[X])$, $V_2 = \{\mathrm{diag}(X^{k_1}, \cdots, X^{k_n}, X^{-k_1}, \cdots, X^{-k_n})\}$,

$$V_3 = TU_{2n}(R[X, X^{-1}]),\quad V_4 = EU_{2n}(R[X,X^{-1}], M[X,X^{-1}]),\quad V = V_1V_2V_3V_4.$$

Lemma 4.6. (1) *If $\alpha \in TU_{2n}(R[X,X^{-1}])$, and $\alpha^\phi \in TU_{2n}(k[X])$, then there is $\beta \in TU_{2n}(R[X])$ such that $\alpha^\phi = \beta^\phi$.*

(2) *For every element α in $TW(k[X])$, there exists $\beta \in TW(R[X])$ such that $\beta^\phi = \alpha$.*

Proof. (1) By the definition of $TU_{2n}(R[X,X^{-1}])$, α can be written as a product of elementary matrices $\rho_{ij}(f)$ where $1 \le i \ne j \le n$ or $1 \le i \le n, n+1 \le j \le 2n$, and $f \in R[X,X^{-1}]$. The coefficients of the term of f containing X^{-1} must be in M. So for $\rho_{ij}(f)$, we can choose f_1 which is the part of f not containing the terms which contain X^{-1} so that $\rho_{ij}(f_1^\phi) = \rho_{ij}(f^\phi)$. For $\rho_{i,\sigma i}(f)$, $f \in C$, the coefficients of each term of f lie in C. Thus we also can choose f_1 as above so that $f_1 \in C$ and $\rho_{i,\sigma i}(f_1^\phi) = \rho_{i,\sigma i}(f^\phi)$. The corresponding product of $\rho_{ij}(f_1)$, denoted by β, satisfies $\beta^\phi = \alpha^\phi$.

(2) Since $TW(k[X]) = D(k[X])W$, W is generated by $\begin{pmatrix} 1 & \\ \pm 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \pm 1 \\ & 1 \end{pmatrix}$, and $D \subset U_{2n}k$, it is not difficult to come to the conclusion.

The following results are proved in just the same way as the corresponding assertions in [7, p.2681].

Lemma 4.7. $\delta V \delta^{-1} \subset V$, $\delta^{-1} V \delta \subset V$.

Lemma 4.8. *For $n \ge 2$ and $a \in R$, $\rho_{ij}(aX)V \subset V$, $\rho_{ij}(aX^{-1})V \subset V$.*

Proposition 4.1. *$EU_{2n}(R[X,X^{-1}]) \subset V$.*

Proposition 4.2. *If $\alpha \in EU_{2n}(R[X,X^{-1}]) \cap U_{2n}(R[X,X^{-1}], M[X,X^{-1}])$, then $\alpha = \gamma_1\gamma_2$ for certain elements $\gamma_1 \in EU_{2n}(R[X], M[X])$ and $\gamma_2 \in EU_{2n}(R[X,X^{-1}], M[X,X^{-1}])$.*

Proof. By Proposition 4.1 and Lemma 4.5, α can be written in the form $\alpha_1\alpha_2\alpha_3\alpha_4$ where $\alpha_i \in V_i$. Since $\alpha \equiv I_{2n} \mod M[X,X^{-1}]$, we have $(a_1^\phi)^{-1} = \alpha_2^\phi\alpha_3^\phi$. If $\alpha_2^\phi = \mathrm{diag}(X^{k_1}, \cdots, X^{k_n}, X^{-k_1}, \cdots, X^{-k_n})$, then $X^{k_1}, \cdots, X^{-k_n}$ are equal to the diagonal elements of the matrix $\alpha_2^\phi\alpha_3^\phi = (a_1^\phi)^{-1}$. Certainly $X^{\pm k_i} \in k[X]$. Hence $k_i = 0$ and $\alpha_2 = I_{2n}$. We have $\alpha_3^\phi = (\alpha_1^\phi)^{-1} \in TU_{2n}(k[X])$. Since $\alpha_3 \in TU_{2n}(R[X,X^{-1}])$, there exists $\beta_1 \in TU_{2n}(R[X])$ such that $\beta_1^\phi = \alpha_3^\phi$ by Lemma 4.6. The matrix $\gamma_2 = (\beta_1^{-1}\alpha_3)\alpha_4$ lies in $EU_{2n}(R[X,X^{-1}], M[X,X^{-1}])$, and the matrix $\gamma_1 = \alpha_1\beta_1$ lies in $EU_{2n}(R[X], M[X])$.

Put $G_1 = EU_{2n}(R[X]) \cap U_{2n}(R[X], M[X])$,

$$G_2 = EU_{2n}(R[X^{-1}]) \cap U_{2n}(R[X^{-1}], M[X^{-1}]),\quad G = G_1G_2,$$

and let H denote the subset of $EU_{2n}(R[X,X^{-1}])$ consisting of all elements h such that $hGh^{-1} \subset G$.

The proofs of the following assertions are the same as those of Lemmas 3.1 and 3.2, Corollary 3.3, and Lemma 3.4 in [4].

Lemma 4.9. *If $n \geq 3$, $H \supset \delta, \delta^{-1}$.*

Lemma 4.10. *If $a \in R$, then $\rho_{ij}(aX)$, $\rho_{ij}(aX^{-1}) \subset H$.*

Lemma 4.11. *$H \supset EU_{2n}(R[X, X^{-1}])$.*

Lemma 4.12. *If $n \geq 3$, then $EU_{2n}(R[X, X^{-1}], M[X, X^{-1}]) \subset G$.*

Proposition 4.3. *Suppose that R is an arbitrary commutative ring, the involution $*$ defined on R satisfies the condition (Δ), and $n \geq 3$. Let $\alpha \in U_{2n}(R[X])$, $\beta \in U_{2n}(R[X^{-1}])$. If $\alpha\beta^{-1} \in EU_{2n}(R[X, X^{-1}])$, then $\alpha \in U_{2n}R \cdot EU_{2n}(R[X])$.*

Proof. Because of Theorem 3.1, we can assume that R is a local ring. The rings $k[X]$ and $k[X^{-1}]$ are Euclidean, and every elementary matrix in $EU_{2n}(k[X])$ (resp. $EU_{2n}(k[X^{-1}])$) has an inverse image in $EU_{2n}(R[X])$ (resp. $EU_{2n}(R[X^{-1}])$) by our hypothesis, i.e., for every $\rho_{ij}(f)$ in $EU_{2n}(k[X])$, there is $\rho_{ij}(f_1)$ in $EU_{2n}(R[X])$ such that $\rho_{ij}(f_1^{\phi}) = \rho_{ij}(f)$. Hence by multiplying α and β by matrices in $EU_{2n}(R[X])$ and $EU_{2n}(R[X^{-1}])$ respectively, we can assume that $\alpha^{\phi} \in U_{2n}(k[X]) \cap D(k)$ and $\beta^{\phi} \in U_{2n}(k[X^{-1}]) \cap D(k)$. Since $\alpha\beta^{-1} \in EU_{2n}(R[X, X^{-1}])$, it follows that $\alpha^{\phi}(\beta^{\phi})^{-1} \in EU_{2n}(k[X, X^{-1}])$, and hence $\alpha^{\phi}(\beta^{\phi})^{-1} \in EU_{2n}k$ (see Lemma 2.6). Then we have $\alpha\beta^{-1} \in EU_{2n}(R[X, X^{-1}] \cap U_{2n}(M[X, X^{-1}])$, $\alpha\beta^{-1}$ can be written in the form $\alpha_1\alpha_2$ where $\alpha_i \in G_i$. We have $\alpha_1^{-1}\alpha = \alpha_2\beta \in U_{2n}(R[X]) \cap U_{2n}(R[X^{-1}]) = U_{2n}R$, and the proof is completed.

Theorem 4.1. *Suppose that $n \geq 3$, $\alpha \in U_{2n}(R[X])$, and f is a monic polynomial. If $\alpha \in EU_{2n}(R[X]_f)$, then $\alpha \in EU_{2n}(R[X])$.*

Proof. Let $f = X^p + a_1X^{p-1} + \cdots + a_p$, and put $Y = X^{-1}$ and $g = 1 + a_1Y + \cdots + a_pY^p$. Then $\alpha \in EU_{2n}(R[X]_f) \subset EU_{2n}(R[X, X^{-1}]_f) = EU_{2n}(R[Y, Y^{-1}]_g) = EU_{2n}(R[Y]_{Y_g})$. Note that $Y \in C$. By Lemma 3.5, there exists $\alpha_1 \in EU_{2n}(R[Y]_Y) = EU_{2n}(R[Y, Y^{-1}])$ and $\beta \in EU_{2n}(R[Y]_g)$ such that $\alpha = \alpha_1\beta$. The matrix $\beta = \alpha_1^{-1}\alpha$ lies in $EU_{2n}(R[Y]_g) \cap EU_{2n}(R[Y]_Y) \subset EU_{2n}(R[Y])$. By Proposition 4.3, $\beta \in U_{2n}R \cdot EU_{2n}(R[Y]) \cap EU_{2n}(R[Y]_g)$. Using Lemma 2.6 (retraction $R[Y]_g \to R$ by $Y \to 0$), we obtain $\beta \in EU_{2n}(R[Y])$. Then $\alpha = \alpha_1\beta \in EU_{2n}(R[X, X^{-1}])$. By Proposition 4.3, $\alpha \in U_{2n}R \cdot EU_{2n}(R[X])$. Again by Lemma 2.6, $\alpha \in EU_{2n}(R[X])$.

§5. Unitary Symbols

Lemma 5.1.[3, Lemma 2.1] *Let R be a local principal ideal ring. Let $M = R\pi$ be the maximal ideal of R with some choice of a generator π. Then*

(1) *$\cap M^i = 0$;*

(2) *for any element $r \in R\backslash\{0\}$ there is a unique integer $j \geq 0$ and a unit u of R such that $r = \pi^j u$;*

(3) *if R is equipped with an involution $*$, $\pi^* = \pi u_1$ for some $u_1 \in R^*$, and $u_1u_1^* = 1$.*

We denote by "$\leftrightarrow$"$_R$ the equivalent relation on Um_2R generated by the following operators:

(S1) $(a, b) \leftrightarrow_R (a, b + ac)$, $c \in C$,

(S2) $(a, b) \leftrightarrow_R (a + bc, b)$, $c \in C$,

(S3) $(1 + ac, b) \leftrightarrow_R (1 + ac, c^*bc)$,

for all $(a, b) \in Um_2R$. If u is a unit in R, we have

(S4) $(a, b) \leftrightarrow_R (ua, u^{-1}b)$.

Note that if $u \in C$, (S4) is a consequence of (S1) and (S2) (see [3, p.38]). We also give another equivalent relation which occurs in linear case, but we will use it for studying Um_4R.

(S5) $(1+ac, b) \leftrightarrow_R (1+ac, bc)$.

The above equivalent relation occurs if one considers the rows of matrices from $U_2R \subset U_4R$ (or $U_4R \subset U_6R$) up to multiplication by elementary unitary matrices and

$$\begin{pmatrix} u & \\ & u^{*^{-1}} \end{pmatrix} + I_2 \text{ in } (\text{diag}(u, u^{*^{-1}}) \cdot EU_4R) \cap U_2R.$$

We consider now the above equivalent relation for polynomial rings over local principal ideal ring.

Lemma 5.2. *Let R be a local principal ideal ring. Then for any $(a,b) \in Um_2(R[X])$ there is a pair $(a',b') \in Um_2(R[X])$ with $(a,b) \leftrightarrow_{R[X]} (a',b')$ and so that a' is a monic polynomial.*

Proof. Let $M = R\pi$ be the maximal ideal of R and $k = R/M$ the residue field. We write $a = a_0 + a_1X + \cdots + a_nX^n$ and $b = b_0 + b_1X + \cdots + b_lX^l$ with $a_i, b_i \in R$. Since (a,b) is unimodular, the pair (a_0, b_0) is unimodular over R. So either a_0 or b_0 has to be a unit in R. We may use (S2) and (S4) to ensure that $a_0 = 1$.

If $l \geq n$, since $a_0b_0^* \in C$, using (S1), we can replace b by $b'_1X + \cdots + b'_lX^l$. Then $a_0b'^*_l \in C$. Again we can replace b by $b'_2X^2 + \cdots + b'_lX^l$. By (S3), b can be replaced by $b'_2 + \cdots + b'_lX^{l-2}$. So we can assume that $l < n$ and $b_0 = 0$.

We shall proceed by induction on the degree n of a. If $n = 0$, then there is nothing to be done (use (S4)).

If $n = 1$, we have $a = 1 + a_1X$ and $b = b_1X$. If $b_1 = 0$, then clearly (a,b) is equivalent to $(1,0)$ and we are done. Otherwise, by Lemma 5.1 we may write $a_1 = \pi^f u_1$ and $b_1 = \pi^g u_2$ with integers $f, g \geq 0$ and u_1, u_2 units in R. If either f or g is zero, we are finished by using (S1)-(S3). Otherwise, using (S3) with $c = \pi$ we may assume that either $g = 1$ or $g = 0$, and $b_1 = \pi^k u_2$ ($k = 0$ or 1). If $k = 0$, we are done by (S2) with $c = u_2^{-1}u_1\pi^f \in C$. If $k = 1$, $a_1b_1^* = \pi^f u_1 \pi^* u_2^* \in C$, either $\pi^{f-1}u_1u_2^* \in C$ when π is not a zerodivisor, or there is a $j > 0$ such that $\pi^j = 0$. In the first case, $u_2^{-1}u_1\pi^{f-1} \in C$, we can use (S2) to make $a = 1$. In the second case, by (S3), we make $b = 0$ and hence $(a,b) \leftrightarrow_{R[X]} (1,0)$.

Now let $n > 1$. In addition, we may assume that $a_n \in M$. By repeatedly using (S1) and (S3) (cancelling the terms b_1X and b_2X^2 repeatedly) we may arrange that all the b_i are multiples of a_n. Then all $a_i (i \geq 1) \in M$ (since the reduction of the row (a,b) modulo M is still unimodular). If all the b_i now happen to be divisible by an even power of π, we may use (S3) to divide them through by it. After this, there are two possibilities for b:

(1) there is a unit amongst the b_i,

(2) one of the b_i is divisible exactly by π, but there is no unit amongst the b_i.

In case (1) we choose k such that k is the largest number so that b_k is a unit. Using (S1) and (S3) with $c = X$, we replace b by $b' = b'_0 + \cdots + b'_nX^n$ with $b'_0, \cdots, b'_n \in M$ and either $b'_n = 0$ and $b'_{n-1} = a_nb_k$ or $b'_n = a_nb_k$. In any case we may reduce the degree of a and keep the property $a_0 = 1$. Thus, our proof in case (1) is completed by induction.

In case (2) we choose the largest k such that b_k is divisible exactly by π. Using (S1) and (S3) with $c = X$, we replace b by $b' = b'_0 + \cdots + b'_nX^n$ with $b'_0, \cdots, b'_n \in M^2$ and either

$b'_n = 0$ and $b'_{n-1} = a_n b_k$ or $b'_n = a_n b_k$. We may assume $a_n b_k \neq 0$, otherwise by (S3) with $c = \pi$ we may replace b by 0. So we divide the coefficients of b' by π^2. After this we see that either $b'_n = 0$ and a_n is a multiple of $\pi b'_{n-1}$ in R or a_n is a multiple of $\pi b'_n$ in R. In any case we may reduce the degree of a and keep $a_0 = 1$ (referring the case of $n = 1$). This finishes the induction step.

Lemma 5.3. *Let R be a local principal ideal ring. Then $(a, b) \leftrightarrow_{R[X]} (1, 0)$ for any $(a, b) \in Um_2(R[X])$ with a monic polynomial a.*

Proof. We proceed by induction on the degree n of the monic polynomial a.

If $n = 0$, then $a = 1$, and we are finished by using (S1). If $n = 1$, by repeatedly using (S1) we may assume that $b \in R$. Reduction modulo M shows that b has to be a unit of R. So we use (S2) and (S4) to complete our proof.

Now suppose $n > 1$, $a = a_0 + \cdots a_{n-1}X^{n-1} + a_n X^n$ and $b = b_0 + \cdots b_{l-1}X^{l-1} + b_l X^l$ with $a_i, b_i \in R$ and $a_n = 1$. In the same way as the proof of Lemma 5.2, we may assume that $l < n$ for the degree l of b. Putting $X = 0$, we see that either a_0 or b_0 has to be a unit in R. By (S2) we may arrange that a_0 is a unit. Using (S4), we make $a_0 = 1$. Now a_n is a unit of R.

Let k be the largest number so that b_k is a unit in R. If $k = 1$, using (S1) we may replace a by $a' = a_0 + \cdots + a'_{n-1}X^{n-1}$ where $a_0 = 1$. Then by Lemma 5.2 we may replace a'_{n-1} by a unit in R and keep the degree $n - 1$ of a. So suppose that $k < l$, then $b_i \in M$ for $i > k$. By $(a, b) \leftrightarrow_{R[X]} (a, bX^2)$ and (S1) we can arrange that either $k = n - 1$ or $k = n - 2$, and $b = b_0 + \cdots + b_{n-2}X^{n-2} + b_{n-1}X^{n-1}$. The case $k = n - 1$ has been dealt with above, so let $k = n-2$. Then $b_{n-1} \in M$. By $(a, b) \leftrightarrow_{R[X]} (a, bX^2)$ and (S1), we replace b by $b_0 + \cdots b'_n X^n$ where $b'_n = b_{n-2} - a_n^{-1} b_{n-1} a_{n-1}$ is a unit. So using (S2) we may reduce the degree of a. Then applying Lemma 5.2, we may make a_{n-1} invertible and $a_0 = 1$, and hence finish the induction step.

Proposition 5.1. *Let R be a local principal ideal ring. Set $A = R[X]$. Then $(a, b) \leftrightarrow_A (1, 0)$ for every $(a, b) \in Um_2 A$.*

Now consider $(a_1, a_2, a_3, a_4) \in Um_4(R[X])$. The equivalent relations (S1)-(S3) only can operate on the pair (a_1, a_3) and (a_2, a_4); (S5) can operate on the pair (a_1, a_2) and (a_3, a_4). We put other four equivalent relations additionally

(S6) $(a_1, a_2, a_3, a_4) \leftrightarrow_A (a_1, a_2 + a_1 c, a_3 - a_4 c^*, a_4)$,

(S7) $(a_1, a_2, a_3, a_4) \leftrightarrow_A (a_1 + a_2 c, a_2, a_3, a_4 - a_3 c^*)$,

(S8) $(a_1, a_2, a_3, a_4) \leftrightarrow_A (a_1, a_2, a_3 + a_2 c, a_4 + a_1 c^*)$,

(S9) $(a_1, a_2, a_3, a_4) \leftrightarrow_A (a_1 + a_4 c, a_2 + a_3 c^*, a_3, a_4)$,

for all $c \in A$.

Proposition 5.2. *Let R be a local principal ideal ring and the involution $*$ defined on R satisfies the condition (Δ). Set $A = R[X]$. Then $(a_1, a_2, a_3, a_4) \leftrightarrow_A (1, 0, 0, 0)$ for every $(a_1, a_2, a_3, a_4) \in Um_4 A$.*

Proof. We only want to sketch the line of the proof. Since $k[X] = R/M[X]$ is Euclidean, we can find a $g \in EU_4 A$ such that $(a_1, a_2, a_3, a_4)g = (1 + f_1, f_2, f_3, f_4)$ where $f_i \in M[X]$, and further the constants of f_i may be zero. This procedure is equivalent to the operation on $v = (a_1, a_2, a_3, a_4)$ by the equivalence relations (S1)-(S9). Then by (S6), (S8), we may assume that the lowest degrees of X in f_2 and f_4 are higher than $1+f_1$. Note that the coefficients of X^0, X^1, X^2 in $(1+f_1)^*f_3$ lie in C, so we can use (S1)-(S4) to operate the pair $(1+f_1, f_3)$. If $(1+f_1, f_3)$ is unimodular, by Proposition 5.1 we are done. Otherwise we may get $f_3=0$ after operation on the pair $(1+f_1, f_3)$ by (S1)-(S4) (check the proof of Lemmas 5.2 and 5.3). Now $v=(1+f_1, f_2, 0, f_4)$. Using (S6) with $c=X^{2l}$, $l>0$, we may replace v by $(1+f_1, f_2', X^{2l}f_4, f_4)$. Using (S3) and (S7), we replace v by $(1+f_1', f_2', f_4, 0)$. If the pair $(1+f_1', f_4)$ is unimodular, we can finish the proof. Otherwise we may get $f_4=0$. Thus $v=(1+f_1', f_2', 0, 0)$ and $(1+f_1', f_2')$ is unimodular. Using (S5), (S6), (S7) and (S4), we complete the proof.

In case R is a local principal ideal ring and if the number of variables is 1, the absolute stable rank (see [5]) of $A=R[X]$ is at most 2. This follows from the fact that the space of maximal ideals of the ring A is the union of two Noetherian subsets of Krull dimension 1. Note that $\begin{pmatrix} T & \\ & T^{*^{-1}} \end{pmatrix} \in EU_6A$ where $T=\begin{pmatrix} 1+at & bt \\ c & d \end{pmatrix} \oplus 1 \in SL_3A$ and $T \oplus \widetilde{T}^{-1} \in EU_4A$ where $T=\begin{pmatrix} 1+at & tbt^* \\ c & 1+dt^* \end{pmatrix}$, $\widetilde{T}=\begin{pmatrix} 1+at & b \\ t^*ct & 1+t^*d \end{pmatrix}$ lie in U_2A (these two matrices correspond to the operations (S5) and (S3) respectively). Referring the proof of Proposition 5.2 and the proof of Proposition 1 in [1], we know that $EU_{2n}A$ acts transitively on $Um_{2n}A$ when $n \geq 3$.

For $(a_1, a_2, a_3, a_4) \in Um_4A$ and $(a, b) \in Um_2A$, we define $\eta(a_1, a_2, a_3, a_4) = \psi(g)EU_{2n}A \in U_{2n}A/EU_{2n}A$ and $\eta(a, b) = \psi(g)EU_{2n}A \in U_{2n}A/EU_{2n}A$, where $n \geq 3$, g is any matrix from U_4A (resp. U_2A) having (a_1, a_2, a_3, a_4) (resp. (a, b)) as the first row and $\psi: U_{2m}A \to U_{2n}A$ $(n > m)$. It is clear that the construction gives a well defined map $\eta: Um_4A \rightharpoonup U_{2n}A/EU_{2n}A$ (resp. $Um_2A \rightharpoonup U_{2n}A/EU_{2n}A$). Applying a result from [3, p.45, p.47] and [8], we have

Lemma 5.4. *Let A be ring and (a_1, a_2, a_3, a_4), $(a_1', a_2', a_3', a_4') \in Um_4A$ with $(a_1, a_2, a_3, a_4) \leftrightarrow_A (a_1', a_2', a_3', a_4')$ (resp. for (a, b) and (a', b')). Then $\eta(a_1, a_2, a_3, a_4) = \eta(a_1', a_2', a_3', a_4')$ and $\eta(a, b) = \eta(a', b')$.*

By Lemma 5.4 and Propositions 5.1 and 5.2, we have

Proposition 5.3. *Let R be a local principal ideal ring and the involution $*$ defined on R satisfies the condition (Δ) and let $n \geq 3$. Then $U_{2n}(R[X]) = \mathrm{diag}(u, u^{*^{-1}}) \cdot EU_{2n}(R[X])$ where u is a unit in R.*

§6. Proofs of Main Results

In this section we shall prove the main results mentioned in the introduction. The following Lemmas are important in our approach.

Lemma 6.1. *Let R be any principal ideal ring, and $S \subset A = R[X]$ be the multiplicative set consisting of all monic polynomials. Then $S^{-1}A$ is also a principal ideal ring* (see [3, Proposition 5.1]).

In the following assertions, let $\sigma_0 \in U_{2n}R$ denote the matrix with the first column $(1, 0, \cdots, 0, 0, \cdots, 0)^t$.

Lemma 6.2. *Let R be a local ring with maximal ideal M and residue field k. Suppose $\sigma_\pm \in U_{2n}(R[X^{\pm 1}])$ and $\sigma_\pm \equiv \sigma_0 \mod (M[X^{\pm 1}])$. If there exists $\alpha \in EU_{2n}(R[X, X^{-1}])$*

such that $\alpha\sigma_+ = \sigma_-$, *then* $\alpha \in EU_{2n}(R[X, X^{-1}]) \cap U_{2n}(M[X, X^{-1}])$, *i.e., there is a* $\gamma \in EU_{2n}(R[X, X^{-1}]) \cap U_{2n}(M[X, X^{-1}])$ *such that* $\gamma\sigma_+ = \sigma_-$.

Proof. Let ϕ denote the canonical projection: $R \to k$. Then $\alpha^\phi \sigma_+^\phi = \sigma_-^\phi$ and $\sigma_\pm^\phi = \sigma_0^\phi$. So $\alpha^\phi = \sigma_-^\phi \cdot (\sigma_+^\phi)^{-1} = I$, $\alpha \in EU_{2n}(R[X, X^{-1}] \cap U_{2n}(M[X, X^{-1}])$.

Lemma 6.3. *Suppose that* R *is an arbitary commutative ring,* $M \in \max(R)$ *and* $n \geq 3$. *Assume that* $\sigma_+ \in U_{2n}(R[X])$ *and there exists* $\alpha \in EU_{2n}(R[X, X^{-1}])$ *such that* $\alpha\sigma_+ \in U_{2n}(R[X^{-1}])$. *Then there exists* $\beta \in EU_{2n}(R[X])$ *such that* $\beta\sigma_+ \in U_{2n}R$.

Proof. By Theorem 3.1 and Corollary 3.2, it suffices to consider the case when R is a local ring. Since the rings $k[X]$ and $k[X^{-1}]$ are Euclidean, it follows that there exists $\beta_\pm \in EU_{2n}(R[X^{\pm 1}])$ such that $\beta_+\sigma_+ \equiv \sigma_0 \mod (M[X])$, $\beta_-\alpha\sigma_+ \equiv \sigma_0 \mod (M[X^{-1}])$. By Lemma 6.2 there exists $\gamma \in EU_{2n}(R[X, X^{-1}]) \cap U_{2n}(M[X, X^{-1}])$ such that $\gamma(\beta_+\sigma_+) = \beta_-\alpha\sigma_+$. By Proposition 4.2 γ^{-1} can be written in the form $\gamma_+\gamma_-$, where $\gamma_\pm \in EU_{2n}(R[X^{\pm 1}])$. We now obtain $\gamma_+^{-1}\beta_+\sigma_+ = \gamma_-\beta_-\alpha\sigma_+$.Hence $\gamma_+^{-1}\beta_+\sigma_+ \in U_{2n}(R[X]) \cap U_{2n}(R[X^{-1}]) = U_{2n}R$.

Proposition 6.1. *Suppose that* $\sigma_+ \in U_{2n}(R[X])$, $\sigma_0 \in U_{2n}R$, f *is a monic polynomial in* $R[X]$ *and* $\alpha \in EU_{2n}(R[X]_f)$ *such that* $\alpha\sigma_+ = \sigma_0$. *Then there exists* $\tau \in EU_{2n}(R[X])$ *satisfying* $\tau\sigma_+ = \sigma_0$.

Proof. As in the proof of Theorem 4.1, put $Y = X^{-1}$, $g(Y) = Y^{\deg g} f(Y^{-1})$. By Proposition 4.2, α^{-1} can be written in the form $\alpha_1\beta$, where $\alpha_1 \in EU_{2n}(R[Y]_Y)$, $\beta \in EU_{2n}(R[Y]_g)$. Then $\alpha_1^{-1}\sigma_+ = \beta\sigma_0 \in U_{2n}(R[Y]_Y) \cap U_{2n}(R[Y]_g) = U_{2n}(R[Y])$. Applying Lemma 6.3 to $\beta\sigma_0$, we can find $\gamma \in EU_{2n}(R[Y])$ such that $\gamma\beta\sigma_0 \in U_{2n}R$. Let ϕ denote the canonical retraction $R[Y]_g \to R$ $(Y \to 0)$ and put $\alpha_2 = ((\gamma\beta)^{-1})^\phi \gamma\alpha_1^{-1} \in EU_{2n}(R[X, X^{-1}])$. It is easy to see that $\alpha_2\sigma_+ = \sigma_0$. If we apply Lemma 6.3 to σ_+, we can find $\beta_1 \in EU_{2n}(R[X])$ such that $\beta_1\sigma_+ \in U_{2n}R$. We now put $\tau = (\alpha_2\beta_1^{-1})^\psi\beta_1$, where $\psi : R[X, X^{-1}] \to R$ is retraction sending X into 1.

Theorem 6.1. *Suppose that* R *is a Noetherian ring,* $A = R[X_1, X_2, \cdots, X_m]$, *and* $n \geq \max(3, \dim R + 2)$. *Then the group* $EU_{2n}A$ *acts transitively on the set of* $Um_{2n}A$.

Proof. Let $v \in Um_{2n}A$ and consider v as the first column of $\sigma \in U_{2n}A$. Let S denote the multiplicative system in A consisting of the polynomials which, for all sufficiently larger ζ, are monic in Y_1 after change of variables $X_1 = Y_1$, $X_2 = Y_2 + Y_1^\zeta$, $\cdots$, $X_m = Y_m + Y_1^{\zeta^{m-1}}$. By Lemma 6.2 in [6], $\dim S^{-1}A \leq \dim R$. Hence by Lemma 2.5, for σ there exists $\alpha \in EU_{2n}(S^{-1}A)$ such that $\alpha\sigma = \sigma_0$, where $\sigma_0 \in U_{2n}(S^{-1}A)$, of which the first column is $(1, 0, \cdots, 0, 0, \cdots, 0)^t$. We can find $f \in S$ such that $\alpha \in EU_{2n}A_f$. Making a change of variables, we assume that f is monic in X_1. Now, by Proposition 6.1 there exists $\gamma \in EU_{2n}A$ such that $\gamma\sigma = \sigma_0$.

Proof of Theorem 1.1. By Theorem 6.1, the canonical homomorphism $U_{2n-2}A \to K_1U(A)$ is surjective. Let us show that the canonical homomorphism $U_{2n}A/EU_{2n}A \to K_1U(A)$ is a monomorphism. Still let S denote the multiplicative system in A consisting of the polynomials which, for sufficiently larger ζ, are monic in Y_1 under the change of variables $X_1 = Y_1$, $X_2 = Y_2 + Y_1^\zeta$, $\cdots$, $X_m = Y_m + Y_1^{\zeta^{m-1}}$; then $\dim S^{-1}A \leq \dim R$. If $\alpha \in U_{2n}A \cap EU(A)$, by Lemma 2.5 $\alpha \in EU_{2n}(S^{-1}A)$. So there is $f \in S$ such that $\alpha \in EU_{2n}A_f$. Hence by Theorem 4.1, $\alpha \in EU_{2n}A$.

Proof of Theorem 1.2. We have

$$U_{2n}(R[X_1,X_2,\cdots,X_m])=U_{2n}R\cdot U_{2n}(R[X_1,X_2,\cdots,X_m],(X_1,X_2,\cdots,X_m))$$

where $(X_1,X_2,\cdots,X_m)$ denotes the ideal generated by $X_1,X_2,\cdots,X_m$.

To prove our result, it suffices to show that $U_{2n}(R[X_1,\cdots,X_m],(X_1,X_2,\cdots,X_m))\subset \begin{pmatrix} u & \\ & {u^*}^{-1}\end{pmatrix} EU_{2n}(R[X_1,X_2,\cdots,X_m])$ where u is a unit in R.

By Corollary 3.4 we are finished if we prove this in case R is a local principal ideal ring. So we now assume that R is a local principal ideal ring. We shall prove by induction on the number $m\geq 0$ of variables that for $n\geq 3$, $U_{2n}(R[X_1,X_2,\cdots,X_m])=\begin{pmatrix} u & \\ & {u^*}^{-1}\end{pmatrix}\cdot EU_{2n}(R[X_1,X_2,\cdots,X_m])$, where u is a unit in R.

For $m=0,1$ we are done in Proposition 5.3. We proceed with the induction step.

For a ring B and an integer $m\geq 0$ we define the ring $\mathcal{X}^m(B)$ inductively in the following way $\mathcal{X}^0(B)=B,\mathcal{X}^1(B)=B[X_1],\cdots,\ \mathcal{X}^{m+1}(B)=S_m^{-1}\mathcal{X}^m(B)[X_{m+1}]$. Here $S_m\subset\mathcal{X}^m(B)$ is the multiplicative set of polynomials which are monic in X_m. Note that for $m\geq 1$ $\mathcal{X}^m(B)$ is a polynomial ring over a ring of fractions of $\mathcal{X}^{m-1}(B)$. We further have $R[X_1,X_2,\cdots,X_{m+1}]\subset\mathcal{X}^{m+1}(R)$.

Since R is a local principal ideal ring, $S^{-1}R[X]$ is still a principal ideal ring[3]. So for $m\geq 1$ the ring $\mathcal{X}^m(R)$ is a polynomial ring over principal ideal ring.

By Corollary 3.4, Theorem 4.1 and Proposition 5.3, we have $U_{2n}(\mathcal{X}^{m+1}(R))=U_{2n}(S_m^{-1}\mathcal{X}^m(R))EU_{2n}(\mathcal{X}^{m+1}(R))$. This implies that an element $\alpha\in U_{2n}(R[X_1,X_2,\cdots,X_m])$ can be written as $\alpha=\alpha_0\alpha_1$ with $\alpha_0\in U_{2n}(S_m^{-1}\mathcal{X}^m(R))$ and $\alpha_1\in EU_{2n}(\mathcal{X}^{m+1}(R))$. Let $\alpha_2\in U_{2n}(S_m^{-1}\mathcal{X}^m(R))$ be the element obtained by putting the variable X_{m+1} in α_1 equal to 0. After multiplying α_1 with an element from $EU_{2n}(S_m^{-1}\mathcal{X}^m(R))$ we assume that $\alpha_2=1$. After this we must have $\alpha_0\in U_{2n}(R[X_1,X_2,\cdots,X_m])$. By induction, it follows that $U_{2n}(R[X_1,X_2,\cdots,X_{m+1}])\subset U_{2m}R\cdot EU_{2n}(\mathcal{X}^{m+1}(R))$ for all $m\geq 0$.

Then we use Theorem 4.1 m times with the order of variables reversed to draw the conclusion.

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