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SOME GEOMETRIC PROPERTIES OF BROWNIAN MOTION ON SIERPINSKI GASKET**

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Abstract

Let $\{X(t), t \geq 0\}$ be Brownian motion on Sierpinski gasket. The Hausdorff and packing dimensions of the image of a compact set are studied. The uniform Hausdorff and packing dimensions of the inverse image are also discussed.

Keywords Brownian motion on Sierpinski gasket, Hausdorff dimension, Packing dimension, Local time.
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§1. Introduction

In [2], Barlow and Perkins constructed a "Brownian motion" taking values in the Sierpinski gasket, a fractal subset of R^2, and studied its properties. This is a point recurrent symmetric diffusion process characterized by local isotropy and homogeneity properties. Recently, Zhou Xianyin studied the Hausdorff measure of the level set of this process. The object of this paper is to consider some other fractal properties.

The structure of this paper is as follows. In section 2, some definitions and lemmas are recalled. In section 3, the Hausdorff and packing dimensions of the image of a compact set are obtained. In section 4, we discuss the uniform Hausdorff and packing dimensions of the inverse image.

We use c, c' to denote unimportant positive constants which may differ from line to line.

§2. Prelimlinaries

Let R^d denote d-dimensional Euclidean space, N denote the set of positive integers and $Z_+ = N \cup \{0\}$. Let E be a subset of R^d. For any $\alpha > 0$, define

$$s^\alpha - m(E) = \liminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} (\text{diam} E_i)^\alpha : E \subset \bigcup_{i=1}^{\infty} E_i \text{ and } \text{diam} E_i \leq \delta \right\}$$

where diam denotes the diameter of a set. The Hausdorff dimension of E, $\dim E$, is defined as follows:

$$\dim E = \inf\{\alpha > 0 : s^\alpha - m(E) = 0\} = \sup\{\alpha > 0 : s^\alpha - m(E) = +\infty\}.$$

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The upper box-counting dimension of E is defined by

$$\Delta(E) = \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{-\log \delta}$$

where $N_\delta(F)$ denotes the minimum number of closed balls of diameter δ needed to cover E. The upper box-counting dimension will not be of interest to use here since it is not σ-stable. Much more relevant is the packing dimension of E which may be defined by

$$\mathrm{Dim}E = \inf\Big\{\sup_{i\geq 1} \Delta(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\Big\},$$

where the infimum is taken over all countable covers of E. It is well known that $\dim E \leq \mathrm{Dim}E$ for any $E \subset R^d$.

Set $e_0 = (0,0)$, $e_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $e_2 = (1,0)$. For any $n \in N$ and $s = (s_1, s_2, \cdots, s_n) \in \{0,1,2\}^n$, we define $\alpha_s = \sum\limits_{k=1}^{n} 2^{-k} e_{s_k}$ and

$$F_s = \{\alpha_s + r_1 e_1 + r_2 e_2; 0 \leq r_1, r_2, r_1 + r_2 \leq 2^{-n}\}.$$

Finally, we define

$$G_n = \{F_s, s \in \{0,1,2\}^n\}, \quad G = \bigcap_{n\in N} \bigcup_{s\in\{0,1,2\}^n} F_s.$$

G is called the Sierpinski gasket.

For any $s \in \{0,1,2\}^n$, let $H_s = G \cap F_s$, where H_s is called an n-triangle. For any $x \in G$, let $\Delta_n(x)$ denote the unique n-triangle containing x whose projection onto the x-axis is close to zero. Let $D_n(x)$ be the union of n-triangles each of which has at least one common vertex point with $\Delta_n(x)$, $\partial D_n(x)$ be the vertices of n-triangles in $D_n(x)$ belongs to only one n-triangle lying in $D_n(x)$, and define $\mathrm{int}(D_n(x)) = D_n(x) \backslash \partial D_n(x)$. The Hausdorff dimension d_f of G is equal to $\frac{\log 3}{\log 2}$. Let $\nu = \frac{\log 2}{\log 5}$, $d_\omega = \frac{1}{\nu}$, $d_s = \frac{2\log 3}{\log 5}$, u be the unique measure on $(R^2, \mathcal{B}(R^2))$ supported on G such that $u(A) = 3^{-n}$ for all $A \in G_n$, $n \in N$ (see [2]).

In [2], Barlow and Perkins give a very good estimate for the transition density of $\{X(t), t \geq 0\}$.

Lemma 2.1.[2] *There is a function $p(t,x,y)$, $(t,x,y) \in (0,\infty) \times G \times G$ such that*

(a) *$p(t,x,y)$ is the transition density of X with respect to u, i.e.,*

$$P_i f(x) = \int_G f(y) p(t,x,y) u(dy) \text{ for all } x \in G, t > 0, f \in C_b(G).$$

(b) *$p(t,x,y) = p(t,y,x)$ for all $x \in G \times G$, $t > 0$.*

(c) *There are positive constants c_1, c_2, c_3, c_4 such that*

$$c_1 t^{-\frac{d_s}{2}} \exp\{-c_2(|x-y|t^{-v})^{\frac{1}{1-v}}\} \leq p(t,x,y)$$
$$\leq c_3 t^{-\frac{d_s}{2}} \exp\{-c_2(|x-y|t^{-v})^{\frac{1}{1-v}}\} \quad \text{for all } t > 0, (x,y) \in G \times G.$$

The next lemma is a version of Gasia lemma.

Lemma 2.2.[2] *Let F be a closed subset of R^d, and let u be a measure on F such that there exist constants $C_1(F)$, $C_2(F)$, d_F so that if*

$$B^F(x,r) = F \cap \{y \in R^d : |x-y| \leq r\},$$

then

$$C_1(F) r^{d_F} \leq u(B^F(x,r)) \leq C_2(F) r^{d_F} \text{ for all } x \in F, r > 0. \tag{2.1}$$

Let P be an increasing function on $[0,\infty)$ with $P(0)=0$, and $\psi: R\to R^+$ be a nonnegative symmetric convex function with $\lim\limits_{x\to+\infty}\psi(x)=+\infty$. Let H be a compact set in F, and let $f: H\to R$ be a measurable function. Suppose that

$$\Gamma=\int_H\int_H\psi\left(\frac{|f(x)-f(y)|}{P(|x-y|)}\right)u(dx)u(dy)<\infty.$$

Then there exists a constant C_F (depending only on $C_1(F)$ and d_F) such that

$$|f(x)-f(y)|\le 8\int_0^{|x-y|}\psi^{-1}\left[\frac{C_F\Gamma}{z^{2d_F}}\right]P(dz) \tag{2.2}$$

for $u\times u$ almost all $(x,y)\in H\times H$. If f is continuous, then (2.2) holds everywhere.

Remark 2.1. If $F=G$, then (2.1) holds with $d_F=d_f=\frac{\log 3}{\log 2}$.

Lemma 2.3.[12] *For any $x\in G$, it is P^x almost surely that if I is an element of G_m $(m=1,2,\cdots)$, then $X^{-1}(I)=\{t\in[0,\infty): X(t)\in I\}$ meets $c\cdot m^{(1-\nu)d_f+2}2^{(1-\frac{d_s}{2})\frac{m}{\nu}}$ of the intevals $[k2^{-\frac{m}{\nu}},(k+1)2^{-\frac{m}{\nu}}], 0\le k\le 2^{\frac{m}{\nu}},\quad k\in Z_+$.*

Remark 2.2. Just as the proof of Lemma 4.2 in [12], by using Markov property, Lemma 2.3 is correct if k changes from $n2^{\frac{m}{\nu}}$ to $(n+1)2^{\frac{m}{\nu}}$, $n=0,1,2,\cdots$.

§3. The Result for Image of a Compact Set

Theorem 3.1. *For any compact subset E of $[0,\infty)$, we have for any $x\in G$*

$$\dim X(E)\le\min\Big(d_f,\frac{\dim E}{\nu}\Big)\qquad P^x\text{a.s.} \tag{3.1}$$

and

$$\mathrm{Dim}X(E)\le\min\Big(d_f,\frac{\mathrm{Dim}E}{\nu}\Big)\qquad P^x\text{a.s.} \tag{3.2}$$

Proof. For any $x\in G$ and $\varepsilon>0$, we know from [2] that $\{X(t),t\ge 0\}$ is P^x a.s. $\nu-\varepsilon$ order Hölder continuous. Then

$$\dim X(E)\le\min\Big(2,\frac{\dim E}{\nu}\Big)\qquad P^x\text{a.s.},$$
$$\mathrm{Dim}X(E)\le\min\Big(2,\frac{\mathrm{Dim}E}{\nu}\Big)\qquad P^x\text{a.s.}$$

It is obviously that $\mathrm{Dim}X(E)\le d_f$. Therefore we have

$$\dim X(E)\le\min\Big(d_f,\frac{\dim E}{\nu}\Big)\qquad P^x\text{a.s.},$$
$$\mathrm{Dim}X(E)\le\min\Big(d_f,\frac{\mathrm{Dim}E}{\nu}\Big)\qquad P^x\text{a.s.}$$

Now we prove the inverse inequality of (3.1).

Case 1. If $\frac{\dim E}{\nu}\le d_f$, for any $\lambda<\frac{\dim E}{\nu}$, it is necessary to prove that there exists a positive measure σ supported on E such that

$$\int_E\int_E E^x(|X(t)-X(s)|^{-\lambda})\sigma(dt)\sigma(ds)<+\infty.$$

Consider $E^x(|X(t)-X(s)|^{-\lambda})$. If $t\ge s$, by strong Markov property, we have

$$E^x(|X(t)-X(s)|^{-\lambda})=E^xE^{X(s)}(|X(t-s)-X(0)|^{-\lambda}); \tag{3.3}$$

if $t<s$, we have

$$E^x(|X(t)-X(s)|^{-\lambda})=E^xE^{X(t)}(|X(s-t)-X(0)|^{-\lambda}). \tag{3.4}$$

Combine (3.3) with (3.4). Then for any $s, t \in [0, +\infty)$,

$$E^x(|X(t) - X(s)|^{-\lambda}) \le \sup_{x \in G} E^x(|X(t-s) - X(0)|^{-\lambda}).$$

By Lemma 2.1

$$\begin{aligned} &E^x(|X(t-s) - X(0)|^{-\lambda}) = E^x(|X(t) - x|^{-\lambda}) \\ \le &\int_G \frac{1}{|y-x|^\lambda} c_3 |t-s|^{-\frac{d_s}{2}} \exp[-c_4(|x-y||t-s|^{-\nu})^{\frac{1}{1-\nu}}] u(dy) \\ \le &\int_{G'} \frac{1}{|t-s|^{\nu\lambda}|z-x|^\lambda} c_3 |t-s|^{-\frac{d_s}{2}} \exp[-c_4 |z-x|^{\frac{1}{1-\nu}}] u'(dz), \end{aligned} \tag{3.5}$$

where $G' = (G - x)|t-s|^{-\nu} + x$, u' is a measure on G' such that

$$u'[(A - x)\cdot |t-s|^{-\nu} + x] = u(A), \text{ for any } A \in G_n, n \in N.$$

By the definition of u (see [2]), we have

$$\begin{aligned} &E^x(|X(t-s) - X(0)|^{-\lambda}) \\ \le &\int_{\{z\in G', |z-x|\le 1\}} \frac{1}{|t-s|^{\nu\lambda}|z-x|^\lambda} c_3|t-s|^{-d_s/2} u'(dz) \\ &+ \int_{\{z\in G', |z-x|>1\}} \frac{1}{|t-s|^{\nu\lambda}|z-x|^\lambda} c_3|t-s|^{-d_s/2} \exp\{-c_4|z-x|^{\frac{1}{1-\nu}}\} u'(dz) \\ = &\sum_{n=0}^\infty \int_{\{z\in G', \frac{1}{2^{n+1}} < |z-x| \le \frac{1}{2^n}\}} \frac{1}{|t-s|^{\nu\lambda}|z-x|^\lambda} c_3|t-s|^{-d_s/2} u'(dz) \\ &+ \sum_{n=0}^\infty \int_{\{z\in G', 2^n < |z-x| \le 2^{n+1}\}} \frac{1}{|t-s|^{\nu\lambda}|z-x|^\lambda} c_3|t-s|^{-d_s/2} \exp[-c_4|z-x|^{\frac{1}{1-\nu}}] u'(dz) \\ = &c_3 \frac{1}{|t-s|^{\nu\lambda}} \sum_{n=0}^\infty (2^{n+1})^\lambda \cdot u'(\frac{1}{2^{n+1}} < |z-x| \le \frac{1}{2^n}) \cdot |t-s|^{-d_s/2} \\ &+ c_3 \frac{1}{|t-s|^{\nu\lambda}} \sum_{n=0}^\infty 2^{-n\lambda} u'(2^n < |z-x| \le 2^{n+1}) \exp\{-c_4 \cdot 2^{\frac{n}{1-\nu}}\} |t-s|^{-d_s/2} \\ \le &c_3 \frac{1}{|t-s|^{\nu\lambda}} \sum_{n=0}^\infty (2^{n+1})^\lambda (\frac{1}{2^n} \cdot |t-s|^\nu)^{d_f} |t-s|^{-d_s/2} \\ &+ c_3 \frac{1}{|t-s|^{\nu\lambda}} \sum_{n=0}^\infty 2^{-n\lambda} (2^{n+1} \cdot |t-s|^\nu)^{d_f} |t-s|^{-d_f/2} \exp\{-c_4 \cdot 2^{\frac{n}{1-\nu}}\} \\ \le &C(\lambda)|t-s|^{-\nu\lambda}. \end{aligned}$$

Since $\frac{\dim E}{\nu} > \lambda$, by Frostman's lemma, there exists a positive measure σ supported on E such that

$$\int_E \int_E \frac{1}{|t-s|^{\nu\lambda}} \sigma(dt)\sigma(ds) < +\infty.$$

Consequently

$$\int_E \int_E E^x[|X(t) - X(s)|^{-\lambda}] \sigma(dt)\sigma(ds) \le C(\lambda) \int_E \int_E \frac{1}{|t-s|^{\nu\lambda}} \sigma(dt)\sigma(ds) < +\infty.$$

Case 2. If $\frac{\dim E}{\nu} > d_f$, we porve that for any $\lambda < d_f$, $\dim X(E) > \lambda$ P^x a.s. But this can be done just in the same way as case 1, thus we complete the proof of Theorem 3.1.

Lemma 3.1. *If E is a compact subset of G and $u(E) > 0$, then $0 < s^{d_f} - m(E) < +\infty$.*

Proof. By the definition of u, we easily know that there exists a constant $c > 0$ such that $\mu(E) = cs^{d_f} - m(E)$. Since E is a compact set, we have $0 < s^{d_f} - m(E) < +\infty$.

Suppose that σ is a bounded Borel measure on G. For any $y \in G$, we define

$$\sigma'(y) = \lim_{m\to\infty} \frac{\sigma(\mathrm{int}D_m(y))}{\mu(\mathrm{int}D_m(y))}.$$

Just as in [9, Chapter 8, Theorem 8.6], it is easy to prove that $\sigma'(y)$ exists finite u a.s.

Lemma 3.2. *Suppose $\sigma \perp u$. Then $\sigma'(y) = +\infty$ σ a.s.*

Proof. Fix $\varepsilon > 0$. Since $\sigma \perp u$, σ is concentrated on an open set V with $u(V) < \varepsilon$. For $n = 1, 2, 3, \cdots$, define E_n to be the set of all $x \in V$ at which $\sigma'(x) < n$. Let K be a compact subset of E_n. If we can show that $\sigma(K) = 0$, the regularity of σ implies that $\sigma(E_n) = 0$ for every n, and this gives the desired result.

Each $x \in K$ lies in $\mathrm{int}(D_{m_x}(x)) \subset V$ so that $\sigma(\mathrm{int}(D_{m_x}(x))) < n \cdot u(\mathrm{int}(D_{m_x}(x))$. Being compact, K is covered by finitely many of these $\mathrm{int}(D_{m_{x_i}}(x_i))$, $1 \le i \le L$. If some point of G lies in five elements of $\{\mathrm{int}D_{m_{x_i}}(x_i), 1 \le i \le L\}$, one of these lies in the union of the other four and can be removed without changing the union. In this way we can remove the superfluous $\mathrm{int}(D_{m_{x_i}}(x_i))$, $1 \le i \le L$, so that no points lie in more than four of $\{\mathrm{int}(D_{m_{x_i}}(x_i)), 1 \le i \le L\}$. Then

$$\begin{aligned}\sigma(K) &\le \sigma\left(\bigcup_{i=1}^{L}(\mathrm{int}(D_{m_{x_i}}(x_i)))\right) \le \sum_{i=1}^{L} \sigma(\mathrm{int}(D_{m_{x_i}}(x_i))) \\ &< n\sum_{i=1}^{L} u(\mathrm{int}(D_{m_{x_i}}(x_i))) \le 4nu\left(\bigcup_{i=1}^{L}\mathrm{int}(D_{m_{x_i}}(x_i))\right) \\ &\le 4nu(V) < 4n\varepsilon.\end{aligned}$$

Since ε is arbitrary, $\sigma(K) = 0$, and the proof is complete.

Theorem 3.2. *For any $x \in G$ and E, a compact subset of $[0, +\infty)$, if $\frac{\dim E}{\nu} > d_f$, then $u(X(E)) > 0$ $\quad P^x$a.s.*

Proof. Since $\dim E > \nu d_f$, by Frostman's lemma, there exists a probability measure σ supported on E such that

$$\int_E \int_E \frac{1}{|s-t|^{\nu d_f}} \sigma(ds)\sigma(dt) < +\infty. \tag{3.6}$$

For any $A \in \mathcal{B}(G)$, define $u_E(A) = \sigma\{t, t \in E, X(t) \in A\}$. It is clear that $u_E(A)$ is a bounded Borel measure on G, P^x a.s.

Let $M = \{y \in G, u'_E(y) = +\infty\}$. By Lemma 3.2, $u_E << u$ if $u_E(M) = 0$, i.e., $X(t) \notin M$ for σ a.s. t, or, in other words, for σ a.s. $t \in E$

$$\lim_{m\to\infty} \frac{1}{u(\mathrm{int}(D_m(X(t))))} u_E(\mathrm{int}(D_m(X(t)))) < +\infty,$$

i.e.,

$$\lim_{m\to\infty} \frac{1}{(\mathrm{int}(D_m(X(t))))} \int_E I_{\{X(s)\in \mathrm{int}(D_m(X(t)))\}} \sigma(ds) < +\infty.$$

Therefore, if we want to prove $u_E << u$ P^x a.s. by the dominated convergence theorem, it is necessary to prove

$$\limsup_{m\to\infty}\frac{1}{4\cdot 3^{-m}}\int_E\int_E E^x[I_{\{X(s)-X(t)|<3\cdot 2^{-m}\}}]\sigma(ds)\sigma(dt)<+\infty.$$

Now, by Markov property,

$$P^x\{|X(s)-X(t)|\le 3\cdot 2^{-m}\}\le\sup_{y\in G}P^y\{|X|s-t|-y)|\le 3\cdot 2^{-m}\}$$
$$\le c_3|t-s|^{-\frac{d_s}{2}}\{3\cdot 2^{-m}\}^{d_f}.$$

By (3.6),

$$\limsup_{m\to\infty}\frac{1}{4\cdot 3^{-m}}\int_E\int_E E^x[I_{\{X(s)-X(t)|<3\cdot 2^{-m}\}}]\sigma(ds)\sigma(dt)$$
$$\le c_3\cdot 3^{d_f}\int_E\int_E\frac{\sigma(ds)\sigma(dt)}{|t-s|^{\frac{d_s}{2}}}<+\infty.$$

Thus we have $u_E << u$ $\quad P^x$a.s. Since $u_E(X(E))\ge\sigma(E)=1$, we have $u(X(E))>0$ P^xa.s. Together with Lemma 3.1, we have

Corollary 3.1. *For any $x\in G$ and E, a compact subset of $[0,\infty)$, if $\frac{\dim E}{\nu}>d_f$, then*

$$0<s^{d_f}-m(X(E))<+\infty\quad P^x\,a.s.$$

§4. The Uniform Dimension Results of Inverse Image

Lemma 4.1. (1) $\lim\limits_{s\to 0}\sup\limits_{x,y\in G}\int_0^s p(t,x,y)dt=0$

(2) *for any $0<\varepsilon<1-\frac{d_s}{2}$, there exists a $C(\varepsilon)>0$ such that for any $x,y\in G$, $h>0$, $\int_0^h|p(t,x,y)-p(t,x,x)|dt\le C(\varepsilon)|x-y|^{\frac{\varepsilon}{\nu}}h^{1-\frac{ds}{2}-\varepsilon}$.*

Proof. (1) It is easy to prove by using Lemma 2.1.

(2) When $x=y$, we have nothing to prove. When $x\ne y$, by Lemma 2.1,

$$\int_0^h|p(t,x,y)-p(t,x,x)|dt$$
$$\le c\int_0^h t^{-\frac{d_s}{2}}\big(1-\exp(-c'[|x-y|t^{-\nu})^{\frac{1}{1-\nu}}]\big)dt$$
$$=c\int_{+\infty}^{\frac{|x-y|}{h^\nu}}\left[\frac{|x-y|^{\frac{1}{\nu}}}{t^{\frac{1}{\nu}}}\right]^{-\frac{d_s}{2}}\big[1-\exp(-c't^{\frac{1}{1-\nu}})\big]\frac{-|x-y|^{\frac{1}{\nu}}}{t^{\frac{1}{\nu}+1}}dt$$
$$=c|x-y|^{d_\omega-d_f}\int_{\frac{|x-y|}{h^\nu}}^{+\infty}\frac{1}{t^{\frac{1}{\nu}+1-d_f}}[1-\exp(-c't^{\frac{1}{1-\nu}})]dt$$
$$\le c|x-y|^{\frac{\varepsilon}{\nu}}h^{1-\frac{d_s}{2}-\varepsilon}\int_0^{+\infty}\frac{1}{t^{1+\frac{\varepsilon}{\nu}}}\big[1-\exp(-c't^{\frac{1}{1-\nu}})\big]dt.$$

Since $d_\omega-d_f=0.73697\cdots$, $\frac{d_\omega}{d_\omega-1}-1=0.75647\cdots$ and $\varepsilon<1-\frac{d_s}{2}$, we have

$$\lim_{t\to 0}\frac{1-\exp(-c't^{\frac{1}{1-\nu}})}{t^{d_\omega-d_f+1}}=0.$$

Therefore

$$\int_0^{+\infty}\frac{1}{t^{1+\frac{\varepsilon}{\nu}}}\big[1-\exp(-c't^{\frac{1}{1-\nu}})\big]dt=C(\varepsilon)<+\infty.$$

Lemma 4.2. *Let L_t^x be the local time of $\{X(t),t\ge 0\}$. Then*

(1) *for each* $x \in G$, *the set of* ω *such that* $L^x_{[0,\infty]}(\omega) = +\infty$ *has* P^x*-probability* 1*;*

(2) *for each* x *and* y, *the set of* ω *such that* $L^y_{[0,\infty]}(\omega) = +\infty$ *has* P^x*-probability* 1*.*

Proof. (1) By the continuity of L^x_t and the construction of G, it is enough to do the case for $x = 0 \triangleq e_0$.

Let

$$\begin{aligned}
\sigma_{0,1} &= 0,\\
\beta_{1,1} &= \inf\{t > \sigma_{0,1}, X(t) \in \partial D_0(0)\backslash\{0\}\},\\
\sigma_{1,1} &= \inf\{t > \beta_{1,1}, X(t) = 0\},\\
&\cdots\\
\beta_{l+1,1} &= f\inf\{t > \sigma_{l,1}, X(t) \in \partial D_0(0)\backslash\{0\}\},\\
\sigma_{l+1,1} &= \inf\{t > \beta_{l+1,1}, X(t) = 0\}, \quad l = 0, 1, 2, \cdots .
\end{aligned}$$

Then $\sigma_{l+1,1}$ ($l \in Z_+$), $\beta_{l,1}$ ($l \in N$) are stopping times and finite P^0 a.s.

Let $\lambda_k = L^0_{[0,\sigma_{k,1})} - L^0_{[0,\sigma_{k-1,1})}$, $k \in N$, and set $L^0_\emptyset = 0$. Thus

$$L^0_{[0,\sigma_{k,1})} = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

By [2, Theorem 1.11] and strong Markov property, $(\sigma_{l+1,1} - \sigma_{l+1}, \lambda_{l+1})$ are P^0-independent and identically distributed for $l = 0, 1, 2, \cdots$

$$E^0\lambda_1 = E^0(L^0_{[0,\sigma_{1,1})}) \geq E^0(L^0_{[0,\beta_{1,1})}) \geq \frac{3}{4} > 0 \text{ by [2]}.$$

Therefore $P^0\{L^0_{[0,\sigma_{l,1})} \to \infty, l \to \infty\} = 1$, this proves (1).

(2) Let $T_y = \text{int}\{t > 0, X(t) = y\}$. We have $P^x\{T_y < \infty\} = 1$. But it is easy to prove that

$$(L^y_{[0,\infty)} = \infty\} = \theta^{-1}_{T_y}\{L^x_{[0,\infty)} = \infty\}.$$

By strong Markov property,

$$P^x(L^y_{[0,\infty)} = \infty\} = P^y\{L^y_{[0,\infty)} = \infty\}.$$

Theorem 4.1. *For any* $x \in G$, P^x a.s.

$$\dim X^{-1}(F) = 1 - \frac{d_s}{2} + v \dim F \textit{ for every compact subset } F \subset G.$$

Proof. Upper bound. For any $0 < \alpha < \frac{d_s}{2}$, $\eta > \dim F$, choose p such that

$$\eta = \frac{p - 1 + \alpha}{v}.$$

There exists a cover $\{B(x_i, \varepsilon_i), i \geq 1\}$ of F, $B(x_i, \varepsilon_i) \cap F \neq \emptyset$ such that

$$\sum_i \varepsilon_i^{\frac{p+\alpha-1}{v}} < +\infty.$$

For any i, choose $n_i \in N$ such that

$$2^{-(n_i+2)} \leq \varepsilon_i < 2^{-(n_i+1)}.$$

Then $B(x_i, \varepsilon_i)$ intersects at most two elements of G_{n_i}, denoted by $I_{i,1}$, $I_{i,2}$. Therefore

$$B(x_i, \varepsilon_i) \cap F \subset I_{i,1} \cup I_{i,2}.$$

Let

$$\psi_i = \{[k2^{-\frac{n_i}{v}}, (k+1)2^{-\frac{n_i}{v}}], k = 0, 1, \cdots, [2^{\frac{n_i}{v}}],$$

and

$$[k2^{-\frac{n_i}{v}},(k+1)2^{-\frac{n_i}{v}}]\cap X^{-1}(I_{i,1}\cup I_{i,2})\neq\emptyset\}.$$

By Lemma 2.3

$$\#\psi_i\leq c2^{(1-\frac{d_s}{2})\frac{n_i}{v}}n_i^{(1-v)d_f+2}.$$

But

$$X^{-1}(F)\cap[0,1]\subset\Big[\bigcup_i X^{-1}(I_{i,1}\cup I_{i,2})\Big]\cap[0,1]\subset\bigcup_i\bigcup_{I\in\psi_i}I.$$

Straight calculus shows that

$$\sum_i\sum_{I\in\psi_i}(\mathrm{diam}I)^p<+\infty.$$

Consequently, we have P^x a.s.

$$\dim(X^{-1}(F)\cap[0,1])\leq p=v\eta+1-\alpha.$$

Since η and α are arbitrary, we have P^x a.s.

$$\dim(X^{-1}(F)\cap[0,1])\leq 1-\frac{d_s}{2}+v\dim F.$$

By Remark 2.2, we have P^x a.s.

$$\dim(X^{-1}(F)\cap[k,k+1])\leq 1-\frac{d_s}{2}+v\dim F\text{ for any }k\in Z_+.$$

The σ-stability of Hausdorff dimension leads P^x a.s.

$$\dim(X^{-1}(F))\leq 1-\frac{d_s}{2}+v\dim F.$$

Lower bound. Take $z\in G$ and let $k\geq 2$ be a positive integer. For distinct $t_1,t_2,\cdots,t_k$ and for $x_1,x_2,\cdots,x_k\in G$, let $p(x_1,x_2,\cdots,x_k,t_1,t_2,\cdots,t_k)$ be the joint density of $(X(t_1),\cdots,X(t_k))$ relative to $u\times u\times\cdots\times u$ at $(x_1,x_2,\cdots,x_k)$.

If $I\subset[0,\infty)$ is a closed interval, $|I|\leq 1$, define the function

$$\begin{aligned}q(x_1,x_2,\cdots,x_k)&=\int_I\cdots\int_I p(x_1,x_2,\cdots,x_k,t_1,t_2,\cdots,t_k)dt_1\cdots dt_k\\&=k!\int\cdots\int_{\substack{t_1<t_2<\cdots<t_k\\ t_1\cdots t_k\in I}}p(t_1,z,x_1)p(t_2-t_1,x_1,x_2)\\&\qquad\cdots p(t_k-t_{k-1},x_{k-1},x_k)dt_1\cdots dt_k.\end{aligned}$$

Then by [12, Lemma 2.1] and the dominated convergence theorem, we have

$$E^z\prod_{i=1}^k L_I^{x_i}=q_I(x_1,\cdots,x_k).\tag{4.1}$$

For any $x,y\in G$, we define the difference operator $\theta_{j,y-x}$ $(1\leq j\leq k)$ as

$$\theta_{j,y-x}q_I(x,x,\cdots,x)=q_I(x,\cdots,\underset{\substack{|\\ j}}{y},\cdots,x)-q_I(x,\cdots,x,\cdots,x).$$

Then by (4.1)

$$E(L_I^y-L_I^x)^k=E(L_I^{x+(y-x)}-L_I^x)^k=\prod_{j=1}^k\theta_{j,y-x}q_I(x,x,\cdots,x).$$

Let $x_1, x_2, \cdots, x_k$ be equal to x. Thus

$$\begin{aligned} q_I(x, x, \cdots, x) &= q_I(x_1, x_2, \cdots, x_k) \\ &= k! \int \cdots \int_{\substack{t_1 < t_2 < \cdots < t_k \\ t_1 \cdots t_k \in I}} p(t_1, z, x_1) p(t_k - t_{k-1}, x_{k-1}, x_k) \\ &\quad \cdot \prod_{j=1}^{\frac{k}{2}-1} p(t_{2j} - t_{2j-1}, x_{2j-1}, x_{2j}) p(t_{2j+1} - t_{2j}, x_{2j}, x_{2j+1}) dt_1 dt_2 \cdots dt_k. \end{aligned}$$

Since the difference operator is linear, we may express

$$\begin{aligned} &\prod_{j=1}^{k} \theta_{j,y-x} q_I(x, x, \cdots, x) \\ &= k! \prod_{j=0}^{\frac{k}{2}-1} \theta_{2j+1,y-x} \theta_{k,y-x} \int \cdots \int_{\substack{t_1 < t_2 < \cdots < t_k \\ t_1 \cdots t_k \in I}} p(t_1, z, x_1) p(t_k - t_{k-1}, x_{k-1}, x_k) \\ &\quad \cdot \prod_{j=1}^{\frac{k}{2}-1} [p(t_{2j} - t_{2j-1}, x_{2j-1}, y) p(t_{2j+1} - t_{2j}, y, x_{2j+1}) \\ &\quad - p(t_{2j} - t_{2j-1}, x_{2j-1}, x) p(t_{2j+1} - t_{2j}, x, x_{2j+1})] dt_1 dt_2 \cdots dt_k. \end{aligned} \tag{4.2}$$

When we apply the operator θ for the variables of the remaining indices, we obtain for (4.2) a sum of $2^{\frac{2}{k}+1}$ terms of the form

$$\begin{aligned} &\pm k! \int \cdots \int_{\substack{t_1 < t_2 < \cdots < t_k \\ t_1 \cdots t_k \in I}} p(t_1, z, y_1) p(t_k - t_{k-1}, y_{k-1}, y_k) \prod_{j=1}^{\frac{k}{2}-1} [p(t_{2j} - t_{2j-1}, y_{2j-1}, y) p(t_{2j+1} \\ &- t_{2j}, y, y_{2j+1}) - p(t_{2j} - t_{2j-1}, y_{2j-1}, x) p(t_{2j+1} - t_{2j}, x, y_{2j+1})] dt_1 dt_2 \cdots dt_k, \end{aligned} \tag{4.3}$$

where $y_{2j+1},\ j = 0, 1, \cdots, \frac{k}{2} - 1,\ y_k$ assume the values x and y. We estimate the typical term in (4.3).

The absolute value is at most equal to

$$\begin{aligned} &\int_I p(t_1, z, y_1) ds \int_0^h p(s, y_{k-1}, y_k) ds \\ &\cdot \prod_{j=1}^{\frac{k}{2}-1} \int_0^h \int_0^h |p(s, y_{2j-1}, y) p(t, y, y_{2j+1}) - p(s, y_{2j-1}, x) p(t, x, y_{2j+1})| ds dt, \end{aligned} \tag{4.4}$$

where $h =$ the length of I.

We first estimate the double integrals in (4.4). By Lemma 4.1 and Lemma 2.1 we have

$$\begin{aligned}&\int_0^h\int_0^h |p(s,y_{2j-1},y)p(t,y,y_{2j+1})-p(s,y_{2j-1},x)p(t,x,y_{2j+1})|dsdt\\ \le& \int_0^h\int_0^h p(s,y_{2j-1},y)|p(t,y,y_{2j+1})-p(t,x,y_{2j+1})|dsdt\\ &+\int_0^h\int_0^h p(t,x,y_{2j+1})|p(s,y_{2j-1},y)-p(t,y_{2j-1},x)|dsdt\\ \le& C(\varepsilon)h^{2(1-\frac{d_s}{2})-\varepsilon}|x-y|^{\frac{\varepsilon}{v}}.\end{aligned}$$

So

$$\prod_{j=1}^k \theta_{j,y-x}q_I(x,x,\cdots,x)=ch^{(1-\frac{d_s}{2}-\frac{\varepsilon}{2})k-2(1-\frac{d_s}{2})+\varepsilon}|x-y|^{\frac{\varepsilon}{2v}k-\frac{\varepsilon}{v}}.$$

Choose k large enough so that

$$\begin{aligned}&(1-\frac{d_s}{2}-\frac{\varepsilon}{2})k-2\Big(1-\frac{d_s}{2}\Big)+\varepsilon\ge\Big(1-\frac{d_s}{2}-\varepsilon\Big)k,\\ &\frac{\varepsilon}{2v}k-\frac{\varepsilon}{v}\ge\frac{\varepsilon}{3v}k,\quad \frac{\varepsilon}{3v}-\frac{2}{k}d_f\ge 0,\quad \varepsilon k>1.\end{aligned}$$

Thus

$$\prod_{j=1}^k \theta_{j,y-x}q_I(x,x,\cdots,x)\le ch^{(1-\frac{d_s}{2}-\varepsilon)k}|x-y|^{\frac{\varepsilon}{3v}k}.$$

For each J which is an closed interval of $[0,\infty)$, $|J|\le 1$, set

$$K(J)=K(J,G,k,\frac{\varepsilon}{3v})=\int_G\int_G\left[\frac{L_J^x-L_J^y}{|x-y|^{\frac{\varepsilon}{3v}}}\right]^k u(dx)u(dy).$$

Since

$$E^zK(J)=\int_G\int_G\frac{E^z(L_J^x-L_J^y|^k)}{|x-y|^{\frac{\varepsilon}{3v}k}}u(dx)u(dy)\le c|J|^{(1-\frac{d_s}{2}-\varepsilon)k},$$

by Lemma 2.2

$$|L_J^x-L_J^y|\le c\cdot 8[K(J)]^{\frac{1}{k}}|x-y|^{\frac{\varepsilon}{3v}\frac{2}{k}d_f}\qquad P^x \text{ a.s.}\tag{4.5}$$

By [12] Remark 2.1

$$E^z[(L_J^0)^k]\le c|J|^{1-\frac{d_s}{2}k}.\tag{4.6}$$

For any $L\in N$, by (4.5), (4.6) and Borel-Cantelli Lemma, we can prove P^z a.s. there exist r. v. $s_1(\omega)$, $s_2(\omega)$ such that

$$K(J)\le|J|^{(1-\frac{d_s}{2}-2\varepsilon)k},\quad \forall J\in D_n, n\ge s_1(\omega),$$

where D_n denotes the family of dyadic intervals in $[0,L]$ with length 2^{-n} and

$$L_J^0\le|J|^{(1-\frac{d_s}{2}-2\varepsilon)},\forall J\in D_n, n\ge s_2(\omega).$$

By (4.5), we have

$$|L_J^x-L_J^y|\le c|J|^{(1-\frac{d_s}{2}-2\varepsilon)}|x-y|^{\frac{\varepsilon}{3v}-\frac{2}{k}d_f}$$

for all $(x,y)\in G\times G$, $J\in D_n$, $n\ge s_1(\omega)$.

For any $x \in G$, let

$$x_1 = \min\{i \in \{0,1,2\}, x \in F_i\},$$
$$\cdots$$
$$x_{m+1} = \min\{i \in \{0,1,2\}, x \in F_{x_1,x_2,\cdots,x_m i}\},$$
$$\cdots$$
$$a_0 = 0,$$
$$\cdots$$
$$a_m = a_{x_1,x_2,\cdots,x_m},$$
$$\cdots$$

Then $a_m \to x \quad (m \to \infty)$. By the continuity of local time, we have

$$L_J^x = L_J^0 + \sum_{m=1}^{\infty} (L_{Jm}^a - L_{Jm-1}^a).$$

Thus if $J \in D_n$, $n \geq \max\{s_1(\omega), s_2(\omega)\}$, then

$$L_J^x \leq L_J^x + \sum_{m=1}^{\infty} |L_{Jm}^a - L_{Jm-1}^a| = c|J|^{(1-\frac{d_s}{2}-2\varepsilon)}. \tag{4.7}$$

In the following, noting Lemma 4.2 and the scaling property of $\{X(t), t \geq 0\}$, just as in [7], we can easily get the desired result.

Theorem 4.2. *For any* $x \in G$*,* P^x a.s.

$$\mathrm{Dim}[X^{-1}(F)] \leq 1 - \frac{d_s}{2} + v \dim F \textit{ for any compact set } F \subset G.$$

Proof. It is necessary to prove that P^x a.s.

$$\Delta(X^{-1}(F) \cap [0,1]) \leq 1 - \frac{d_s}{2} + v\Delta(F),$$

where Δ is the upper box-counting dimension.

For any $n \in N$, let $M(2^{-n}, F)$ be the smallest number of closed balls of radius 2^{-n} needed to cover F and $\{B(x_i, 2^{-n}), i = 1, 2, \cdots, M(2^{-n}, F)\}$ be such a covering. Then each $B(x_i, 2^{-n})$ intersects at most 7 elements in G_n, denoted by $I_{i,j}$, $j = 1, 2, \cdots, 7$. Let

$$\psi_i = \left\{ \begin{array}{l} [k2^{-\frac{n}{\nu}}, (k+1)2^{-\frac{n}{\nu}}], k = 0, 1, \cdots, [2^{\frac{n}{\nu}}] \\ \text{and } [k2^{-\frac{n}{\nu}}, (k+1)2^{-\frac{n}{\nu}}] \cap X^{-1}\Big(\bigcup_{i=1}^{7} I_{i,j}\Big) \neq \emptyset \end{array} \right\}.$$

By Lemma 2.3

$$\#\psi_i \leq c \cdot 2^{(1-\frac{d_s}{2})\frac{n}{\nu}} n^{(1-\nu)d_f+2}.$$

Since

$$X^{-1}(F) \cap [0,1] \subset \Big[\bigcup_i X^{-1}\Big(\bigcup_{j=1}^{7} I_{i,j}\Big)\Big] \cap [0,1] \subset \bigcup_{i=1}^{M(2^{-n},F)} \bigcup_{I \in \psi_i} I,$$

we have

$$M(2^{-\frac{n}{\nu}}, X^{-1}(F) \cap [0,1]) \leq M(2^{-n}, F) c \cdot 2^{(1-\frac{d_s}{2})\frac{n}{\nu}} n^{(1-\nu)d_f+2}$$

and

$$\Delta(X^{-1}(F)\cap[0,1]) \le \limsup_{n\to\infty} \frac{\log[M(2^{-n},F)c2^{(1-\frac{d_s}{2})\frac{n}{\nu}}n^{(1-\nu)d_f+2}}{\log 2^{\frac{n}{\nu}}}$$
$$= 1-\frac{d_s}{2}+v\Delta(F).$$

Corollary 4.1. *For any* $x \in G$*,* P^x a.s.

$$\dim X^{-1}(y) = 1-\frac{d_s}{2},$$
$$\mathrm{Dim} X^{-1}(y) = 1-\frac{d_s}{2} \text{ for every } y\in G.$$

Proof. By Theorems 4.1 and 4.2, noting that $\mathrm{Dim}(E) \ge \dim E$, $\forall E \subset R_+$, we can easily obtain the result.

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