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DISTORTION THEOREM FOR BIHOLOMORPHIC MAPPINGS IN TRANSITIVE DOMAINS (IV)

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Abstract

The distortion theorem for biholomorphic starlike mappings (with respect to origin) in bounded symmetric domains are given. The distortion theorem for locally biholomorphic convex mappings in bounded symmetric domains are given also.

Keywords Biholomorphic mapping, Transitive domian, Distortion theorem.
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§1. Introduction

Let $M \subset \mathbf{C}^n$ be a bounded symmetric domain. Let G be a Lie group consisting of some holomorphic automorphisms of M and acting transitively on M, K be the isotropy subgroup of G which fixes the point $m \in M$. Then M is a bounded realization of symmetric space G/K. Let ξ be the holomorphic diffeomorphism of G/K onto M. Moreover, let $M_0 \subset \mathbf{C}^n$ be the Harish-Chandra canonical realization of G/K, and ξ_0 be the holomorphic diffeomorphism of G/K onto M_0. We assume $\xi_0\xi^{-1}(m) = 0$. Let $K_M(z, \bar{z})$, $K_{M_0}(z, \bar{z})$ be the Bergman kernel functions of M and M_0 respectively. Let $\Psi_z \in G$ which transits the point $z \in M$ to m, i.e., $\Psi_z(z) = m$, and φ_z be the inverse of Ψ_z, i.e., $\varphi_z \in G$ and $\varphi_z(m) = z$.

Let $\mathcal{G}$ be the Lie algebra of $G, \mathcal{K}$ be the subalgebra of $\mathcal{G}$ corresponding to K. Then we have Cartan decomposition of $\mathcal{G} = \mathcal{K} + \mathcal{P}$. Suppose that $\mathcal{A}$ is a maximal Abelian subspace in $\mathcal{P}$ and A is the analytic subgroup in G corresponding to $\mathcal{A}$ in $\mathcal{G}$. We can choose a basis of $\mathcal{A}$, say $X_1, \cdots X_q$, where $q = \dim\mathcal{A}$ =rank G/K, and for any $X \in \mathcal{A}$ there exists a unique expression $X = x_1X_1 + \cdots + x_qX_q$. For any $z \in M_0$, there exist $k \in K$ and $X \in \mathcal{A}$, such that

$$z = \xi_0 \left(k \exp X \cdot O\right) = (\tanh x_1, \cdots, \tanh x_q, 0, \cdots, 0)\, \tilde{k}, \tag{1.1}$$

where O is the identity coset in G/K, and $k \in K \to \tilde{k}' \in U_n$ is the unitary representation of K and U_n is the unitary group.

First we will prove the following theorem.

Theorem 1.1. *Let $M \subset \mathbf{C}^n$ be a bounded symmetric domain and $S(M)$ denote the family of normalized biholomorphic starlike mappings* (*with respect to origin*) *from M into*

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$\mathbf{C}^n$. *Then there exists a positive constant* $C(S(M))$, *such that for any* $f \in S(M)$ *the following inequality*

$$\sqrt{\frac{K_M\left(z,\bar{z}\right)}{K_M\left(m,\bar{m}\right)}}\lim_{\theta\to b}\inf_{\in\partial M_0}\left(\frac{K_{M_0}\left(\theta,\bar{\theta}\right)}{K_{M_0}\left(\eta,\bar{\eta}\right)}\right)^{C(S(M))}$$

$$\leq |\det J_f(t)| \leq \sqrt{\frac{K_M\left(z,\bar{z}\right)}{K_M\left(m,\bar{m}\right)}}\lim_{\theta\to b\in}\sup_{\partial M_0}\left(\frac{K_{M_0}\left(\theta,\bar{\theta}\right)}{K_{M_0}\left(\eta,\bar{\eta}\right)}\right)^{C(S(M))} \tag{1.2}$$

holds, where $z=\xi(ka\cdot O)$, $\theta=\xi_0(a_1\cdot O)$, $\eta=\xi_0(a_1a\cdot O)$, $k\in K, a, a_1\in A$. *Furthermore,* $C(S(M))$ *is bounded by*

$$\frac{1}{2}\leq C(S(M))\leq\frac{7n-2}{4}. \tag{1.3}$$

A holomorphic mapping on M is locally biholomorphic if $\det J_f(z)\neq 0$ for all $z\in M$.

Let $d(a,z)$ be the Bergman distance between a and z, and let $D(a,\rho)$ denote

$$\{w\in M\mid d(a,w)<\rho\}.$$

Suppose that f is a locally biholomorphic mapping on M, and denote

$$\sup\{r\mid f \text{ is biholomorphic on } D(z,r)\}$$

by $\rho(z,f)$, and $\inf\{\rho(z,f)|\ z\in M\}$ by $\rho(f)$.

We call f a uniformly locally biholomorphic mapping if $\rho(f)>0$. Moreover, if these exist $0<r\leq\rho(f)$ such that f is a convex mapping on $D(z,r)$, then we call f a uniformly locally biholomorphic convex mapping.

Suppose that $\Omega\subset M$ is an open set in $\mathbf{C}^n$, $m\in\Omega$. Let $f: M\to\mathbf{C}^n$ be a locally biholomorphic mapping. If for any $z\in M$, $f\circ\varphi_z$ is a biholomorphic convex mapping on Ω, then we call f an $\Omega-$uniformly locally biholomorphic convex mapping.

If Ω is a geodesic ball centered at m, then the definition of $\Omega-$uniformly locally biholomorphic convex mapping is coincide with the definition of uniformly locally biholomorphic convex mapping.

The family of $\Omega-$uniformly locally biholomorphic convex mappings is an invariant family (cf. [1]).

We will prove the following theorem.

Theorem 1.2. *Let* $M\subset\mathbf{C}^n$ *be a bounded symmetric domain, and* $f: M\to\mathbf{C}^n$ *be a normalized* $\Omega-$*uniformly locally biholomorphic convex mapping. Suppose that* $\tilde{\Omega}=\xi_0\cdot\xi^{-1}(\Omega)$, $\tilde{\Omega}$ *contains the largest ball centered at the origin with a radius* r, *and is in the smallest ball centered at the origin with a radius* R. *Then*

$$\sqrt{\frac{K_M(z,\bar{z})}{K_M(m,\bar{m})}}\prod_{j=1}^{q}\left(\frac{1-|\tan hx_j|}{1+|\tan hx_j|}\right)^{A_j}$$

$$\leq|\det J_f(z)|\leq\sqrt{\frac{K_M(z,\bar{z})}{K_M(m,\bar{m})}}\prod_{j=1}^{q}\left(\frac{1-|tanhx_j|}{1+|tanhx_j|}\right)^{A_j}, \tag{1.4}$$

where $x_j, i=1,\cdots,q$, *are as defined in* (1.1), *and* $A_j, i=1,\cdots,q$, *are constants which* *satisfy the inequalities*

$$1 \leq \frac{1}{4} \sum_{p=q+1}^{2n} |\alpha_p(X_j)| + \frac{1}{2} \leq A_j \leq nr^{-2}R, \tag{1.5}$$

where $\alpha_{q+1}, \cdots, \alpha_{2n}$ *are the positive roots (counting the multiplicity) of the adjoint representations of* $\mathcal{A}$ *at* $\mathcal{G}$.

§2. On the Determinant of Jacobian

In [1], we studied transitive domain $M \subset \mathbf{C}^n$, including both bounded and unbounded. Denote the Jacobian of Ψ_z by J_{Ψ_z}. If M is unbounded, we assume $|\det J_{\Psi_k}(m)| = 1$ for all $k \in K$ and define

$$K_M(z, \bar{z}) = c \det J_{\Psi_z}(z) \overline{\det J_{\Psi_z}(z)},$$

where c is a constant. If M is bounded, $K_M(z, \bar{z})$ is the Bergman kernel function of M. We denote $K(m, \bar{m})^{-1} \left(\frac{\partial}{\partial z} K(z, \bar{z})\right)_{z=m}$ by Cp.

For any point z in M, we can connect z and m by an analytic curve $a(\rho) = (a_1(\rho), \cdots, a_n(\rho))$, $(0 \leq \rho \leq 1)$ such that $a(0) = m$, $a(1) = z$. If M is starlike with respect to m, we can take $a(\rho) = m + \rho(z - m)$.

In [1], we proved

Theorem A. *Assume that* f *is a biholomorphic mapping on transitive domain* M *which maps* M *into* $\mathbf{C}^n$. *Let* $a(\rho) = (a_1(\rho), \cdots, a_n(\rho))$, $F(w) = (f(\varphi_a(w)) - f(a)) \left(J_f(a)^{-1}\right)' \left(J_{\varphi_a}(m)^{-1}\right)'$,

$$d_{ij}(\rho) = \left(d_{ij}^{(1)}(\rho), \cdots, d_{ij}^{(n)}(\rho)\right) = \frac{1}{2} \frac{\partial^2 F(w)}{\partial w_i \partial w_j}\bigg|_{w=m}, \quad 1 \leq i, j \leq n,$$

$(J_{\Psi_a}(a))' = (u_{p\ell}(a))$. *Then*

$$\begin{aligned} \log\det J_f(z) - \log\det J_f(m) = {} & \frac{1}{2}\left(\log K_M(z, \bar{z}) - \log K_M(m, \bar{m})\right) \\ & + \sum_{\ell,k=1}^{n} \int_0^1 \frac{da_k}{d\rho} u_{k\ell}(\rho) \left(2\sum_{j=1}^{n} d_{\ell j}^{(j)}(\rho) - c_\ell\right) d\rho \\ & + i\mathrm{Im}\left\{\sum_{j=1}^{n} \frac{da_j}{d\rho} \frac{\partial}{\partial a_j} k(a, \bar{a}) \cdot K(a, \bar{a})^{-1} d\rho\right\}. \end{aligned} \tag{2.1}$$

Actually, Theorem A holds even if M is a locally biholomorphic mapping or f is a holomorphic mapping at the point where $\det J_f(z) \neq 0$ (assume $\det J_f(m) \neq 0$).

We give another version of Theorem A.

Theorem 2.1. *Assumptions are the same as Theorem* A *and* M *is a bounded symmetric* *domain. Then*

$$\log\det J_f(z)-\log\det J_f(m)=\frac{1}{2}\left(\log K_M(z,\bar z)-\log k_M(m,\bar m)\right)$$
$$+\sum_{p=1}^{q}\sum_{\ell=1}^{n}x_p\int_0^1 k_{p\ell}\left(2\sum_{j=1}^{n}d_{\ell_j}^{(j)}(\rho)-c_\ell\right)d\rho$$
$$+i\mathrm{Im}\left\{\int_0^1\sum_{j=1}^{n}\frac{da_j}{d\rho}\frac{\partial}{\partial a_j}K\left(a,\bar a\right)\cdot K(a,\bar a)d\rho\right\},\tag{2.2}$$

where $(k_{p\ell})$ is a unitary matrix.

Proof. in [3], we already proved that $\frac{\det J_f(z)}{\sqrt{K_M(z,\bar z)/K_M(m,\bar m)}}$ is a biholomorphic invariant. We have

$$\frac{\det J_f(z)}{\sqrt{K_M(z,\bar z)\,/K_M(m,\bar m)}}=\frac{\det J_{\tilde f}\left(\xi_0\circ\xi^{-1}(z)\right)}{\sqrt{K_{M_0}\left(\xi_0\circ\xi^{-1}(z),\overline{\xi_0\circ\xi^{-1}(z)}\right)/K_{M_0}(0,0)}},$$

where $\tilde f$ is the normalization of $f\circ\xi\circ\xi_0^{-1}$. We just need to estimate

$$2\sum_{i,k,\ell=1}^{n}\int_0^1\frac{da_k}{d\rho}u_{k\ell}(\rho)d_{\ell j}^{(i)}(\rho)dp$$

on M_0 only.

Let $M_{0,R}=\left\{(x,y)\in\mathbf{R}^{2n}\,|z=x+iy\in M_0\subset\mathbf{C}^n\right\}$, and the real mapping $\xi_{0,R}$ corresponding to ξ_0 be

$$\xi_{0,R}\Big(k\exp\Big(\sum_{j=1}^{q}x_iX_j\Big)\cdot O\Big)=(\tanh x_1,\cdots\tanh x_q,0,\cdots,0)\,P(k)\in M_{0,R}.$$

Let $g\in G$, Ψg denote the holomorphic automorphism corresponding to g. If $g=k_0\exp Y$, $k_0\in K$, $Y\in\mathcal{P}$, then $g(\rho)=k_0\exp\rho Y=k_0\exp Adk(\rho X)$ where $k\in K$, $X\in\mathcal{A}$. Then

$$\xi_0(g(\rho)\cdot O)=a(\rho)\in M_0,\quad \Psi_{g(\rho)}=\Psi_{a(\rho)},$$
$$J_{\Psi g(\rho)}\left(\xi_0(g(\rho)\cdot O)\right)=J_{\Psi_a}(a)=(u_{k\ell}(a))',$$
$$\Psi_{g(\rho)}(z)=\Psi_{g(\rho)}\left(\xi_0(g(1)\cdot O)\right)=\xi_0\left(g(\rho)^{-1}g(1)\cdot O\right)$$

and $\Psi_{g(\rho)}\left(\xi_0(g(\rho)\cdot O)\right)$, where $z=\xi_0(g(1)\cdot O)$, i.e., $a(1)=z$.

Let $a(\rho)=x(\rho)+iy(\rho)\in M_0$. Then $(x(\rho),y(\rho))\in M_{0,R}$. Obviously, $k_0\exp\rho Y\exp tY\cdot O$ is a curve in G/K which contains the point $k_0\exp\rho Y\cdot O$. $g(\rho)^{-1}$ transfers this curve as $\exp tY\cdot O$ which contains the point 0. The corresponding curves in M_0 are the curves $\xi_0\left(k_0\exp\rho Y\exp tY\cdot 0\right)$ and $\xi_0\left(k_0\exp tY\cdot O\right)$, these two curves contain the point $\xi_0(\exp tY\cdot 0)$ and $\xi_0(e\cdot O)=0$ respectively. Thus $\Psi_{g(\rho)}$ maps the curve $a(\rho+t)$ to the curve $\xi_0(\exp tY\cdot 0)$, $\frac{da(\rho)}{d\rho}$ is the tangent of the curve $a(\rho+t)$ at the point $a(\rho)$ and $\frac{d\xi_0(\exp tY\cdot O)}{dt}\,|_{t=0}$ is the tangent of the curve $\xi_0(\exp tY\cdot O)$ at the point 0. We have

$$\frac{da(\rho)}{dp}\left(J_{\Psi_{g(\rho)}}\left(\xi_0(g(\rho)\cdot O)\right)\right)'=\frac{d\xi_0(\exp tY\cdot O)}{dt}\Big|_{t=0}.$$

It is easily to verify

$$\frac{d\xi_0(\exp tY\cdot O)}{dt}\Big|_{t=0}=(x_1,\cdots,x_q,0,\cdots,0)\,\tilde k,$$

where $\tilde{k}=(k_{j\ell})$ is a unitary matrix. Noticing

$$\left(J_{\Psi_{g(p)}}\left(\xi_0(g(\rho)\cdot O)\right)\right)'=\left(J_{\Psi_{a(p)}}(a(\rho))\right)'=(u_{k\ell}),$$

we get on M_0,

$$\sum_{k=1}^{n}\frac{da_k(\rho)}{d\rho}u_{k\ell}(p)=\sum_{j=1}^{q}x_jk_{j\ell}. \tag{2.3}$$

By (2.3), we obtain (2.2) immediately.

§3. Distortion of Biholomorphic Mapping

Moreover, we can prove the following distortion theorem.

Theorem 3.1. *If the assumptions of Theorem 2.1 are satisfied, then*

$$\sqrt{\frac{K_M(z,\bar{z})}{K_M(m,\bar{m})}}\lim_{\theta\to b\in}\inf_{\partial M_0}\left(\frac{K_{M_0}(\theta,\bar{\theta})}{K_{M_0}(\eta,\bar{\eta})}\right)^{c(f,M_0)}$$

$$\le|\det J_f(z)|\le\sqrt{\frac{K_M(z,\bar{z})}{K_M(m,\bar{m})}}\lim_{\theta\to b\in}\sup_{\partial M_0}\left(\frac{K_{M_0}(\theta,\bar{\theta})}{K_{M_0}(\eta,\bar{\eta})}\right)^{c(f,M_0)} \tag{3.1}$$

holds, where $z=\xi(ka\cdot O)$, $\theta=\xi_0(a_1\cdot O)$, $\eta=\xi_0(a_1a\cdot O)$, $k\in K$, $a,a_1\in A$,

$$c(f,M_0)=\sup\left\{\left|2\int_0^1\sum_{j=1}^{n}d_{\ell j}^{(j)}(\rho)d\rho\right|\Big/\left(\frac{1}{2}\sum_{i=q+1}^{2n}|\alpha_j(X_\ell)|+1\right),\ \ell=1,\cdots,q,\ z\in M_0\right\},$$

$d_{\ell j}(\rho)$ *are given in Theorem A, and* $\alpha_{q+1},\cdots,\alpha_{2n}$ *are the positive roots* (*counting the multiplicity*) *of the adjoint representations of* $\mathcal{A}$ *at* $\mathcal{G}$.

Proof. By Theorem 2.1, formula (2.2) and $\det J_f(m)=1$, we have

$$\log|\det J_f(z)|=\frac{1}{2}\log\frac{K_M(z,\bar{z})}{K_M(m,\bar{m})}+\mathrm{Re}\sum_{l,p=1}^{n}x_p\int_0^1k_{p\ell}\left(2\sum_{j=1}^{n}d_{\ell_j}^{(j)}(\rho)-c_\ell\right)d\rho.$$

Since $\frac{\det J_f(z)}{\sqrt{K_M(z,\bar{z})/K_M(m,\bar{m})}}$ is a biholomorphic invariant, we just need to estimate

$$\sum_{\ell,p=1}^{n}x_p\int_0^1k_{p\ell}\left(2\sum_{j=1}^{k}d_{\ell j}^{(j)}(\rho)-c_\ell\right)d\rho$$

on M_0 only. But at $M_0, c_\ell=0$ we have

$$\left|\frac{\det J_f(z)}{\sqrt{K_M(z,\bar{z})\,/K_M(m,\bar{m})}}\right|=e^{\mathrm{Re}\left\{\sum\limits_{j,\ell,p=1}^{n}2x_p\int_0^1k_{p\ell}d_{\ell j}^{(j)}(\rho)d\rho\right\}}.$$

Since $e^{2x_j}=\frac{1+\tanh x_j}{1-\tanh x_j}$, the previous equality is equivalent to

$$\left|\frac{\det J_f(z)}{\sqrt{K_M(z,\bar{z})\,/K_M(m,\bar{m})}}\right|=\prod_{p=1}^{q}\left(\frac{1+\tanh x_p}{1-\tanh x_p}\right)^{\sum\limits_{j,\ell=1}^{n}\mathrm{Re}\int_0^1k_{p\ell}d_{\ell j}^{(j)}(\rho)d\rho} \tag{3.2}$$

On the other hand, we already proved that (cf. [3])

$$\frac{K_{M_0}(z,\bar{z})}{K_{M_0}(0,0)}=\prod_{j=1}^{q}\frac{1}{1-\lambda_j^2}\prod_{j=q+1}^{2n}\frac{e^{\alpha_j(X)}-e^{-\alpha_j(X)}}{2\alpha_j(\tanh X)}, \tag{3.3}$$

where $\lambda_j = \tanh x_j, j = 1, \cdots, q$,

$$\tanh X = \tanh\left(\sum_{j=1}^{q} x_j X_j\right) = \sum_{j=1}^{q} (\tanh x_j)\, X_j = \sum_{j=1}^{q} \lambda_j X_j.$$

By the Dynkin diagram of restricted root systems of the adjoint representation of $\mathcal{A}$ at real semi-simple Lie algebra of $\mathcal{G}$, there are only the following possibilities for $\alpha_j(X)$ (cf. [4]): for $X = x_1X_1 + \cdots + x_qX_q \in \mathcal{A}$, we have

1) $\alpha_j(X) = x_p + x_\ell,\ p \neq \ell,\ 1 \le p, \ell \le q$;
2) $\alpha_j(X) = x_p - x_\ell,\ p \neq \ell,\ 1 \le p, \ell \le q$;
3) $\alpha_j(X) = x_p$;
4) $\alpha_j(X) = 2x_p$.

For the case (1),

$$\alpha_j(X_i) = \begin{cases} 1, & \text{when } i = p \text{ or } \ell; \\ 0, & \text{otherwise;} \end{cases}$$

then we have

$$\begin{aligned}\frac{e^{\alpha_j(X)} - e^{-\alpha_j(X)}}{\alpha_j(\tanh X)} &= \frac{e^{x_p + x_\ell} - e^{-x_p - x_\ell}}{2\,(\lambda_p + \lambda_\ell)} \\ &= \left(\frac{1}{1-\lambda_p^2}\right)^{\frac{1}{2}} \left(\frac{1}{1-\lambda_\ell^2}\right)^{\frac{1}{2}} = \prod_{p=1}^{q} \left(\frac{1}{1-\lambda_p^2}\right)^{\frac{1}{2}|\alpha_j(X_p)|}.\end{aligned}$$

Similarly for the cases 2), 3), 4), we also have

$$\frac{e^{\alpha_j(X)} - e^{-\alpha_j(X)}}{2\alpha_j(\tanh X)} = \prod_{p=1}^{q} \left(\frac{1}{1-\lambda_p^2}\right)^{\frac{1}{2}|\alpha_j(X_p)|}.$$

From (3.3), we get

$$\begin{aligned}\frac{K_{M_0}(z,\bar{z})}{K_{M_0}(0,0)} &= \prod_{p=1}^{q} \frac{1}{1-\lambda_p^2} \prod_{j=q+1}^{2n} \prod_{p=1}^{q} \left(\frac{1}{1-\lambda_p^2}\right)^{\frac{1}{2}|\alpha_j(X_p)|} \\ &= \prod_{p=1}^{q} \left(\frac{1}{1-\lambda_p^2}\right)^{\frac{1}{2}\sum\limits_{j=q+1}^{2n}|\alpha_j(X_p)|+1}. \end{aligned} \tag{3.4}$$

Let

$$z = \xi_0(ka \cdot O) = \xi_0\left(k \exp \sum_{j=1}^{q} x_j X_j \cdot O\right), \quad a = \exp \sum_{j=1}^{q} x_j X_j \in A;$$

$$\theta = \xi_0(a_1 \cdot O) = \xi_0\left(\exp \sum_{j=1}^{q} y_j X_j \cdot O\right), \qquad a_1 = \exp \sum_{j=1}^{q} y_j X_j \in A;$$

$$\eta = \xi_0(a_1 a \cdot O) = \xi_0\left(\exp\left(\sum_{j=1}^{q} (x_j + y_j)\, X_j\right) \cdot O\right).$$

Then by (3.4) we have

$$\frac{K_{M_0}(\theta,\bar{\theta})}{K_{M_0}(\eta,\bar{\eta})} = \prod_{p=1}^{q} \left(\frac{1-\nu_p^2}{1-\mu_p^2}\right)^{\frac{1}{2}\sum\limits_{j=q+1}^{2n}|\alpha_j(X_p)|+1},$$

where $\nu_p = \tanh(x_p + y_p)$, $\mu_p = \tanh y_p$. Obviously,

$$\sup_{\theta \to b \in \partial M_0} \lim \frac{K_{M_0}(\theta, \bar{\theta})}{K_{M_0}(\eta, \bar{\eta})} = \prod_{p=1}^{q} \left(\frac{1 + |\tanh x_p|}{1 - |\tanh x_p|} \right)^{\frac{1}{2} \prod_{j=q+1}^{2n} |\alpha_j(X_p)|+1} \tag{3.5}$$

and

$$\inf_{\theta \to b \in \partial M_0} \lim \frac{K_{M_0}(\theta, \bar{\theta})}{K_{M_0}(\eta, \bar{\eta})} = \prod_{p=1}^{q} \left(\frac{1 - |\tanh x_p|}{1 + |\tanh x_p|} \right)^{\frac{1}{2} \prod_{j=q+1}^{2n} |\alpha_j(X_p)|+1}. \tag{3.6}$$

From (3.2), (3.5) and (3.6), we have (3.1).

§4. Proof of Theorem 1.1

We now give a proof of Theorem 1.1 based on Theorem 3.1. Gong Sheng[5] proved the following result.

If M_0 is the bounded symmetric domain, it is the Harish-Chandra canonical realization of symmetric space G/K. If $\varphi(z)$ is defined as: $\varphi(z) = r$, if $z = rz_0$, $z_0 \in \partial M_0$, and f is a normalized biholomorphic starlike (with respect to origin) mapping on M_0 which maps M_0 into $\mathbf{C}^n$, then

$$|f(z)| \le \frac{\sqrt{q}\varphi(z)}{(1 - \varphi(z))^2}, \tag{4.1}$$

where q =rank G/K.

Let $\{\phi_\nu(z)\}$ be the othonormal system on M_0. Then the Bergman kernel function $K_{M_0}\left(z, \bar{\xi}\right) = \sum_{\nu=0}^{\infty} \phi_\nu(z)\overline{\phi_\nu(\xi)}$ and

$$F(z) = f(rz) = \int_{M_0} f(r\xi) K_{M_0}\left(z, \bar{\xi}\right) d\xi = \sum_{\nu=0}^{\infty} c_\nu(r)\phi_\nu(z),$$

where $c_\nu(r) = \int_{M_0} f(r\xi)\overline{\phi_\nu(\xi)} d\xi$. By Parseval's equality, $\sum_{\nu=0}^{\infty} |c_\nu(r)|^2 = \int_{M_0} |f(r\xi)|^2 d\xi$. Let $w = rz$, $f(z) = (f_1(z), \cdots, f_n(z))$, $c_\nu(r) = \left(c_\nu^{(1)}(r), \cdots, c_\nu^{(n)}(r)\right)$. Then

$$\frac{\partial F(z)}{\partial z} = r\frac{\partial f}{\partial w}(w) = \sum_{\nu=0}^{\infty} c_\nu(r)\frac{\partial \phi_\nu(z)}{\partial z},$$

and $r\frac{\partial f_\ell}{\partial w_j}(w) = \sum_{\nu=0}^{\infty} c_\nu^{(\ell)}(r)\frac{\partial \phi_\nu(z)}{\partial z_j}$. By the Schwarz inequality,

$$\begin{aligned} r\left|\frac{\partial f_\ell}{\partial w_j}(w)\right| &\le \|f_\ell(r\xi)\|_2 \left(\sum_{\nu=0}^{\infty} \frac{\partial}{\partial z_j}\phi_\nu(z)\frac{\partial}{\partial \bar{z}_j}\overline{\phi_\nu(z)}\right)^{\frac{1}{2}} \\ &= \|f_\ell(r\xi)\|_2 \left(\frac{\partial^2}{\partial z_j \partial \bar{z}_j} K_{M_0}(z, \bar{z})\right)^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

By (3.4), it is easy to verify that

$$\frac{\partial}{\partial \bar{z}_j} K_{M_0}(z, \bar{z}) = O\left(K_{M_0}(z, \bar{z}) \prod_{\ell=1}^{q} \left(\frac{1}{2}\sum_{j=q+1}^{2n} |\alpha_j(x_\ell)| + 1\right) \frac{2\lambda_\ell}{1 - \lambda_\ell^2}\bar{k}_{j\ell}\right),$$

where $z=(\lambda_1,\cdots,\lambda_q,0,\cdots,0)\,\tilde{k}$, $\quad \tilde{k}=(k_{j\ell})$. Similarly, we get

$$\frac{\partial^2 K_{M_0}(z,\bar{z})}{\partial z_j\partial\bar{z}_j}=O\left(K_{M_0}(z,\bar{z})\prod_{p=1}^{q}\frac{1}{\left(1-\lambda_p^2\right)^2}\right). \tag{4.3}$$

On the other hand, by (4.1),

$$\begin{aligned}\|f(r\xi)\|_2^2&=\int_{M_0}|f(r\xi)|^2\,d\xi\le q\int_{M_0}\frac{(\varphi(r\xi))^2}{(1-\varphi(r\xi))^4}d\xi\\&\le c\int_{-1}^{1}\cdots\int_{-1}^{1}\frac{(\varphi(r\lambda))^2\,d\lambda}{(1-\varphi(r\lambda))^4},\end{aligned} \tag{4.4}$$

where $\lambda=(\lambda_1,\cdots,\lambda_q,0,\cdots,0)$, $\quad d\lambda=d\lambda_1,\cdots,d\lambda q$.

By the definition of φ, the right hand side of (4.4) is $O\left((1-r)^{-3}\right)$.

Let $z=rz_0$, $z_0\in\partial M_0$, $w=rz=(\eta_1,\cdots,\eta_q,0,\cdots,0)\,\tilde{k}$. Then

$$\varphi(w)=r^2=\sup\{|\eta_j|\,j=1,\cdots,q\}$$

and

$$\frac{1}{1-r^4}\le\frac{1}{1-\sup\left\{\eta_j^2\right\}}\le\prod_{j=1}^{q}\frac{1}{1-\eta_j^2}.$$

We have

$$\|f(r\xi)\|_2=O\left(\left(\prod_{j=1}^{q}\frac{1}{1-\eta_j^2}\right)^{\frac{3}{2}}\right). \tag{4.5}$$

From (4.2), (4.3) and (4.5), we have

$$\left|\frac{\partial f_\ell}{\partial w_j}(w)\right|=O\left(\left(\prod_{j=1}^{q}\frac{1}{1-\eta_j^2}\right)^{\frac{3}{2}}(K_{M_0}(z,\bar{z}))^{\frac{1}{2}}\prod_{p=1}^{q}\frac{1}{1-\lambda_p^2}\right).$$

But $\frac{1}{1-\lambda_p^2}=O\left(\frac{1}{1-\eta_p^2}\right)$, so we get

$$|\det J_f(w)|=O\left(K_{M_0}(z,\bar{z})^{\frac{n}{2}}\left(\prod_{j=1}^{q}\frac{1}{1-\eta_j^2}\right)^{\frac{5n}{2}}\right).$$

Since $\frac{1}{2}\sum\limits_{j=q+1}^{2n}|\alpha_j(x_p)|+1\ge 2$, (3.4) implies

$$\frac{K_{M_0}(w,\bar{w})}{K_{M_0}(0,0)}\ge\left(\prod_{p=1}^{q}\frac{1}{1-\eta_p^2}\right)^2$$

and $|\det J_f(w)|=O\left(K_{M_0}(w,\bar{w})^{\frac{7n}{4}}\right)$. Since

$$K_{M_0}(w,\bar{w})=O\left(\sup_{\theta\to b\in}\lim_{\partial M}\frac{K_{M_0}(\theta,\bar{\theta})}{K_{M_0}(\eta,\bar{\eta})}\right),$$

we have

$$|\det J_f(w)|=O\left(\sqrt{\frac{K_{M_0}(w,\bar{w})}{K_{M_0}(0,0)}}\sup_{\theta\to b\in}\lim_{\partial M}\left(\frac{K_{M_0}(\theta,\bar{\theta})}{K_{M_0}(\eta,\bar{\eta})}\right)^{\frac{7n-2}{4}}\right).$$

This argument proves the right hand side of inequality (1.3). The left hand side of inequality (1.3) is obviously, because otherwise the left hand side of (1.2) will tend to infinity for every point as z tends to the boundary of M_0.

§5. A Theorem on Bounded Domain

Theorem 5.1. *Let $U \subset \mathbf{C}^n$ be a bounded domain and B_d denote a ball centered at origin with radius d. If $B_r \subset U \subset B_R$ let S_U denote the family of normalized biholomorphic convex mapping on U which maps U into $\mathbf{C}^n$. If $f \in S_U$ and $f(z) = z + \sum_{i,j=1}^{n} d_{ij} z_i z_j + \cdots$, then*

$$|d_{ij}| \le Rr^{-2}, \quad i, j = 1, 2, \cdots, n. \tag{5.1}$$

Proof. The mapping $\Phi(z) = f^{-1}\left(\frac{1}{2} f(zV) + f(z)\right)$ maps B_r into U and $\Phi(0) = 0$ if $V \in U_n$, $f \in S_U$.

If $\Phi(z) = zJ_\phi(0) + \sum \Phi_{ij} z_i z_j + \cdots$, then

$$\begin{aligned} f(\Phi(z)) &= \Phi(z) + \sum d_{ij} (\Phi(z))_i (\Phi(z)_j + \cdots \\ &= zJ_\phi(0) + \sum \Phi_{ij} z_i z_j + \sum d_{ij} (zJ_\phi(0)_i (zJ_\phi(0))_j + \cdots . \end{aligned}$$

Since $f(zV) = zV + \sum d_{ij} (zV)_i (zV)_j + \cdots$ and $f(\Phi(z)) = \frac{1}{2}(f(zV) + f(z))$, we have

$$\begin{aligned} & zJ_\phi(0) + \sum \Phi_{ij} z_i z_j + \sum d_{ij} (zJ_\phi(0))_i (zJ_\phi(0))_j + \cdots \\ &= \frac{z}{2}(V + I) + \frac{1}{2}\left(\sum d_{ij} (zV)_i (zV)_j + \sum d_{ij} z_i z_j\right) + \cdots \end{aligned}$$

if $z \in B_r$. Thus, $J_\phi(0) = \frac{1}{2}(V + I)$ and $\sum \phi_{ij} z_i z_j = \sum d_{ij} \left(\frac{z}{2}(I - V)\right)_i \left(\frac{z}{2}(I + V)\right)_j$. The mapping

$$\Phi(z) = z \cdot \frac{1}{2}(I + V) + \sum d_{ij} \left(\frac{z}{2}(I - V)\right)_i \left(\frac{z}{2}(I - V)\right)_j + \cdots \tag{5.2}$$

is a biholomorphic mapping on B_r which maps B_r into U, and $|\Phi(z)| < R$ if $z \in B_r$.

Let $\xi = \frac{1}{2} z(I + V)$, $\eta = \frac{1}{2} z(I - V)$. Then $|\xi|^2 + |\eta|^2 = |z|^2 < r^2$ and $\mathrm{Re}\xi\bar{\eta}' = 0$.

Let $\xi = 0, \eta = r_1 e^{i\theta} e_j$, where $0 < r_1 < r$ and e_j is the unit vector such that all the components are zero except the j-th component is 1. Then by (5.2),

$$\Phi(z) = r_1^2 e^{2i\theta} d_{ij} + \cdots \in U \subset B_R. \tag{5.3}$$

Multiplying (5.3) by $e^{-2i\theta}$, integrating with respect to θ from 0 to 2π and dividing by 2π, we get

$$r_1^2 d_{jj} \in B_R, \qquad r_1 < r. \tag{5.4}$$

Let $\xi = 0$, $\eta = r_1 e^{it} e^{i\theta} e_k + r_1 e^{-it} e^{i\theta} e_j, k \ne j$, where $2r_1^2 \le r^2$, and e_k is the unit vector such that all the components are zero except the k-th component is 1. Then by (5.2),

$$\Phi(z) = r_1^2 e^{2i\theta} \left(e^{2it} d_{kk} + 2d_{jk} + e^{-2it} d_{jj}\right) + \cdots \in U \subset B_R. \tag{5.5}$$

Multiplying (5.5) by $e^{-2i\theta}$, integrating with respect to θ from 0 to 2π, dividing by 2π, and then integrating with respect to t from 0 to 2π and dividing by 2π, we get

$$2r_1^2 d_{jk} \in B_R, \quad 2r_1^2 < r^2. \tag{5.6}$$

Letting $r_1 \to r$ at (5.4), $2r_1^2 \to r^2$ at (5.6), we get (5.1).

Corollary 5.1. *Assumptions are the same as Theorem* 5.1*, and* U *is the unit ball in* $\mathbf{C}^n$*. Then*

$$|d_{ij}| \le 1 \tag{5.7}$$

for all $i, j = 1, 2, \cdots, n$.

This estimation is precise, because the mapping $F(z) = \left(\frac{z_1}{1-z_1}, \cdots, \frac{z_n}{1-z_1}\right)$ is a normalized biholomorphic convex mapping on the unit ball in $\mathbf{C}^n$; this mapping makes equality hold in (5.7).

§6. Proof of Theorem 1.2

We start from formula (3.2). As we have already pointed out in [2],

$$\sup\left\{\left|\sum_{j\cdot\ell=1}^{n} k_{p\ell} d_{\ell j}^{(j)}\right|, p = 1, \cdots, n,\ f \in S_{M_0}\right\}$$

is the same as

$$\sup\left\{\left|\sum_{j=1}^{n} d_{\ell j}^{(j)}\right|, \ell = 1, \cdots, n,\ f \in S_{M_0}\right\}$$

if $(k_{p\ell}) = \tilde{k}$, and $S(M_0)$ is an invariant family. By Theorem 5.1,

$$\sup\left\{\left|\sum_{j=1}^{n} d_{\ell j}^{(j)}\right|, \ell = 1, \cdots, n,\ f \in S_{M_0}\right\} \le n r^{-2} R$$

since $S(M_0)$ is the family of normalized Ω-uniformly locally biholomorphic convex mappings. We get the right hand side inequality of (1.5).

In order to prove the left hand side inequality of (1.5), we consider $\frac{\det J_{\tilde{f}}(\tilde{z})}{\sqrt{K_{M_0}(\tilde{z},\bar{\tilde{z}})/K_{M_0}(0,0)}}$, where $\tilde{f}$ is the normalization of $f \circ \xi \circ \xi_0^{-1}$ and $\tilde{z} = \xi_0 \circ \xi^{-1}(z) \in M_0$. By (3.4), we have

$$\prod_{\ell=1}^{q} \frac{(1 - |\tanh x_\ell|)^{A_\ell - \frac{1}{4}\sum\limits_{i=q+1}^{2n}|\alpha_j(X_\ell)| - \frac{1}{2}}}{(1 + |\tanh x_\ell|)^{A_\ell + \frac{1}{4}\sum\limits_{i=q+1}^{2n}|\alpha_j(X_\ell)| + \frac{1}{2}}} \le \left|\det J_{\tilde{f}}(\tilde{z})\right|. \tag{5.8}$$

Since $\det J_{\tilde{f}}(\tilde{z})$ is a holomorphic function on M_0 and $\det J_{\tilde{f}}(\tilde{z}) \ne 0$ for any $\tilde{z} \in M_0$, $\left|\det J_{\tilde{f}}(\tilde{z})\right|$ attends its minimum value on the boundary of M_0. Each $A_\ell - \frac{1}{4}\sum\limits_{j=q+1}^{2n}|\alpha_j(X_\ell)| - \frac{1}{2}$ must be ≥ 0, otherwise the left hand side of (5.8) would tend to infinity as $\tilde{z}$ approaches the boundary. This proves the left hand side inequality of (1.5).

References

[1] Gong Sheng & Zheng Xuean, Distortion theorem for biholomorphic mappings in transitive domains I, International symposium in memory of L.K. Hua, Vol. II, Springer-Verlag, 1991, 111-122.

[2] Zheng Xuean & Gong Sheng, Distortion theorem for biholomorphic mappings in transitive domains II, *Chin. Ann. of Math.,* **13B:**4 (1992), 471-484.

[3] Zheng Xuean & Gong Sheng, Distortion theorem for biholomorphic mappings in transitive domains III, *Chin. Ann of Math.,* **14B:**3 (1993), 367-386.

[4] Helgason, S., Differential geometry, Lie groups, and symmetric spaces, Academic Press, 1978.

[5] Gong Sheng, Convex and Starlike mappings in several complex variables, Science Press, Beijing, China (to appear).