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LOCAL C-COSINE FAMILY THEORY AND APPLICATION**

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Abstract

This paper introduces the concept of local **C**-cosine family and inverstigate its basic properties. In particular, a characterization of the complete infinitesimal generator of a local **C**-cosine family is obtained. As an application of this theory to the second order abstract Cauchy problems, a characterization of the local **C**-well-posedness of these problems is given in terms of the local **C**-cosine family theory.

Keywords Local **C**-cosine family, Asymptotic **C**-cosine resolvent, Complete generator, Second order Cauchy problem.
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§1. Introduction

Tanaka[15] introduced the concept of exponentially bounded **C**-cosine family and gave a characterization of its complete generator. Delaubenfels[6] gave a way to construct some exponentially bounded **C**-cosine family by making use of operator functional calculus which is different from Laplace transforms and applied the results obtained by him to improperly posed second order abstract Cauchy problems. However, as it is seen from Examples 4.1 and 4.2 in this paper, there exist **C**-cosine families which are neither exponentially bounded nor defined on $(-\infty, \infty)$. In order to characterize them, in this paper we introduce the concept of the local **C**-cosine family and attempt to give systematical study of the local **C**-cosine family.

Arendt[1], Neubrander[12] pointed out that the characterization of exponentially bounded n-times integrated semigroups unifies the classical characterization of $\mathbf{C}_0$-semigroups, cosine families or exponential distribution semigroups. However, in contrast with this case, by the analysis for Examples 4.1 and 4.2, the local **C**-cosine family theory does not be unified by local **C**-semigroup and the local integrated semigroup theory in [17]. In particular, as shown in Example 4.1, for the complete infinitesimal generator of a local **C**-cosine family, its resolvent set or **C**-resolvent set can be empty. Therefore these cases can not be treated in the classical method. Moreover, from applications to the second order abstract linear Cauchy problems in Examples 4.1 and 4.2, we see that the local **C**-cosine family theory is better than the local **C**-semigroup and the local integrated semigroup theory in [17].

In §2 we introduce the concept of the local **C**-cosine family on a Banach space and give a few basic properties of the complete infinitesimal generator of a local **C**-cosine family.

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Section 3 deals with a characterization of the complete infinitesimal generator in terms of its asymptotic **C**-cosine resolvent. In §4 we apply the local **C**-cosine family theory to the second order abstract linear Cauchy problems and give two typical examples.

§2. Definition and Properties of Local C-Cosine Family

Let X be a Banach space. We denote by $B(X)$ the set of all bounded linear operators on X to X. Let $T \in (0, \infty]$ and $C \in B(X)$. Throughout this paper we assume that C is injective and has dense range $R(C)$.

Definition 2.1. *A one-parameter family $\{C(T); |t| < T\}$ in $B(X)$ is called a local C-cosine family on X if it satisfies the following conditions:*

(i) $C(0) = C$,

(ii) $(C(t+s) + C(t-s))C = 2C(t)C(s)$ *for* $|t \pm s|, |s|, |t| < T$.

(iii) $C(t)x : (-T, T) \to X$ *is continuous for every* $x \in X$.

From the definition, we have

Proposition 2.1. *If $\{C(t); |t| < T\}$ is a local* **C***-cosine family on X, then*

(i) $C(t) = C(-t)$, *for* $|t| < T$;

(ii) *for* $|t| < T, C(t)$ *and* C *mutually commute.*

For a local **C**-cosine family $\{C(t); |t| < T\}$ on X, we define a linear operator G as follows:

$$Gx = \frac{d^2}{dt^2}(C(t)C^{-1}x)|_{t=0} \tag{2.1}$$

with $D(G) = \{x \in R(C); C^{-1}x \in \underset{0<\delta<T}{\cup} C^2(\delta)\}$, where $C^2(\delta) = \{x \in X; C(t)x : (-\delta, \delta) \to X$ is twice continuously differentiable$\}$.

Afterwards we denote $\frac{d^r}{dt^r}C(t)x|_{t=s}$ by $C^{(r)}(s)x, r = 0, 1, 2 \cdots$.

Lemma 2.1. *Let $\{C(t) : |t| < T\}$ be a local* **C***-cosine family on X. For $0 < s, r < T, x \in X$, let $w = C^{-1}\int_0^s \int_0^r C(u)C(v)xdudv$. Then for $|t| < T - s - r$,*

$$\frac{d^2}{dt^2}C(t)w = \frac{C}{4}(C(t+r+s)x - C(t+s-r)x - C(t+r-s)x + C(t-s-r)x).$$

Proof. For $|t| < T - r - s$, we have

$$\begin{aligned} C(t)w &= \frac{1}{2}\int_0^r \int_0^s (C(t+u) + C(t-u)C(v)x)dudv \\ &= \frac{1}{2}\int_0^r \left(\int_t^{t+s} C(u)C(v)xdu + \int_{t-s}^t C(u)C(v)xdu\right)dv \\ &= \frac{1}{2}\int_0^r \int_{t-s}^{t+s} C(u)C(v)xdudv. \end{aligned}$$

Therefore.

$$\begin{aligned} \frac{d}{dt}C(t)w &= \frac{1}{2}\int_0^r (C(t+s) - C(t-s)C(v)xdv \\ &= \frac{C}{4}\int_0^r (C(t+s+v) + C(t+s-v) \\ &\quad - C(t-s+v) - C(t-s-v))xdv \\ &= \frac{C}{4}\left(\int_{t+s-r}^{t+s+r} C(v)xdv - \int_{t-s-r}^{t-s+r} C(v)xdv\right) \end{aligned}$$

and

$$\frac{d^2}{dt^2}C(t)w = \frac{C}{4}(C(t+s+r)x - C(t+s-r)x - C(t+s-r)x + C(t-s-r)x).$$

Proposition 2.2. *Linear operator G is closable with $\overline{D(G)} = X$ and for $x \in D(G), |t| < T$,*

$$\frac{d^2}{dt^2}C(t)x = GC(t)x = C(t)Gx. \tag{2.2}$$

Proof. Let $x \in X$ and $0 < 2a < T$. Suppose $x_a = \int_0^a \int_0^a C(u)C(v)xdudv$. Then by Lemma 2.1, for $t < T - 2a$, we have

$$\frac{d^2}{dt^2}C(t)C^{-1}x_a = \frac{C}{4}(C(t+2a)x + C(t-2a)x - 2C(t)x).$$

So, $x_a \in D(G)$. Since $a^{-2}x_a \to C^2x$, as $a \to 0^+$, we have $\overline{D(G)} \supset \overline{R(C^2)} = X$. Thus, G is densely defined. Now, we prove that G is closable. Let $x \in D(G), |t| < T$. Supposing $h < 2^{-1}(T-t)$, by Definition 2.1, we have

$$\begin{aligned}\frac{1}{4h^2}(C(t+2h) + (C(t-2h) - 2C(t))x &= \frac{1}{2h^2}C(t)(C(2h) - C)C^{-1}x \\ &= \frac{1}{2h^2}(C(2h) - C)C^{-1}C(t)x.\end{aligned}$$

Letting $h \to 0$, we obtain

$$\frac{d^2}{dt^2}C(t)x = GC(t)x = C(t)Gx, \quad \text{for} \quad x \in D(G), |t| < T.$$

Since $C(t) = C(-t)$ for $|t| < T$, we have $C^{(1)}(0)x = 0$ for $x \in D(G)$. Thus, for $x \in D(G), |t| < T$,

$$C(t)x - Cx = \int_0^t \int_0^s C(r)Gxdrds. \tag{2.3}$$

Let $x_n \in D(G)$ and $x_n \to 0, Gx_n \to y$, as $n \to \infty$. Substituting x by x_n in (2.3) and letting $n \to \infty$, we get $\int_0^t \int_0^s C(r)ydrds = 0$. Also since

$$Cy = \lim_{t \to 0} \frac{2}{t^2} \int_0^t \int_0^s C(r)ydrds = 0,$$

we get $y = 0$. Therefore, G is closable.

By Proposition 2.2, we can give the following definition.

Definition 2.2. *Let $\{C(t); |t| < T\}$ be a local **C**-cosine family. G is defined by (2.1). Then $\overline{G}$ is called the complete infinitesimal generator of $\{C(t); |t| < T\}$.*

Arguing as in the proof of [16, Theorem 2.1] we can prove

Proposition 2.3. *Let $\{C(t); |t| < T\}$ be a local **C**-cosine family on X. Assume that $\overline{G}$ is its complete infinitesimal generator. Then $CD(\overline{G})$ is a core of $\overline{G}$, i.e., $\overline{\overline{G}|_{CD(G)}} = \overline{G}$.*

Let linear operator $L_\tau(\lambda)$ be defined by

$$L_\tau(\lambda)x = \int_0^\tau e^{-\lambda t} \int_0^t C(s)xdsdt, \quad \text{for} \quad x \in X \quad \text{and} \quad 0 < \tau < T, \tag{2.4}$$

where $\{C(t); |t| < T\}$ is a local **C**-cosine family on X.

Proposition 2.4. *Let $\overline{G}$ be the complete infinitesimal generator of a local **C**-cosine family $\{C(t); |t| < T\}$. Then*

(i) *$C(t)x \in D(\overline{G})$ and $C(t)\overline{G}x = \overline{G}C(t)x$ for $x \in D(\overline{G})$ and $|t| < T$;*

(ii) *$L_\tau(\lambda)x \in D(\overline{G})$ and $(\lambda^2-\overline{G})L_\tau(\lambda)x = Cx - e^{-\lambda\tau}C(\tau)x - \lambda e^{-\lambda\tau}\int_0^\tau C(t)xdt$ for $x \in X$ and $\tau \in (0,T)$;*

(iii) *$\overline{G}L_\tau(\lambda)x = L_\tau(\lambda)\overline{G}x$ for $x \in D(\overline{G})$ and $L_\tau(\lambda)L_\tau(\mu)x = L_\tau(\mu)L_\tau(\lambda)x$ for $x \in X$;*

(iv) *for $x \in X, L_\tau(\lambda)x$ is infinitely differentiable in λ, and there exists a constant $M_\tau > 0$, depending on τ, such that*

$$\Big\|\frac{\lambda^{n+2}}{(n+1)!}(\frac{d^n}{d\lambda^n})L_\tau(\lambda)\Big\| \le M_\tau$$

for $\lambda > 0$ and $n \in N_0$, where N_0 denotes the set of nonnegative integers;

(v) *for $0<\beta<\tau<T$, there exists a constant $M_{\tau,\beta}>0$, depending on τ,β, such that*

$$\Big\|\frac{\lambda^{n+1}}{n!}\frac{d^n}{d\lambda^n}(\lambda L_\tau(\lambda))\Big\| \le M_{\tau,\beta}, \quad \text{for} \quad \frac{n+1}{\lambda} \in (0,\beta], n \in N_0.$$

Proof. (i) can be easily deduced from (2.2). Now, we prove (ii). Let G be defined by (2.1). For $x \in D(G), |t_0|, |t_0+t_1| < T-\tau$, we have

$$\begin{aligned}
&\frac{C(t_0+t_1)-C(t_0)}{t_1}L_\tau(\lambda)x \\
&= \frac{C}{2t_1}\int_0^\tau e^{-\lambda t}\int_0^t (C(t_0+t_1+s)x + C(t_0+t_1-s)x - C(t_0+s)x - C(s-t_0)x)dsdt \\
&= \frac{C}{2t_1}\int_0^\tau e^{-\lambda t}\Big(\int_{t_0+t}^{t_0+t+t_1} C(s)xds - \int_{t_0}^{t_0+t_1} C(s)xds \\
&\quad + \int_{-t_0-t_1}^{-t_0} C(s)xds - \int_{t-t_0-t_1}^{t-t_0} C(s)xds\Big)dt.
\end{aligned}$$

So

$$\frac{d}{dt_0}C(t_0)L_\tau(\lambda)x = \frac{C}{2}\int_0^\tau e^{-\lambda t}(C(t_0+t)x - C(t-t_0)x)dt$$

and

$$\frac{d^2}{dt_0^2}C(t_0)L_\tau(\lambda)x = \frac{C}{2}\int_0^\tau e^{-\lambda t}(\frac{d}{dt_0}C(t_0+t)x - \frac{d}{dt_0}C(t-t_0)x)dt.$$

Thus

$$\begin{aligned}
&C^{(2)}(0)L_\tau(\lambda)x \\
&= C\int_0^\tau e^{-\lambda t}C^{(1)}(t)xdt \\
&= C(-Cx + e^{-\lambda\tau}C(\tau)x + \lambda\int_0^\tau e^{-\lambda t}C(t)xdt) \\
&= C(-Cx + e^{-\lambda\tau}C(\tau)x + \lambda e^{-\lambda\tau}\int_0^\tau C(t)xdt + \lambda^2\int_0^\tau e^{-\lambda t}\int_0^t C(s)xdsdt).
\end{aligned}$$

Therefore $L_\tau(\lambda)x \in D(\overline{G})$ and

$$(\lambda^2-\overline{G})L_\tau(\lambda)x = Cx - e^{-\lambda\tau}C(\tau)x - \lambda e^{-\lambda\tau}\int_0^\tau C(t)xdt, \quad \text{for} \quad x \in D(G).$$

This implies (ii) by Proposition 2.2 and the closedness of $\overline{G}$. Also (iii) can be deduced from (i) and (ii). In the following we prove (iv). Let $M_\tau^0 = \max_{[0,\tau]}\|C(t)\|$ and $M_\tau = 2M_\tau^0$. Since for $n \in N_0, x \in X$,

$$\begin{aligned}
&\Big\|\frac{\lambda^{n+2}}{(n+1)!}\frac{d^n}{d\lambda^n}((\frac{1}{\lambda}e^{-\lambda\tau})\int_0^\tau C(t)xdt)\Big\| \\
&\le \Big\|\frac{\lambda^{n+2}}{(n+1)!}\sum_{i=0}^{n} C_n^i\cdot i!\lambda^{-i-1}\tau^{n-i}e^{-\lambda\tau}\int_0^\tau C(t)xdt\Big\| \\
&\le M_\tau^0\left[\frac{1}{(n+1)!}\sum_{i=0}^{n}\frac{n!}{(n-i)!}(\lambda\tau)^{n+1-i}e^{-\lambda\tau}\right]\|x\| \\
&\le M_\tau^0\|x\|
\end{aligned}$$

and similarly

$$\Big\|\frac{\lambda^{n+2}}{(n+1)!}\frac{d^n}{d\lambda^n}\Big(\frac{1}{\lambda}\int_0^\tau e^{-\lambda t}C(t)xdt\Big)\Big\| \le M_\tau^0\|x\|,$$

we have

$$\begin{aligned}
\Big\|\frac{\lambda^{n+2}}{(n+1)!}\frac{d^n}{d\lambda^n}L_\tau(\lambda)x\Big\| &= \Big\|\frac{\lambda^{n+2}}{(n+1)!}\frac{d^n}{d\lambda^n}\Big(\int_0^\tau e^{-\lambda t}\int_0^t C(s)xdsdt\Big)\Big\| \\
&= \Big\|\frac{\lambda^{n+2}}{(n+1)!}\frac{d^n}{d\lambda}\Big(-\frac{1}{\lambda}e^{-\lambda\tau}\int_0^\tau C(t)xdt + \frac{1}{\lambda}\int_0^\tau e^{-\lambda t}C(t)xdt\Big)\Big\| \\
&\le M_\tau\|x\|.
\end{aligned}$$

Finally, we prove (v). For $x \in X, n \in N_0$, we have

$$\begin{aligned}
&\Big\|\frac{\lambda^{n+1}}{n!}(\lambda L_\tau(\lambda)x)^{(n)}\Big\| \\
&= \Big\|\frac{\lambda^{n+1}}{n!}(\lambda L_\tau^{(n)}(\lambda) + nL_\tau^{(n-1)}(\lambda))x\Big\| \\
&= \Big\|(-1)^n\frac{\lambda^{n+2}}{n!}\int_0^\tau e^{-\lambda t}t^n\int_0^t C(s)xdsdt \\
&\quad + (-1)^{n-1}\frac{\lambda^{n+1}}{(n-1)!}\int_0^\tau e^{-\lambda t}t^{n-1}\int_0^t C(s)dxdt\Big\| \\
&= \Big\|(-1)^{n-1}\frac{\lambda^{n+1}}{n!}e^{-\lambda\tau}\tau^n\int_0^\tau C(s)xds + (-1)^n\frac{\lambda^{n+1}}{n!}\int_0^\tau e^{-\lambda t}t^nC(t)xdt\Big\| \\
&\le M_\tau^0\frac{(\lambda\tau)^{n+1}}{n!}e^{-\lambda\tau}\|x\| + M_\tau^0\frac{\lambda^{n+1}}{n!}\int_0^\tau e^{-\lambda t}t^ndt\|x\| \\
&\le \left(M_\tau^0\frac{(\lambda\tau)^{n+1}}{n!}e^{-\lambda\tau} + M_\tau^0\right)\|x\|.
\end{aligned}$$

Thus, since $F(t) = \frac{1}{n!}(t\tau)^{n+1}e^{-t\tau}$ is decreasing on $[\frac{n+1}{\tau}, \infty)$, we get

$$\|\frac{\lambda^{n+1}}{n!}(\lambda L_\tau(\lambda)x)^{(n)}\| \le \Big(M_\tau^0\frac{(\frac{n+1}{\beta}\tau)^{n+1}}{n!}e^{-(n+1)(\tau/\beta)} + M_\tau^0\Big)\|x\|$$

for $x \in X, n \in N_0$ and $\frac{n+1}{\lambda} \in [0, \beta]$. Write $a_n = \frac{(n+1)^{n+1}}{n!}\left(\frac{\tau}{\beta}\right)^{n+1}e^{-(n+1)\tau/\beta}$. Then

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\Big(1+\frac{1}{n}\Big)^{n+1}\Big(\frac{\tau}{\beta}\Big)e^{-\tau/\beta} = \frac{\tau}{\beta}e^{1-\tau/\beta} < 1, \tag{2.5}$$

and so $N_{\tau,\beta} = \sup\{a_n; n \in N_0\} < \infty$. Setting $M_{\tau,\beta} = M_\tau^0(1 + N_{\tau,\beta})$, we see that (v) holds. The proof is complete.

§3. Characterization of Local C-Cosine Family

The purpose of this section is to give a characterization of the complete infinitesimal generator of a local **C**-cosine family. For this reason we first introduce the concept of asymptotic **C**-cosine resolvent.

Definition 3.1. *Let A be a closed linear operator in $X, \tau \in (0,T)$ and $\beta \in (0,\tau)$. A family $\{L_\tau(\lambda) : \lambda > a\}$ in $B(X)$ is called an asymptotic* **C***-cosine resolvent of A if it satisfies the following conditions:*

(a$_1$) *for $x \in X, \lambda > a,$ $L_\tau(\lambda)x$ is an infinitely differentiable function of λ;*

(a$_2$) *for $x \in X$ and $\lambda, \mu > a, L_\tau(\mu)L_\tau(\lambda)x = L_\tau(\lambda)L_\tau(\mu)x$;*

(a$_3$) *$L_\tau(\lambda)x \in D(A)$ and $(\lambda^2 - A)L_\tau(\lambda)x = Cx + V_\tau(\lambda)x$ for $x \in X$ and $\lambda > a$, where $V_\tau(\lambda)x$ is infinitely differentiable for $\lambda > a$ and there exists a constant $M_{\tau,\beta} > 0$, depending on τ, β, such that*

$$\Big\|\frac{d^n}{d\lambda^n}V_\tau(\lambda)x\Big\| \le M_{\tau,\beta}\tau^n e^{-\lambda\tau}\|x\| \tag{3.1}$$

for $x \in X,$ $\lambda > \max\{a, \frac{n}{\beta}\}$ and $n \in N_0$;

(a$_4$) *$AL_\tau(\lambda)x = L_\tau(\lambda)Ax$ for $x \in D(A)$ and $\lambda > a$.*

Theorem 3.1. *A closed linear operator A in X is the complete infinitesimal generator of a local* **C***-cosine family $\{C(t) : |t| < T\}$ if and only if it satisfies the following conditions:*

(i) *$D(A)$ is dense in X;*

(ii) *for every $\tau \in (0,T)$ there exists an asymptotic* **C***-cosine resolvent $\{L_\tau(\lambda); \lambda > a\}$ and a constant $M_\tau > 0$, depending on τ, such that*

$$\Big\|\frac{\lambda^{n+2}}{(n+1)!}\frac{d^n}{d\lambda^n}L_\tau(\lambda)\Big\| \le M_\tau, \quad \textit{for} \quad n \in N_0, \frac{n+1}{\lambda} \in (0,\tau] \quad \textit{and} \quad \lambda > a; \tag{3.2}$$

(iii) *for $\tau \in (0,T), \beta \in (0,\tau)$, there exists a constant $M_{\tau,\beta} > 0$, depending on τ, β, such that*

$$\Big\|\frac{\lambda^{n+1}}{n!}\frac{d^n}{d\lambda^n}(\lambda L_\tau(\lambda))\Big\| \le M_{\tau,\beta}, \quad \textit{for} \quad \lambda > \max\{a, \frac{n}{\beta}\} \quad \textit{and} \quad n \in N_0; \tag{3.3}$$

(iv) *$CD(A)$ is a core A.*

In order to prove the sufficiency of Theorem 3.1, we first prove some lemmas.

Lemma 3.1. *Let A be a closed linear operator and satisfy the conditions* (i) *and* (ii) *in Theorem* 3.1 *and $\tau \in (0,T)$. Then*

(b$_1$) *for $x \in D(A), Cx \in D(A)$ and $ACx = CAx$;* (3.4)

(b$_2$) *$L_\tau(\lambda)Cx = CL_\tau(\lambda)x$ for $x \in X$ and $\lambda > a$;* (3.5)

(b$_3$) *$L_\tau^{(n+1)}(\lambda)Cx = V_\tau^{(n+1)}(\lambda)L_\tau(\lambda)x - V_\tau(\lambda)L_\tau^{(n+1)}(\lambda)x - 2\lambda(n+1)L_\tau^{(n)}(\lambda)L_\tau(\lambda)x - n(n+1)L_\tau^{(n-1)}(\lambda)L_\tau(\lambda)$, for $x \in X$ and $n \ge 1$;* (3.6)

(b$_4$) *for $x \in X, n \in N_0$,*

$$\lim_{\lambda\to\infty}\frac{(-1)^n}{(n+1)!}\lambda^{n+2}L_\tau^{(n)}(\lambda)x = Cx, \tag{3.7}$$

where $L_\tau^{(n)}(\lambda)x = \frac{d^n}{d\lambda^n}L_\tau(\lambda)x,$ $V_\tau^{(n)}(\lambda)x = \frac{d^n}{d\lambda^n}V_\tau(\lambda)x$.

Proof. First we prove (b$_1$) and (b$_2$). Let $\beta \in (0,\tau)$. By (a$_3$) and (ii), for $x \in D(A), \lambda >$

$\max\{a, 1/\beta\}$, we have

$$\begin{aligned}\|\lambda^2 L_\tau(\lambda)x - Cx\| &\le \|L_\tau(\lambda)Ax\| + \|V_\tau(\lambda)x\| \\ &= \frac{M_\tau}{\lambda^2}\|Ax\| + M_{\tau,\beta}e^{-\lambda\tau}\|x\|.\end{aligned}$$

Therefore, $\lim\limits_{\lambda\to\infty} \lambda^2 L_\tau(\lambda)x = Cx$ for $x \in D(A)$ and, since $\|\lambda^2 L_\tau(\lambda)\| \le M_\tau$ for $\lambda > \max\{a, 1/\tau\}$ and $\overline{D(A)} = X$, we obtain

$$\lim_{\lambda\to\infty} \lambda^2 L_\tau(\lambda)x = Cx, \qquad \text{for} \quad x \in X. \tag{3.8}$$

Thus (a_4) and (3.8) imply that (b_1) holds, since A is closed. Moreover, we can obtain (b_2) from (a_2) and (3.8). Now, we prove (b_3). Let $\lambda > a$, from $(\lambda^2 - A)L_\tau(\lambda)x = Cx + V_\tau(\lambda)x$ for $x \in X$, we have

$$(\lambda^2 - A)L_\tau^{(n+1)}(\lambda)x + 2(n+1)\lambda L_\tau^{(n)}(\lambda)x + n(n+1)L_\tau^{(n-1)}(\lambda)x = V_\tau^{(n+1)}(\lambda)x. \tag{3.9}$$

Multiplying (3.9) by $L_\tau(\lambda)$ and then using (a_3) and (b_2), we get

$$\begin{aligned}&L_\tau^{(n+1)}(\lambda)(Cx + V_\tau(\lambda)x) + 2(n+1)\lambda L_\tau^{(n)}(\lambda)L_\tau(\lambda)x + n(n+1)L_\tau^{(n-1)}(\lambda)L_\tau(\lambda)x \\ &= V_\tau^{(n+1)}(\lambda)L_\tau(\lambda)x.\end{aligned}$$

Thus, (b_3) holds. In the following we prove (3.7) by induction with respect to n. First, (3.8) implies that (3.7) holds for $n = 0$. We now prove that (3.7) holds for $n = 1$. Suppose $\beta \in (0, \tau), x \in D(A)$ and $\lambda > \max\{a, 2/\beta\}$. Differentiating the following equality

$$(\lambda^2 - A)L_\tau(\lambda)x = Cx + V_\tau(\lambda)x$$

and then multiplying $\lambda^3 L_\tau(\lambda)$, we get

$$2(\lambda^2 L_\tau(\lambda)^2 x + \lambda^3 L_\tau^{(1)}(\lambda)\lambda^2 L_\tau(\lambda)x - \lambda^3 AL_\tau^{(1)}(\lambda)L_\tau(\lambda)x = \lambda^3 V_\tau^{(1)}(\lambda)L_\tau(\lambda)x.$$

Also by (b_1) and (b_2), we have

$$\begin{aligned}&\|-\frac{1}{2}\lambda^3 L_\tau^{(1)}(\lambda)Cx - C^2x\| \\ &= \|-\frac{1}{2}\lambda^3 L_\tau^{(1)}(\lambda)(Cx - \lambda^2 L_\tau(\lambda)x) - \frac{1}{2}\lambda^3 L_\tau^{(1)}(\lambda)\lambda^2 L_\tau(\lambda)x \\ &\quad - (\lambda^2 L_\tau(\lambda))^2 x + (\lambda^2 L_\tau(\lambda))^2 x - C^2 x\| \\ &\le \|\frac{\lambda^3}{2}L_\tau^{(1)}(\lambda)\| \cdot \|Cx - \lambda^2 L_\tau(\lambda)x\| + \frac{1}{2}\|\lambda^3 L_\tau^{(1)}(\lambda)L_\tau(\lambda)Ax\| \\ &\quad + \frac{1}{2}\|\lambda^3 V_\tau^{(1)}(\lambda)L_\tau(\lambda)x\| + \|(\lambda^2 L_\tau(\lambda))^2 - C^2x\|.\end{aligned}$$

So for $x \in D(A), \lambda > \max\{a, 2/\beta\}$, by (3.8), the condition (ii) and (a_3),

$$\begin{aligned}&\|-\frac{1}{2}\lambda^3 L_\tau^{(1)}(\lambda)Cx - C^2x\| \\ &\le M_\tau\|Cx - \lambda^2 L_\tau(\lambda)x\| + \frac{M_\tau^2}{\lambda^2}\|Ax\| + \frac{\lambda}{2}M_\tau \cdot M_{\tau,\beta}e^{-\lambda\tau}\|x\| \\ &\quad + \|(\lambda^3 L_\tau(\lambda))^2 - C^2x\| \to 0 \quad \text{as} \quad \lambda \to \infty.\end{aligned}$$

Thus, for $x \in D(A)$,

$$\lim_{\lambda\to\infty} (-\frac{1}{2})\lambda^3 L_\tau^{(1)}(\lambda)Cx = C^2x$$

and so (3.7) holds for $n=1$ since $\overline{CD(A)}=X$ and $\|\frac{1}{2}\lambda^3 L_\tau^{(1)}(\lambda)\|\le M_\tau$. Now, supposing (3.7) is true for $n\le k$, we prove that it is true for $n=k+1$. Suppose $x\in X$, $\lambda>\max\{a,(k+1)/\beta\}, \beta\in(0,\tau)$. It follows from (3.6) that

$$\begin{aligned}\Delta_k x &= \frac{(-1)^{k+1}}{(k+2)!}\lambda^{k+3}L_\tau^{(k+1)}(\lambda)Cx - C^2x\\ &= \frac{(-1)^{k+1}}{(k+2)!}\lambda^{k+3}(V_\tau^{(k+1)}(\lambda)x - L_\tau^{(k+1)}(\lambda)V_\tau(\lambda)x)\\ &\quad - \frac{(-1)^{k+1}}{(k+2)!}\cdot 2(k+1)\lambda^{k+4}L_\tau^{(k)}(\lambda)L_\tau(\lambda)x\\ &\quad - \frac{(-1)^{k+1}}{(k+2)!}k(k+1)\lambda^{k+3}L_\tau^{(k-1)}(\lambda)L_\tau(\lambda)x - C^2x\\ &= \frac{(-1)^{k+1}}{(k+2)!}\lambda^{k+3}(V_\tau^{(k+1)}(\lambda)x - L_\tau^{(k+1)}(\lambda)V_\tau(\lambda)x)\\ &\quad + \frac{2k+2}{k+2}\Big(\frac{(-1)^k}{(k+1)!}\lambda^{k+2}L_\tau^{(k)}(\lambda)\lambda^2L_\tau(\lambda)x - C^2x\Big)\\ &\quad + \frac{k}{k+2}\Big(\frac{(-1)^{k-1}}{k!}\lambda^{k+1}L_\tau^{(k-1)}(\lambda)\lambda^2L_\tau(\lambda)x - C^2x\Big).\end{aligned}$$

So, by (a_3) and the condition (ii) we have

$$\begin{aligned}\|\Delta_k x\| &\le \frac{M_{\tau,\beta}}{(k+2)!}(\lambda\tau)^{k+3}\tau^{-2}\tau^{-\lambda\tau}\|x\| + M_\tau\cdot M_{\tau,\beta}e^{-\lambda\tau}\|x\|\\ &\quad + \frac{2k+2}{k+2}\|\frac{(-1)^k}{(k+1)!}\lambda^{k+2}L_\tau^{(k)}(\lambda)(\lambda^2L_\tau(\lambda)x - Cx)\|\\ &\quad + \frac{2k+2}{k+2}\|\frac{(-1)^k}{(k+1)!}\lambda^{k+2}L_\tau^{(k)}(\lambda)Cx - C^2x\|\\ &\quad + \frac{k}{k+2}\|\frac{(-1)^{k-1}}{k!}\lambda^{k+1}L_\tau^{(k-1)}(\lambda)(\lambda^2L_\tau(\lambda)x - Cx)\|\\ &\quad + \frac{k}{k+1}\|\frac{(-1)^{k-1}}{k!}\lambda^{k+1}L_\tau^{(k-1)}(\lambda)Cx - C^2x\|.\end{aligned}$$

Thus, the induction hypothesis and the condition (ii) yield

$$\lim_{\lambda\to\infty}\frac{(-1)^{k+1}}{(k+2)!}\lambda^{k+3}L_\tau^{(k+1)}(\lambda)Cx = C^2x \quad \text{for} \quad x\in X.$$

This implies that (3.7) is true for $n=k+1$ and $x\in X$, since

$$\overline{R(C)} = X \quad \text{and} \quad \|\frac{\lambda^{k+3}}{(k+2)!}L_\tau^{(k+1)}(\lambda)\|\le M_\tau.$$

Lemma 3.2. *Suppose that the conditions* (i) *and* (ii) *of Theorem* 3.1 *hold. Then*

$$\lim_{\lambda\to\infty}\frac{(-1)^n}{n!}\lambda^{n+1}\frac{d^n}{d\lambda^n}(\lambda L_\tau(\lambda))x = Cx \quad \textit{for} \quad x\in X \quad \textit{and} \quad n\in N_0. \tag{3.10}$$

Proof. It follows from (3.7) that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{(-1)^n}{n!}\lambda^{n+1}\frac{d^n}{d\lambda^n}(\lambda L_\tau(\lambda))x\\
&=\lim_{\lambda\to\infty}\frac{(-1)^n}{n!}\lambda^{n+2}L_\tau^{(n)}(\lambda)x+\lim_{\lambda\to\infty}\frac{(-1)^n}{(n-1)!}\lambda^{n+1}L_\tau^{(n-1)}(\lambda)x\\
&=(n+1)\lim_{\lambda\to\infty}\frac{(-1)^n}{(n+1)!}\lambda^{n+2}L_\tau^{(n)}(\lambda)x-n\lim_{\lambda\to\infty}\frac{(-1)^{n-1}}{n!}\lambda^{n+1}L_\tau^{(n-1)}(\lambda)x\\
&=(n+1)Cx-nCx\\
&=Cx.
\end{aligned}$$

Fix $\tau\in(0,T)$ arbitrarily. For $n>|a|\tau$, we define families $\{C_{n,\tau}(t):|t|\le\tau\}$ and $\{S_{n,\tau}(t);|t|\le\tau\}$ in $B(X)$ by

$$C_{n,\tau}(t)x=\begin{cases}\frac{(-1)^n}{n!}\lambda^{n+1}\frac{d^n}{d\lambda^n}(\lambda L_\tau(\lambda))x|_{\lambda=n/t}, & 0<t\le\tau,\\ Cx, & t=0,\\ C_{n,\tau}(-t)x, & -\tau\le t<0\end{cases}$$

and

$$S_{n,\tau}(t)x=\begin{cases}\frac{(-1)^n}{n!}\lambda^{n+1}\frac{d^n}{d\lambda^n}L_\tau(\lambda)x|_{\lambda=n/t}, & 0<t\le\tau,\\ 0, & t=0,\\ -S_{n,\tau}(-t)x, & -\tau\le t<0.\end{cases}$$

We suppose that all the following lemmas satisfy the conditions of Theorem 3.1.

Lemma 3.3. *$C_{n,\tau}(t)$ and $S_{n,\tau}(t)$ are strongly continuous on $[-\tau,\tau]$.*

Proof. $C_{n,\tau}(t)$ are strongly continuous on $[-\tau,\tau]$ by Lemma 3.2.For $\lambda>\max\{a,(n+1)/t\}$,

$$\Big\|\frac{\lambda^{n+2}}{(n+1)!}L_\tau^{(n)}(\lambda)\Big\|\le M_\tau,$$

it follows that $\|S_{n,\tau}(t)\|\le\frac{n+1}{n}|t|$. Thus, we see that $S_{n,\tau}(t)$ is strongly continuous on $[-\tau,\tau]$.

We will prove that $C_{n,\tau}(t)$ uniformly converges to a strongly continuous bounded linear operator family $C_\tau(t)$ on $[-\beta,\beta](o<\beta<\tau)$, and $\{C_\tau(t):|t|<\beta\}$ is a **C**-cosine family on X. By using this fact, we construct a **C**-cosine family $\{C(t):|t|<T\}$ with its complete infinitesimal generator A. For this reason, we first found some lemmas.

Lemma 3.4. *Suppose $x\in D(A^2)$ and $\beta\in(0,\tau)$. Then for $0<t<\beta$ and $n\ge2$,*

$$\frac{d}{dt}C_{n,\tau}(t)Cx=\frac{2n+2}{n}S_{n,\tau}(t)ACx-\frac{n+1}{n}S_{n-1,\tau}\Big(\frac{n-1}{n}t\Big)ACx+M_{n,\tau}(t)x$$

and $\|M_{n,\tau}(t)x\|$ uniformly converges to 0 on $(0,\beta]$ as $n\to\infty$.

Proof. Let

$$\begin{aligned}
P_n(t)x&=\frac{(-1)^{n-1}}{n!}(n/t)^{n+3}(\frac{1}{t})[V_\tau^{(n+1)}(n/t)L_\tau(n/t)x-V_\tau(n/t)L_\tau^{(n+1)}(n/t)x],\\
Q_n(t)x&=[\frac{(-1)^n}{n!}(n/t)^{n+2}L_\tau^{(n)}(n/t)\frac{2n+2}{t}+\frac{(-1)^n}{(n+1)!}(n/t)^{n+1}L_\tau^{(n-1)}(n/t)\frac{n+1}{t}]V_\tau(n/t)x,\\
R_n(t)x&=[\frac{(-1)^n}{n!}(n/t)^nL_\tau^{(n)}(n/t)\frac{2n+2}{t}+\frac{(-1)^n}{(n-1)!}(n/t)^{n-1}L_\tau^{(n-1)}(n/t)\frac{n+1}{t}]L_\tau(n/t)A^2x,
\end{aligned}$$

Using (a_3) and (3.6), we get

$$
\begin{aligned}
&\frac{d}{dt}C_{n,\tau}(t)Cx \\
&= \frac{(-1)^n}{n!}(n/t)^{n+2}(-\frac{n}{t^2})L_\tau^{(n+1)}(n/t)Cx + \frac{(-1)^n}{n!}(n+2)(-\frac{n}{t^2})L_\tau^{(n)}(n/t)Cx \\
&+ \frac{(-1)^n}{(n-1)!}(n/t)^{n+1}(-\frac{n}{t^2})L_\tau^{(n)}(n/t)Cx \\
&+ \frac{(-1)^n}{(n-1)!}(n/t)^n(n+1)(-\frac{n}{t^2})L_\tau^{(n-1)}(n/t)Cx \\
&= \frac{(-1)^{n-1}}{n!}(n/t)^{n+3}(\frac{1}{t})[V_\tau^{(n+1)}(n/t)L_\tau(n/t) - V_\tau(n/t)L_\tau^{(n+1)}(n/t)x \\
&- \frac{2n(n+1)}{t}L_\tau^{(n)}(n/t)L_\tau(n/t)x - n(n+1)L_\tau^{(n-1)}(n/t)L_\tau(n/t)Cx] \\
&+ \frac{(-1)^n}{n!}(n/t)^{n+2}\frac{n+2}{t}L_\tau^{(n)}(n/t)Cx + \frac{(-1)^{n-1}}{n!}(n/t)^{n+3}L_\tau^{(n)}(n/t)Cx \\
&+ \frac{(-1)^{n-1}}{n!}(n+1)(n/t)^{n+2}L_\tau^{(n-1)}(n/t)Cx \\
&= P_n(t)x + \frac{(-1)^n}{n!}(n/t)^{n+2}L_\tau^{(n)}(n/t)\Big[\frac{2n+2}{t}(n/t)^2L_\tau(n/t)x - \frac{2n+2}{t}Cx\Big] \\
&+ \frac{(-1)^n}{(n-1)!}(n/t)^{n+1}L_\tau^{(n-1)}(n/t)\Big[\frac{n+1}{t}(n/t)^2L_\tau(n/t)x - \frac{n+1}{t}Cx\Big] \\
&= P_n(t)x + \frac{(-1)^n}{n!}(n/t)^{n+2}L_\tau^{(n)}(n/t)\frac{2n+2}{t}(L_\tau(n/t)Ax + V_\tau(n/t)x) \\
&+ \frac{(-1)^n}{(n-1)!}(n/t)^{n+1}L_\tau^{(n-1)}(n/t)\frac{n+1}{t}(L_\tau(n/t)Ax + V_\tau(n/t)x) \\
&= P_n(t)x + Q_n(t) + \frac{(-1)^n}{n!}(n/t)^nL_\tau^{(n)}(n/t)\frac{2n+2}{t}(L_\tau(n/t)A^2x + ACx + V_\tau(n/t)Ax) \\
&+ \frac{(-1)^n}{(n-1)!}(n/t)^{n-1}L_\tau^{(n-1)}(n/t)\frac{n+1}{t}(L_\tau(n/t)A^2x + ACx + V_\tau(n/t)Ax) \\
&= P_n(t)x + Q_n(t)x + Q_n(t)Ax + R_n(t)x + \frac{2n+2}{n}S_{n,\tau}(t)ACx \\
&- \frac{n+1}{n}S_{n-1,\tau}(\frac{n-1}{n}t)ACx.
\end{aligned}
$$

Now, let $M_{n,\tau}(t) = P_n(t)x + Q_n(t)(x+Ax) + R_n(t)x$. Then, using ($a_3$) and the condition (ii) of Theorem 3.1, for $0 < t \le \beta$, we have

$$
\begin{aligned}
\|M_{n,\tau}(t)x\| &\le \|P_n(t)x\| + \|Q_n(t)(x+Ax)\| + \|R_n(t)x\| \\
&\le \frac{M_\tau \cdot M_{\tau,\beta}}{n!\cdot n}(\frac{n\tau}{t})^{n+2}\tau^{-1}e^{-\frac{n\tau}{t}}\|x\| + M_\tau \cdot M_{\tau,\beta}\frac{n+1}{t}e^{-\frac{n\tau}{t}}\|x\| \\
&+ M_\tau \cdot M_{\tau,\beta}\Big(\frac{2(n+1)^2}{t} + \frac{n(n+1)}{t}\Big)e^{-\frac{n\tau}{t}}\|x+Ax\| \\
&+ M_\tau^2(t/n)^2\Big[2(\frac{n+1}{n})^2t + \frac{n+1}{n}t\Big]\|A^2x\|.
\end{aligned}
$$

Moreover (2.5) shows that the first three terms on the left of the above inequality uniformly converges to 0 on $(0,\beta)$ as $n \to \infty$. Therefore, $M_{n,\tau}(t)x$ uniformly converges to 0 on $(0,\beta]$ as $n \to \infty$.

Using arguments similar to those in the proof of Lemma 3.4, we have

Lemma 3.5. *Let $x \in D(A), \beta \in (0, \tau)$. For $t \in (0, \beta]$, we have*

$$\frac{d}{dt} S_{n,\tau}(t)Cx = C_{n,\tau}(t)Cx + N_{n,\tau}(t)x$$

and $N_{n,\tau}(t)x$ uniformly converges to 0 on $(0, \beta]$ as $n \to \infty$.

Lemma 3.6. *Let $x \in D(A), \beta \in (0, \tau)$. Then $S_{n,\tau}(t)Cx - S_{n-1,\tau}(\frac{n-1}{n})Cx$ uniformly converges to 0 on $(0, \beta]$ as $n \to \infty$.*

Proof. Let

$$F(t)Cx = \frac{(-1)^{n-1}}{(n-1)!}(n/t)^{n-1} L_\tau^{(n-1)}(n/t)Cx.$$

Then

$$\begin{aligned} F'(t)Cx &= \frac{(-1)^{n-1}}{(n-1)!}(n/t)^{n-2}\Big(-\frac{(n-1)n}{t^2}\Big) L_\tau^{(n-1)}(n/t)Cx \\ &\quad + \frac{(-1)^{n-1}}{(n-1)!}(n/t)^{n-1}\Big(-\frac{n}{t^2}\Big) L_\tau^{(n)}(n/t)Cx \\ &= S_{n,\tau}(t)Cx - \frac{n-1}{n} S_{n-1,\tau}\Big(\frac{n-1}{n}t\Big)Cx. \end{aligned}$$

Also since $F(t)Cx = \frac{t}{n} S_{n,\tau}(t)Cx$, using Lemma 3.5, we get

$$F'(t)Cx = \frac{1}{n} S_{n,\tau}(t)Cx + \frac{t}{n}(N_{n,\tau}(t)x + C_{n,\tau}(t)Cx).$$

Thus,

$$S_{n,\tau}(t)Cx - S_{n-1,\tau}\Big(\frac{n-1}{n}t\Big)Cx = \frac{t}{n-1}(N_{n,\tau}(t)x + C_{n,\tau}(t)Cx).$$

From condition (iii), for any $n \in N_0, t \in (0, \beta], \|C_{n,\tau}(t)\| \le M_{\tau,\beta}$; the proof is complete.

From Lemmas 3.4–3.6 and the fact that $\|S_{n,\tau}(t)\| \le \frac{n+1}{n}|t|$ for $t \in [-\tau, \tau]$, we obtain

Lemma 3.7. *Let $x \in D(A^2), \beta \in (0, \tau)$. Then for $0 < t \le \beta$ and $n \ge 2$,*

$$\frac{d}{dt} C_{n,\tau}(t)Cx = S_{n,\tau}(t)ACx + M_{n,\tau}(t)x,$$

where $M_{n,\tau}(t)x$ uniformly converges to 0 on $(0, \beta]$ as $n \to \infty$.

In the following, we prove Theorem 3.1.

Proof of Theorem 3.1. The necessity follows from Propositions 2.3 and 2.4. Now, we prove the sufficiency.

In the following we will use the fact that for $t \in (0, \beta]$,

$$\|C_{n,\tau}(t)\| \le M_{\tau,\beta} \quad \text{and} \quad \|S_{n,\tau}(t)\| \le 2\beta M_\tau, \tag{3.11}$$

where $\beta \in (0, \tau)$. The proof is divided into four steps.

Step 1. We prove that $C_{n,\tau}(t)x$ converges to a strongly continuous function on $(-\tau, \tau)$ as $n \to \infty$. Let $0 < s < \beta < \tau$ and take $\varepsilon > 0$ satisfying $0 < \varepsilon \le s \le \tau - \varepsilon < \tau$ and suppose $n, m > |a|\tau$ and $x \in D(A^2)$. By Lemmas 3.5 and 3.7,

$$\frac{d}{ds} C_{n,\tau}(s)Cx = S_{n,\tau}(s)ACx + M_{n,\tau}(s)x \tag{3.12}$$

and

$$\frac{d}{ds} S_{n,\tau}(s)Cx = C_{n,\tau}(s)Cx + N_{n,\tau}(s)x, \tag{3.13}$$

where $M_{n,\tau}(s)x$ and $N_{n,\tau}(s)x$ uniformly converge to 0 on $(0, \beta]$ as $n \to \infty$.

So,

$$\begin{aligned}&\frac{d}{ds}C_{n,\tau}(s)C_{m,\tau}(t-s)Cx\\&=(S_{n,\tau}(s)AC+M_{n,\tau}(s))C_{m,\tau}(t-s)x-C_{n,\tau}(s)(S_{m,\tau}(t-s)ACx+M_{m,\tau}(t-s)x).\end{aligned}$$

Integrating this equality from ε to $t-\varepsilon$ and then letting $\varepsilon\downarrow 0$, we obtain

$$\begin{aligned}&C_{n,\tau}(t)C^2x-C_{m,\tau}(t)C^2x\\&=\int_0^t S_{n,\tau}(s)C_{m,\tau}(t-s)ACxds-\int_0^t C_{n,\tau}(s)S_{m,\tau}(t-s)ACxds\\&\quad+\int_0^t M_{n,\tau}(s)C_{m,\tau}(t-s)xds-\int_0^t M_{m,\tau}(t-s)C_{n,\tau}(s)xds\\&=-\int_0^t S_{n,\tau}(s)dS_{m,\tau}(t-s)ACx-\int_0^t S_{n,\tau}(s)N_{m,\tau}(t-s)ACxds\\&\quad-\int_0^t C_{n,\tau}(s)S_{m,\tau}(t-s)ACxds+\int_0^t M_{n,\tau}(s)C_{m,\tau}(t-s)xds\\&\quad-\int_0^t M_{m,\tau}(t-s)C_{n,\tau}(s)xds\\&=\int_0^t S_{m,\tau}(t-s)N_{n,\tau}(s)ACxds-\int_0^t S_{n,\tau}(s)N_{m,\tau}(t-s)ACxds\\&\quad+\int_0^t M_{n,\tau}(s)C_{m,\tau}(t-s)xds-\int_0^t M_{m,\tau}(t-s)C_{n,\tau}(s)xds.\end{aligned}$$

Thus from (3.11), for $x\in C^2D(A^2)$ and $t\in(0,\beta], C_{n,\tau}(t)x$ uniformly converges on $(0,\beta]$ as $n\to\infty$. Also since $C^2D(A^2)$ is dense in X, it follows that for $x\in X$, $\lim\limits_{n\to\infty}C_{n,\tau}(t)x=C_\tau(t)x$ uniformly for $t\in[0,\beta]$ and $C_\tau(t)x$ is continuous on $[0,\beta]$. By the arbitrariness of β, we see that $C_\tau(t)x$ is continuous on $(0,\tau)$ for $x\in X$. Moreover, by the definition of $C_{n,\tau}(t)$ we have $C_\tau(t)x=C_\tau(-t)x$ for $x\in X$ and $t\in(-\tau,0)$.

Step 2. We prove that for $|s|,|t+s|,|t-s|<\tau$ and $x\in X$,

$$[C_\tau(t+s)+C_\tau(t-s)]Cx=2C_\tau(t)C_\tau(s)x. \tag{3.14}$$

Now, suppose $\beta\in(0,\tau),|t+s|,|t-s|,|s|,|t|\le\beta$ and $x\in D(A^2)$. Let

$$F(r)x=[C_{n,\tau}(t+r)C_{n,\tau}(s-r)+C_{n,\tau}(t-r)C_{n,\tau}(s-r)]Cx,\quad\text{for}\quad 0\le r\le s.$$

Then, by (3.12) and (3.13), we get

$$\begin{aligned}\frac{d}{dr}F(r)x=P_n(r)x&+[S_{n,\tau}(t+r)C_{n,\tau}(s-r)-C_{n,\tau}(t+r)S_{n,\tau}(s-r)]ACx\\&-[C_{n,\tau}(s-r)S_{n,\tau}(t-r)+C_{n,\tau}(t-r)S_{n,\tau}(s-r)]ACx,\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}P_n(r)x=M_{n,\tau}(t+r)C_{n,\tau}(s-r)x&-M_{n,\tau}(s-r)C_{n,\tau}(t+r)x\\&-M_{n,\tau}(t-r)C_{n,\tau}(s-r)x-M_{n,\tau}(s-r)C_{n,\tau}(t-r)x.\end{aligned}$$

Let

$$\begin{aligned}Q_n(r)x&=S_{n,\tau}(s-r)[N_{n,\tau}(t+r)+N_{n,\tau}(r-t)]ACx,\\R_n(r)x&=N_{n,\tau}(s-r)[C_{n,\tau}(t+r)+C_{n,\tau}(r-t)]ACx.\end{aligned}$$

Integrating (3.15) from ε to s and then letting $\varepsilon \downarrow 0$, we get

$$\begin{aligned}
&C_{n,\tau}(t+s)C^2x + C_{n,\tau}(t-s)C^2x - 2C_{n,\tau}(s)C_{n,\tau}(t)Cx \\
&= \int_0^s P_n(r)xdr + \int_0^s C_{n,\tau}(s-r)[S_{n,\tau}(t+r) + S_{n,\tau}(r-t)]ACxdr \\
&\quad - \int_0^s S_{n,\tau}(s-r)[C_{n,\tau}(t+r) + C_{n,\tau}(r-t)]ACxdr \\
&= \int_0^s P_n(r)xdr + \int_0^s C_{n,\tau}(s-r)[S_{n,\tau}(t+r) + S_{n,\tau}(r-t)]ACxdr \\
&\quad + \int_0^s Q_n(r)xds - S_{n,\tau}(s-r)[S_{n,\tau}(r+t) + S_{n,\tau}(r-t)]ACx|_{r=0}^{r=s} \\
&\quad + \int_0^s R_n(r)xdr - \int_0^s C_{n,\tau}(s-r)[S_{n,\tau}(t+r) + S_{n,\tau}(r-t)]ACxdr \\
&\quad + \int_0^s R_n(r)xdr - \int_0^s C_{n,\tau}(s-r)[S_{n,\tau}(t+r) + S_{n,\tau}(r-t)]ACxdr \\
&= \int_0^s [P_n(r)x + Q_n(r)x + R_n(r)x]dr.
\end{aligned}$$

Thus, for $x \in C^2D(A^2)$ and $|s|, |t|, |t+s|, |t-s| \leq \beta$, we have

$$[C_\tau(t+s) + C_\tau(t-s)]Cx = 2C_\tau(t)C_\tau(s)x.$$

Since $C^2D(A^2)$ is dense in X and $\beta \in (0,\tau)$ is arbitrary, we see that (3.14) holds.

Step 3. First we show that for $x \in D(A)$ and $|t| < \tau$,

$$C_\tau(t)x \in D(A), AC_\tau(t)x = C_\tau(t)Ax. \tag{3.16}$$

This can be deduced from (a_4) and (3.4) and the fact that A is closed. Next we prove that for $x \in X, |t| < \tau$,

$$\int_0^t \int_0^s C_\tau(r)xdrds \in D(A)$$

and

$$C_\tau(t)x - Cx = A\int_0^t \int_0^s C_\tau(r)xdrds \tag{3.17}$$

and

$$S_\tau(t) = \int_0^t C_\tau(s)xds, \tag{3.18}$$

where $S_\tau(t)x = \lim\limits_{n\to\infty} S_{n,\tau}(t)$ for $|t| < \tau$ and $x \in X$.

Let $\beta \in (0,\tau)$ and $|t| \leq \beta$ and $x \in D(A^2)$. By Lemmas 3.5 and 3.7, we have $S_\tau(t)Cx = \int_0^t C_\tau(t)Cxdr$ and $C_\tau(t)Cx - C^2x = \int_0^t S_\tau(r)ACxdr$ for $|t| < \beta$ and $x \in CD(A^2)$. By the closedness of A, we have

$$C_\tau(t)Cx - C^2x = A\int_0^t \int_0^s C_\tau(t)Cxdrds$$

and

$$S_\tau(t)Cx = \int_0^t C_\tau(s)xds.$$

Thus by the arbitrariness of $\beta \in (0,\tau)$ and $\overline{CD(A^2)} = X$, we see that (3.17) and (3.18) hold.

Step 4. We define a local **C**-cosine family and prove that A is its complete infinitesimal generator.

Define $C(t)$ on $(-T,T)$ by

$$C(t)x = C_\tau(t)x \quad \text{for} \quad t \in (-\tau,\tau), \tau \in (0,T) \quad \text{and} \quad x \in X.$$

Then $C(t)$ is well defined. In fact, let $\tau_1, \tau_2 \in (0,T), |t| \le \min(\tau_1,\tau_2)$. For $x \in D(A^2)$, by the results of Step 3, we have

$$\begin{aligned}
&C_{\tau_1}(t)x - C_{\tau_2}(t)x \\
&= \int_0^t \frac{d}{ds}[C_{\tau_1}(t-s)C_{\tau_2}(s)]xds \\
&= -\int_0^t AS_{\tau_1}(t-s)C_{\tau_2}(s)xds + \int_0^t C_{\tau_1}(t-s)AS_{\tau_2}(s)xds \\
&= -AS_{\tau_1}(t-s)S_{\tau_2}(s)x|_0^t - A\int_0^t S_{\tau_2}(s)C_{\tau_1}(t-s)xds \\
&\quad + A\int_0^t C_{\tau_1}(t-s)S_{\tau_2}(s)xds \\
&= 0.
\end{aligned} \tag{3.19}$$

Also since $\overline{D(A)} = X$, (3.19) holds for $x \in X$. This means that $C(t)$ is well-defined. Moreover, from (3.10) and (3.14), $\{C(t); |t| < T\}$ is a local **C**-cosine family.

Finally, let G be the operator defined by (2.1). Then we need to show $A = \overline{G}$. Let $x = Cy$, where $y \in D(A)$. From (3.17), we have

$$C(t)C^{-1}x - x = \int_0^t\int_0^s C(r)Aydrds = A\int_0^t\int_0^s C(r)ydrds, \quad \text{for} \quad |t| < \tau.$$

So, $C(t)C^{-1}x$ is twice continuously differentiable on $(-\tau,\tau)$. Thus, $x \in D(G)$ and $Gx = CAy = ACy = Ax$. Therefore, $A|_{CD(A)} = G|_{CD(A)} \subset \overline{G}$ and so $A \subset \overline{G}$ by the condition (iv). Conversely, let $x \in D(\overline{G})$. Then there exists a sequence $\{x_n\}$ in $D(G)$ such that $x_n \to x$ and $Cx_n \to \overline{G}x$ as $n \to \infty$. Since $x_n \in D(G) \subset R(C)$, we have $L_\tau(\lambda)x_n \in R(C)$ by (3.5) and

$$\begin{aligned}
GL_\tau(\lambda)x_n &= (C(t)C^{-1}L_\tau(\lambda)x_n)^{(2)}|_{t=0} \\
&= 2\lim_{h\to 0}\frac{C(h)C^{-1}L_\tau(\lambda)x_n - L_\tau(\lambda)x_n}{h^2} \\
&= L_\tau(\lambda)\cdot 2\lim_{h\to 0} h\frac{C(h)C^{-1}x_n - x_n}{h^2} \\
&= L_\tau(\lambda)Gx_n,
\end{aligned}$$

where we used the fact that $L_\tau(\lambda)C(t)x = C(t)L_\tau(\lambda)x$ by (a_2). In the above equality letting $n \to \infty$, we get $L_\tau(\lambda)x \in D(\overline{G})$ and

$$\overline{G}L_\tau(\lambda)x = L_\tau(\lambda)\overline{G}x \quad \text{for} \quad x \in D(\overline{G}). \tag{3.20}$$

Since $L_\tau(\lambda)x \in D(A)$ for $x \in X$ by (a_3) and $A \subset \overline{G}$, it follows that $AL_\tau(\lambda)x = \overline{G}L_\tau(\lambda)x$ for $x \in X$. By (3.19), we have $\lambda^2 L_\tau(\lambda)\overline{G}x = \overline{G}(\lambda^2 L_\tau(\lambda))x = A(\lambda^2 L_\tau(\lambda)x)$ for $x \in D(\overline{G})$. Also since A is closed, (3.8) implies $Cx \in D(A)$ and $ACx = C\overline{G}x = \overline{G}Cx$. This means that $\overline{G}|_{CD(\overline{G}} = A|_{CD(\overline{G}} \subset A$ and so $\overline{G} \subset A$ by Proposition 2.3. Therefore $\overline{G} = A$.

§4. The Abstract Cauchy Problem and Examples

Let A be a linear operator in a Banach space X. Now, we consider the abstract Cauchy problem on $(-T, T)$,

$$(\mathrm{ACP}, \mathrm{T}) \qquad \begin{cases} \ddot{u}(t) = Au(t), & -T < t < T, \\ u(0) = x, & \dot{u}(0) = y, \end{cases}$$

where $T \in (0, \infty]$.

In order to clarify the relationship of Cauchy problem (ACP,T) and local **C**-cosine family, we introduce the following concept.

Definition 4.1. *Cauchy problem* (ACP,T) *is called* **C***-well-posed if for every* $x, y \in CD(A)$ *there exists a unique solution* $u(t, x, y)$ *to* (ACP,T) *with initial value* $u(0) = x$ *and* $\dot{u}(0) = y$ *such that* $\|u(t, x, y)\| \leq M(t)(\|C^{-1}x\| + \|C^{-1}y\|)$ *for* $t \in (-T, T)$*, where* $M(t)$ *is bounded on every compact subinterval of* $(-T, T)$.

Theorem 4.1. *Let A be a linear operator on X. A is the complete infinitesimal generator of a local* **C***-cosine family* $\{C(t) : -T < t < T\}$ *on X if and only if the following conditions hold:*

(i) *A is closed and* $\overline{D(A)} = X$*;*

(ii) $Cx \in D(A), ACx = CAx$ *for* $x \in D(A)$*;*

(iii) $CD(A)$ *is a core of A;*

(iv) (ACP, T) *is* **C***-well-posed.*

Proof. Necessity. By Propositions 2.3 and 2.4, we get (i)-(iii). For $x, y \in CD(A)$, letting

$$u(t) = C(t)C^{-1}x + \int_0^t C(s)C^{-1}yds, \qquad -T < t < T, \tag{4.1}$$

we see that $u(t)$ is a solution to (ACP, T) by (2.4). To verify the uniqueness, suppose that $u(t)(-T < t < T)$ is a solution to (ACP,T) with $u(0) = 0, \dot{u}(0) = 0$. It is clear that $v(t) = \int_0^t u(s)ds$ is a solution to (ACP,T) with $v(0) = \dot{v}(0) = 0$ and $v(t) \in D(A)$ by the closedness of A and the continuity of $Au(t)$. Define function

$$F(s) = C(t-s)v'(s) + A\int_0^{t-s} C(r)v(s)dr \quad \text{for} \quad 0 \leq s \leq t < T.$$

Since $AC(t)x = C(t)Ax, A\int_0^t C(s)xds = \frac{d}{dt}C(t)x$ for $x \in D(A)$ and $t \in (-T, T)$, we get

$$\begin{aligned} F'(s) &= C(t-s)Av(s) - \frac{d}{ds}C(t-s)v'(s) - C(t-s)Av(s) + \int_0^{t-s} C(r)Av(s)ds \\ &= 0 \quad (0 \leq s \leq t). \end{aligned}$$

So, $F(t) \equiv Cv'(t) = Cu(t) \equiv F(0) = 0$. Since **C** is injective, we see $u(t) \equiv 0$ on $[0, T]$. The same is true for $t \in (-T, 0)$. Thus we prove the uniqueness of solution to (ACP,T). Also from (4.1) we can conclude that (ACP,T) is **C**-well-posed.

Sufficiency. For $x \in CD(A)$, let $u(t, x)(|t| < T)$ be a solution to (ACP,T) with initial value $u(0) = x, \dot{u}(0) = 0$. Write $C(t)x = Cu(t, x)$. Clearly, $Cu(t, x)$ and $u(t, Cx)$ are both the solutions to (ACP,T) with initial value $u(0) = Cx$ and $\dot{u}(0) = 0$. Thus the uniqueness of solution implies that $C(t)x = Cu(t, x) = u(t, Cx)$ and

$$CC(t)x = Cu(t, Cx) = C(t)Cx \qquad \text{for} \qquad x \in CD(A) \quad \text{and} \quad |t| < T. \tag{4.2}$$

For given $x \in X$, since $\overline{D(A)} = X$, there exists $x_n \in D(A)$ such that $x_n \to x$ as $n \to \infty$. Thus, the **C**-well-posedness of (ACP,T) implies that $C(t)x_n = u(t, Cx_n)$ converges uniformaly on compact set of $(-T, T)$. Let $C(t)x = \lim_{n\to\infty} C(t)x_n$ for $x \in X$, where $x_n \in D(A)$ and $x_n \to x$ as $n \to \infty$. Then $C(x)$ is a strongly continuous bounded linear operator function for $t \in (-T, T)$. Also, from (4.2) and $\overline{CD(A)} = X$, we have

$$C(t)Cx = CC(t)x \qquad \text{for} \qquad x \in X \quad \text{and} \quad |t| < T. \tag{4.3}$$

Moreover, for $|t|, |s|, |t+s|, |t-s| < T$ and $x \in C^2D(A)$, emulating the calculation in [7, p.89], we have

$$u(t-s, x) + u(t+s, x) = 2u(t, u(s, x))$$

where $u(t, x)(|t| < T)$ denotes the solution to (ACP,T) with initial value $u(0) = s$ and $\dot{u}(0) = 0$. Thus,

$$C[(t+s) + C(t-s)]Cx = 2C(t)C(s)x. \tag{4.4}$$

Also since $C(t)$ is bounded on $(-T, T)$ and $\overline{C^2D(A)} = X$, (4.4) holds for $x \in X$ and $|t|, |s|, |t+s|, |t-s| < T$. Thus, we obtain a local **C**-cosine family $\{C(t); |t| < T\}$ on X. Let $\overline{G}$ be the complete infinitesimal generator of $\{C(t); |t| < T\}$, where G is defined by (2.1). We need to show $A = \overline{G}$. Let $x \in CD(A)$ and $u(t, x)(|t| < T)$ be the solution to (ACP,T) with initial value $u(0) = x$ and $\dot{u}(0) = 0$. Then $u(t, x) = C(t)C^{-1}x(|t| < T)$. So, $\frac{d^2}{dt^2}C(t)C^{-1}x|_{t=0} = u(t, x)|_{t=0} = Au(0, x) = Ax$. Thus, $x \in D(\overline{G})$ and $Ax = Gx$. Therefore $A|_{CD(A)} \subset \overline{G}$. By condition (ii) we have $A \subset \overline{G}$. Now, we prove $\overline{G} \subset A$. Let

$$L_\tau(\lambda)x = \int_0^\tau e^{-\lambda t} \int_0^t C(r)x\,dr\,dt, \quad x \in X \quad \text{and} \quad \tau \in (0, T). \tag{4.5}$$

We prove that for $x \in X$ and $\tau \in (0, T)$, $L_\tau(\lambda)x \in D(A)$ and

$$AL_\tau(\lambda)x = \lambda^2 L_\tau(\lambda)x - Cx + e^{-\lambda\tau}C(\tau)x + \lambda e^{-\lambda\tau}\int_0^\tau C(t)x\,dt. \tag{4.6}$$

In fact, for $x \in CD(A)$, we have $C(t)x \in CD(A)$ by (4.3) for $t \in (0, \tau)$. So $AC(t)x = \overline{G}C(t)x = C(t)\overline{G}x$ for $x \in CD(A)$. From the closedness of A we deduce that

$$AL_\tau(\lambda)x = L_\tau(\lambda)\overline{G}x \quad \text{for} \quad x \in CD(A).$$

So, by Proposition 2.4 (i), (iii) and (4.5) we see that (4.6) holds for $x \in CD(A)$. Thus from $\overline{CD(A)} = X$ and the closedness of A, it follows that $L_\tau(\lambda)x \in D(A)$ and (4.6) holds for $x \in X$. Letting $x \in D(\overline{G})$, by Porposition 2.4 (ii), (iii) and (4.6), we have $A(\lambda^2 L_\tau(\lambda)x) = \lambda^2 L_\tau(\lambda)Gx$. Also by Proposition 2.4 (i) and (3.8), we get $Cx \in D(A)$ and $ACx = \overline{G}Cx$ or $\overline{G}|_{CD(G)} \subset A$. Observing that $CD(G)$ is a core of $\overline{G}$, we see $A \supset \overline{G}$. Thus $A = \overline{G}$.

Example 4.1. Let m be Lebesgue measure on $\mathcal{C}$, the complex plane. On $L^2(\mathcal{C}, m)$, let $(Af)(\mu) = \mu f(\mu)$ for $f \in D(A) = \{g \in L^2(\mathcal{C}, m); Ag \in L^2(\mathcal{C}, m)\}$. We now define a bounded linear operator family $\{C(t)\}_{t\in R}$ by $C(t) = \frac{1}{2}(e^{tA} + e^{-tA})e^{-|A|^2}$ for $t \in R$. It is easy to see that $\{C(t)\}$ is a **C**-cosine family on $L^2(\mathcal{C},\ m)$ with $C = C(0) = e^{-|A|^2}$. Since for $|t| > 1$,

$$\|C(t)\| = \sup\{\frac{1}{2}|e^{\mu t} + e^{-\mu t}|e^{-|\mu|^2}; \mu \in \mathcal{C}\} \geq \frac{1}{2}(e^{t^2} - 1),$$

$\{C(t)\}_{t\in R}$ is not exponentially bounded. Moreover, it is easy to show that $G = A^2$ is the complete infinitesimal generator of $\{C(t)\}_{t\in R}$ with $\rho_C(G) = \rho(G) = \phi$, where $\rho_C(G) = \{\lambda \in$

$\mathcal{C}$; $(\lambda - G)$ is injective, $R(\lambda - G) \supset R(C)$ and $(\lambda - G)^{-1}C$ is a bounded linear operator} is the **C**-resolvent set G and $\rho(G)$ the resolvent set of G.

Set $\mathcal{A} = \begin{pmatrix} 0 & I \\ G & 0 \end{pmatrix}$. Then $D(\mathcal{A})$ is dense in $L^2(\mathcal{C},m) \oplus L^2(\mathcal{C},m)$ since $D(G)$ is dense in $L^2(\mathcal{C},m)$. Moreover, it is easy to check that $\rho(\mathcal{A}) = \phi$ from $\rho(G) = \phi$. Therefore, by Proposition 4.5 in [17], $\mathcal{A}$ does not generate a local integrated semigroup on $L^2(\mathcal{C},m) \oplus L^2(\mathcal{C},m)$.

In order to make arrangements for the following Example 4.2, we first establish a lemma.

Lemma 4.1. *Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Assume that $A \in B(H)$ satisfies $A = \{a_n\}_{n=1}^{\infty}$ with $a_n \neq a_m$ as $n \neq m$, i.e., $Ax = \sum\limits_{n=1}^{\infty} a_n x_n e_n$ for $x = \sum\limits_{n=1}^{\infty} x_n e_n \in H$. We have*

(i) *if $C \in B(H)$ satisfies $CA = AC$, then $(Ce_n, e_m) = 0$ as $n \neq m$, i.e., $C = \{c_n\}_{n=1}^{\infty}$;*

(ii) *if $\mathcal{B} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in B(H \oplus H)$ satisfies $\mathcal{B}\mathcal{A} = \mathcal{A}\mathcal{B}$, where $\mathcal{A} = \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}$, then $B_1 = B_4 = \{b_{1n}\}_{n=1}^{\infty}, B_3 = \{b_{2n}\}_{n=1}^{\infty}$ and $B_2 = \{a_n b_{2n}\}_{n=1}^{\infty}$.*

Proof. (i) Let $c_{nj} = (Ce_n, e_j),\ n, j = 1, 2, \cdots$. Then $Cx = \sum\limits_{n,j=1}^{\infty} c_{jn} x_j e_n$. Thus

$$CAx = \sum_{n,j=i}^{\infty} c_{jn}(Ax)_j e_n = \sum_{n,j=1}^{\infty} c_{jn} a_j x_j e_n$$

and

$$ACx = A \sum_{n,j=1}^{\infty} c_{in} x_j e_n = \sum_{n,j=1}^{\infty} c_{jn} x_j a_n e_n,$$

and so from $AC = CA$,

$$\sum_{j=1}^{\infty} c_{jn} x_j a_n = \sum_{j=1}^{\infty} c_{jn} a_j x_j, \quad n = 1, 2, \cdots.$$

Setting $x = (x_j) = e_m, m = 1, 2, \cdots$, we obtain $c_{mn} a_m = c_{mn} a_n, n, m = 1, 2, \cdots$. Therefore $c_{mn} = 0$ as $n \neq m$ since $a_n \neq a_m$ as $n \neq m$. So (i) holds.

(ii) We have

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}\begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix} = \begin{pmatrix} B_2 & B_1 A \\ B_4 & B_3 A \end{pmatrix}, \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} AB_3 & AB_4 \\ B_1 & B_2 \end{pmatrix}.$$

Thus from $\mathcal{B}\mathcal{A} = \mathcal{A}\mathcal{B}$, we obtain $B_1 = B_4, B_2 = AB_3 = B_3 A$. Therefore $B_3 = \{b_{2n}\}_{n=1}^{\infty}$ by (i) and $B_2 = AB_3 = \{a_n b_{2n}\}_{n=1}^{\infty}$. Also since $\mathcal{A}^2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\mathcal{B}\mathcal{A}^2 = \mathcal{A}^2\mathcal{B}$ from $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$ we have $B_1 A = AB_1$, and so $B_1 = B_4 = \{b_{1n}\}_{n=1}^{\infty}$ by (i). The proof is complete.

Example 4.2. Let H be the Hilbert space of all sequences $x = \{x_n\}_{n=1}^{\infty}$ of complex numbers such that $\sum\limits_{m=1}^{\infty} |x_m|^2 < \infty$, with the norm $\|x\| = \big(\sum\limits_{m=1}^{\infty} \|x_m\|^2\big)^{1/2}$. Let $T > 2$ and set

$$a_m = \frac{m}{T} + i\big\{\big(\frac{e^m}{m}\big)^2 - \big(\frac{m}{T}\big)^2\big\}^{1/2} \quad \text{for} \quad m \in N,$$

the set of all nature numbers. We define $C(t)$ by

$$C(t)x = \Big\{|a_m|^{-1}\Big(\frac{e^{a_m t} + e^{-a_m t}}{2}\Big)x_m\Big\}_{m=1}^{\infty} \quad \text{for} \quad x = \{x_m\} \in H.$$

Let $C = C(0) = \{|a_m|^{-1}\}_{m=1}^{\infty}$. It is clear that $C \in B(H)$ is injective with $\overline{R(C)} = H$. Observing $|a_m| = m^{-1}e^m$ since $Te^m > m^2$ for $T > 2$ and $m \in N$, we have

$$\frac{m}{2}\Big(e^{m(\frac{|t|}{T}-1)} - e^{-m(\frac{|t|}{T}+1)}\Big) \le \frac{1}{|a_m|}\Big|\frac{e^{a_m t} + e^{-a_m t}}{2}\Big| \le \frac{m}{2}\Big(e^{m(\frac{|t|}{T}-1)} + e^{-m(\frac{|t|}{T}+1)}\Big).$$

Therefore we see that

$$\|C(t)\| = \sup\Big\{|a_m|^{-1}\Big|\frac{e^{\alpha_m t} + e^{-\alpha_m t}}{2}\Big|; m \in N\Big\} < \infty$$

if and only if $|t| < T$. Moreover, it is easy to show that $\{C(t); |t| < T\}$ satisfies (ii) of Definition 2.1, while (iii) of Definition 2.1 can be obtained by a calculation similar to that in [17, p.76]. Hence $\{C(t); |t| < T\}$ is a local **C**-cosine family. Clearly, it cannot be extended to $|t| \ge T$.

A simple calculation shows that $G = \{a_m^2\}_{m=1}^{\infty}$ is the complete infinitesimal generator of $\{C(t); |t| < T\}$ with $D(G) = \{\{a_m^{-2}x_m\}; \{x_m\} \in H\}$. For $x \in CD(G) = \{\{a_m^{-3}f_m\}; \{f_m\} \in H\}$ and $y \in CD(G^{1/2}) = \{\{a_m^{-2}f_m\}; \{f_m\} \in H\}$, the Cauchy problem

$$\begin{cases} \frac{d^2u}{dt^2} = Gu, & t \in (-T, T), \\ u(0) = x, & \dot{u}(0) = y \end{cases} \tag{4.7}$$

has a unique solution

$$u(t) = C(t)C^{-1}x + \int_0^t C(s)C^{-1}y ds$$

which satisfies

$$\|u(t)\| \le M(t)(\|C^{-1}x\| + \|C^{-1}y\|),$$

where $M(t)$ is a locally bounded positive function $(0, T)$ with $M(t) \to \infty$ as $t \to T$.

Let $\mathcal{A} = \begin{pmatrix} 0 & I \\ G & 0 \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in B(H \oplus H)$ be injective. If $\mathcal{A}$ generates a local $\mathcal{B}$-semigroup $T(t)$ on $H \oplus H$ ([17]), then $\mathcal{B}\mathcal{A}u = \mathcal{A}\mathcal{B}u$ for $u \in D(\mathcal{A})$. Obversing $G^{-1} = \{a_m^{-2}\} \in B(H)$ (so $\rho(G) \ne \phi$), we have $\mathcal{A}^{-1} = \begin{pmatrix} 0 & G^{-1} \\ I & 0 \end{pmatrix}$ and $\mathcal{B}\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{B}$. Thus by Lemma 4.1, $B_1 = B_4 = \{b_{1m}\}, B_3 = \{b_{2m}\}$ and $B_2 = \{a_m^{-2}b_{2m}\}$, so

$$\mathcal{B} = \left\{\begin{pmatrix} b_{1m} & a_m^{-2}b_{2m} \\ b_{2m} & b_{1m} \end{pmatrix}\right\}.$$

Moreover, it is easy to check that

$$T(t) = \left\{\begin{pmatrix} b_{1m} & a_m^{-2}b_{2m} \\ b_{2m} & b_{1m} \end{pmatrix}\begin{pmatrix} \frac{1}{2}(e^{a_m t} + e^{-a_m t}) & \frac{1}{2a_m}(e^{a_m t} - e^{-a_m t}) \\ \frac{a_m}{2}(e^{a_m t} - e^{-a_m t}) & \frac{1}{2}(e^{a_m t} + e^{-a_m t}) \end{pmatrix}\right\}.$$

Therefore $T(t)$ is a strongly continuous $\mathcal{B}$-semigroup on $[0, T]$ only if there exists $M > 0$ such that $b_{1m} = \alpha_{1m}a_m^{-2}$, $b_{2m} = \alpha_{2m}a_m^{-1}$ with $|\alpha_{jm}| \le M, j = 1, 2$, and $(\alpha_{1m}^2 - \alpha_{2m}^2) \ne 0$ for $m = 1, 2, \cdots$. By the local **C**-semigroup theory in [17] for

$$\begin{aligned}(x, y)^T \in \mathcal{B}D(\mathcal{A}) &= \left\{\begin{pmatrix} \alpha_{1m}a_m^{-2} & \alpha_{2m}a_m^{-3} \\ \alpha_{2m}a_m^{-1} & \alpha_{1m}a_m^{-2} \end{pmatrix}\begin{pmatrix} a_m^{-2}f_m \\ g_m \end{pmatrix}; (f_m), (g_m) \in H\right\} \\ &= \left\{\begin{pmatrix} \alpha_{1m}a_m^{-4}f_m + \alpha_{2m}a_m^{-3}g_m \\ \alpha_{2m}a_m^{-3}f_m + \alpha_{1m}a_m^{-2}g_m \end{pmatrix}; (f_m), (g_m) \in H\right\},\end{aligned}$$

the Cauchy problem

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, & t \in (0,T), \\ u_1(0) = x, & u_2(0) = y \end{cases} \tag{4.8}$$

has a unique solution

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = T(t)\mathcal{B}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

which satisfies

$$\left\|\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right\| \le M(t)\|\mathcal{B}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\|_{H\oplus H}$$

such that $u_1(t)$ is a solution of (4.7). We show that $\mathcal{B}D(\mathcal{A}) \subset CD(G) \oplus CD(G^{1/2})$. In fact, for any

$$\begin{pmatrix} \alpha_{1m}a_m^{-4}f_m + \alpha_{2m}a_m^{-3}g_m \\ \alpha_{2m}a_m^{-3}f_m + \alpha_{1m}a_m^{-2}g_m \end{pmatrix} \in \mathcal{B}D(\mathcal{A}),$$

the equation

$$\begin{cases} a_m^{-3}x_m = \alpha_{1m}a_m^{-4}f_m + \alpha_{2m}a_m^{-3}g_m, \\ a_m^{-2}y_m = \alpha_{2m}a_m^{-3}f_m + \alpha_{1m}a_m^{-2}g_m \end{cases} \tag{4.9}$$

has a unique solution $\{x_m\} = \{\alpha_{1m}a_m^{-1}f_m + \alpha_{2m}g_m\} \in H$ and $\{y_m\} = \{\alpha_{2m}a_m^{-1}f_m + \alpha_{1m}g_m\} \in H$ for $\{f_m\}, \{g_m\} \in H$, so $\mathcal{B}D(\mathcal{A}) \subseteq CD(G) \oplus CD(G^{1/2})$. Conversely, for given $\{\{a_m^{-3}x_m\}, \{a_m^{-2}y_m\}\} \in CD(G) \oplus CD(G^{1/2})$, the equation (4.9) has a unique solution

$$\{f_m\} = \{(\alpha_{1m}^2 - \alpha_{2m}^2)^{-1}a_m(\alpha_{1m}x_m - \alpha_{2m}y_m)\}$$

and

$$\{g_m\} = \{(\alpha_{1m}^2 - \alpha_{2m}^2)^{-1}(\alpha_{1m}y_m - \alpha_{2m}x_m)\}.$$

Thus $\{f_m\}, \{g_m\} \in H$ for any $\{x_m\}, \{y_m\} \in H$ if and only if there exists $N > 0$ such that $|(\alpha_{1m}^2 - \alpha_{2m}^2)^{-1}a_m\alpha_{jm}| \le N$. So

$$|a_m| \le \min\Big\{N\big|\alpha_{2m}\frac{\alpha_{2m}}{\alpha_{1m}} - \alpha_{1m}\big|, \quad N\big|\alpha_{1m}\frac{\alpha_{1m}}{\alpha_{2m}} - \alpha_{2m}\big|\Big\}.$$

Hence $|\alpha_{1m}\alpha_{2m}^{-1}| \to \infty$ and $|\alpha_{2m}\alpha_{1m}^{-1}| \to \infty$ since $|\alpha_{jm}| \le M, j = 1,2, m = 1,2,\cdots$, a contradiction. Therefore

$$\mathcal{B}D(\mathcal{A}) \subsetneqq CD(G) \oplus CD(G^{1/2}).$$

From the above, the local **C**-cosine family theory cannot be unified by the local **C**-semigroup and the local integrated semigroup theory in [17]. In particular, from application to the second order Cauchy problems, we see that the local **C**-cosine family theory is better than the local **C**-semigroup and the local integrated semigroup theory.

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