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QUASI-CONVEX MULTIOBJECTIVE GAME—SOLUTION CONCEPTS, EXISTENCE AND SCALARIZATION**

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Abstract

This paper deals with the solution concepts, scalarization and existence of solutions for multiobjective generalized game. The scalarization method used in this paper can characterize completely the solutions and be applied to prove the existence of solutions for quasi-convex multiobjective generalized game. On the other hand, a new concept of security strategy is introduced and its existence is proved. At last, some relations between these solutions are established.

Keywords Multiobjective generalized game, Pareto-optimal absolute security strategy, Pareto-optimal weakly efficient security strategy.

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§1. Introduction

A game with vector payoff is called a multiobjective game. Such games are generalizations of classical games and have attracted limited attention in the game theory literature. Recently, certain results have been obtained. Shapley[5] defined the concept of equilibrium of multiobjective game and presented its scalarization. Nieuwenhuis[4] and Tanaka[6] provided some possible generalizations of the notions of minmax, maxmin and saddle points for vector-valued function. Ghose and Prasad[1] gave the concept of Pareto-optimal security strategy and its scalarization. Nevertheless, all of these results are concerned with multiobjective matrix game or convex multiobjective games in the sense of classical Pareto efficiency.

In this paper, we will deal with multiobjective generalized game. A closed pointed convex cone is used as domination structure, rather than the positive orthant of Euclidian space. The cross constraints are admitted and given by set-valued mappings. Besides the solution concepts defined by other authors, a new concept of security strategy is introduced. We present a different scalarization and prove the existence of these solutions and some relations between them.

In Section 2 we review some definitions and results in the multiobjective programming. Sections 3,4,5 are devoted to the equilibrium, the Pareto-optimal absolute security strategy and the Pareto-optimal weakly efficient security strategy respectively.

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§2. Preliminary

For later development we review multiobjective programming briefly.

Let $\Re^p$ be a p-dimensional Euclidian space, $D \subset \Re^p$ a closed pointed convex cone with nonempty interior $\mathrm{Int}D, <, \leqslant$ are cone-orders generated on $\Re^p$ by D, i.e., for $y, y' \in \Re^p$

$$y < y' \ \text{ if } \ y' - y \in \mathrm{Int}D, \quad y \leqslant y' \ \text{ if } \ y' - y \in D.$$

For $Y \subset \Re^p$, $y^* \in Y$ is called a weakly efficient point of Y (w.r.t. D), if there is no $y \in Y$ such that $y < y^*$. The set of all weakly efficient points is denoted by $\mathrm{WMin}(Y; D)$.

Let $X \subset \Re^n$, $f\colon X \twoheadrightarrow \Re^p$ be a set-valued mapping. The multiobjective programming, with X, f, D as feasible set, objective function and domination structure respectively, is denoted by

$$(\mathrm{VP})\colon \begin{array}{l} \mathrm{WMin}(f(x); D) \\ \text{s.t. } x \in X. \end{array}$$

$x^* \in X$ is called a weakly efficient solution of (VP), if $f(x^*) \cap \mathrm{WMin}(f(X); D) \neq \emptyset$.

Lemma 2.1. *If X is compact and $f\colon X \twoheadrightarrow \Re^p$ is an upper semi-continuous set-valued mapping with compact value, then the set of weakly efficient solutions is nonempty.*

Proof. By the property of u.s.c set-valued mapping, $f(X)$ is compact. So $\mathrm{WMin}\,(f(X); D) \neq \emptyset$, and there is $x^* \in X$ such that $f(x^*) \cap \mathrm{WMin}(f(X); D) \neq \emptyset$. x^* is a weakly efficient solution of (VP).

In the remainder of this section we assume that f is point-valued.

For $a \in \Re^p$, $d \in \mathrm{Int}D$, define function $\varphi(\cdot; a, d)\colon \Re^p \to \Re$ by $\varphi(y; a, d) = \inf\{t \in \Re | y \in a + td - D\}$. It is obvious that $\varphi(\cdot; a, d)$ is well defined and continous.

Lemma 2.2. *For multiobjective programming* (VP)*, suppose that there is $a \in \Re^p$ such that $f(X) \subset a + \mathrm{Int}\,D$. Then $x^* \in X$ is a weakly efficient solution of* (VP) *if and only if there is $d \in \hat{D} = \{d \in \mathrm{Int}D | \|d\| = 1\}$, x^* is an optimal solution of the following scalar programming:*

$$(\mathrm{SP})\colon \begin{array}{l} \mathrm{Min}\ h(x) \triangleq \varphi(f(x); a, d) \\ \text{s.t. } x \in X. \end{array}$$

Proof. "$\Rightarrow$" Let $d = (f(x^*) - a)/\|f(x^*) - a\| \in \hat{D}$. Then $h(x^*) = \|f(x^*) - a\|$. If x^* is not a solution of (SP), then there is $\bar{x} \in X$ satisfying $h(\bar{x}) < h(x^*)$. Let $t = \frac{1}{2}(h(\bar{x}) + h(x^*))$. Then $f(\bar{x}) \in a + td - D$. Therefore $f(\bar{x}) \leqslant a + td < a + h(x^*)d = f(x^*)$, a contradiction.

"$\Leftarrow$" If x^* is not a weakly efficient solution of (VP), then there is $\bar{x} \in X$ satisfying $f(\bar{x}) < f(x^*)$. It is easy to verify that $h(\bar{x}) < h(x^*)$ for any $d \in \hat{D}$. Hence x^* is not a solution of (SP).

A vector-valued function $f\colon \Re^n \to \Re^p$ is called $D-$quasi-convex, if for any $y \in \Re^p$ the level set $L_f(y) = \{x \in \Re^p | f(x) \leqslant y\}$ is a convex set.

Lemma 2.3. *If $f\colon \Re^n \to \Re^p$ is D-quasi-convex, then the composite function $h = \varphi(f; a, d)$ defined above is quasi-convex in usual sense.*

Proof. It follows immediately from $L_h(t) = \{x \in \Re^n | h(x) \leqslant t\} = \{x | f(x) \in a + td - D\} = L_f(a + td)$.

In fact, the inverse proposition of Lemma 2.2 is true too.

If D can be represented by $D = \{y \in \Re^p | \langle c^i, y \rangle \geqslant 0, i \in I\}$, where $\{c^i | i \in I\} \subset \Re^p$ and I is an index set, then $\varphi(y; a, d) = \sup\{\langle c^i, y - a\rangle / \langle c^i, d\rangle | i \in I\}$. In particular, for $D = \Re^p_+ = \{y = (y_1, \cdots, y_p) | y_i \geqslant 0, i = 1, \cdots, p\}$, $\varphi(y; a, d) = \mathrm{Max}\{(y_i - a_i)/d_i | i = 1, \cdots, p\}$ where $a = (a_1, \cdots, a_p)$, $d = (d_1, \cdots, d_p)$.

We need the following theorem proved by Berge [2].

Lemma 2.4. *Suppose that $X \subset \Re^n$, $Y \subset \Re^p$, Y is compact, $g: X \times Y \to \Re$ is continuous and $\theta: X \twoheadrightarrow Y$ is a continuous set-valued mapping with compact value. Define $\mu: X \twoheadrightarrow Y$ and $G: X \to \Re$ as $\mu(x) = \{y \in \theta(x) | g(x, y) = \underset{y' \in \theta(x)}{\mathrm{Min}}\, g(x, y')\}$, $G(x) = \underset{y \in \theta(x)}{\mathrm{Min}}\, g(x, y)$. Then μ and G are u.s.c and compact-valued. Furthermore, if $G(x)$ is bounded, then G is continuous.*

§3. The Equilibrium of Multiobjective Generalized Game

A multiobjective generalized game (MOGG) can be described as follows. There are n players, $N = \{1, \cdots, n\}$ is the set of players. For $i \in N$, $X_i \subset \Re^{n(i)}$ and

$$f_i = (f_{i1}, \cdots, f_{ip(i)}): X = \prod_{i \in N} X_i \to \Re^{p(i)}$$

are the strategy set and vector payoff function of player i respectively. The domination structure of player i is a cone $D_i \subset \Re^{p(i)}$. For a generalized game, strategy $x_i \in X_i$ can not be chosen independently and is restricted by the strategies taken by other players. Let $\theta_i: X_{-i} = \prod_{j \in N \setminus \{i\}} X_j \twoheadrightarrow X_i$ be a set-valued mapping, player i can take a strategy $x_i \in \theta_i(x_{-i})$ when other players take strategies $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{-i}$. Every player want to minimize his vector payoff w.r.t his domination structure. MOGG, with N, $\{X_i\}$, $\{f_i\}$, $\{D_i\}$, $\{\theta_i\}$ as set of players, strategy sets, vector payoffs, domination structures and constraints respectively, is denoted by $\Gamma = \{N, X_i, f_i, D_i, \theta_i\}$.

When all $p(i) = 1$, Γ is an abstract economy (Debreu).

When all $\theta_i(x_{-i}) \equiv X_i$, Γ is a conventional multiobjective game (MOG), and denoted by $\Gamma = \{N, X_i, f_i, D_i\}$.

In this paper we make following assumptions: For each $i \in N$

i) X_i is compact,

ii) f_i is continuous,

iii) θ_i is a continuous set-valued mapping with compact-value,

iv) D_i is a closed pointed convex cone with nonempty interior.

Definition 3.1. *$x^* = (x_1^*, \cdots, x_n^*) \in X$ is called an equilibrium of Γ, if for each $i \in N$, $x_i^* \in \theta_i(x_{-i}^*)$ and $f_i(x^*) \in \mathrm{WMin}(f_i(\theta_i(x_{-i}^*), x_{-i}^*); D_i)$.*

Take $a = (a_i, \cdots, a_n) \in \Re^N \triangleq \prod_{i \in N} \Re^{p(i)}$ such that $f_i(X) \subset a_i + \mathrm{Int}D_i$ for each $i \in N$. For any $d = (d_1, \cdots, d_n) \in \hat{D} \triangleq \prod_{i \in N} \hat{D}_i$ ($\hat{D}_i = \{d_i \in \mathrm{Int}D_i | \|d_i\| = 1\}$), define a generalized game with scalar payoff $\Gamma(d) = \{N, X_i, f_i(\cdot; d_i), \Re_+, \theta_i\}$, where $f_i(\cdot; d_i): X \to \Re$ is given by $f_i(x; d_i) = \varphi(f_i(x); a_i, d_i)$.

The equilibrium of MOGG Γ can be scalarized as follows:

Theorem 3.1. *$x^* = (x_1^*, \cdots, x_n^*)$ is an equilibrium of Γ if and only if there is $d \in \hat{D}$ such that x^* is an equilibrium of $\Gamma(d)$.*

Proof. $x^* = (x_1^*, \cdots, x_n^*)$ is an equilibrium of Γ

$\Leftrightarrow f_i(x^*) \in \mathrm{WMin}(f_i(\theta_i(x^*_{-i}), x^*_{-i}); D_i)$ for $i \in N$

$\Leftrightarrow$ There is $d_i \in \hat{D}_i$ such that x^*_i is an optimal solution of the following programming: $\underset{x_i \in \theta_i(x^*_{-i})}{\mathrm{Min}} f_i(x_i, x^*_{-i}; d_i)$ for $i \in N$ (by Lemma 2.2).

$\Leftrightarrow x^*$ is an equilibrium of $\Gamma(d)$ for $d = (d_1, \cdots, d_n) \in \hat{D}$.

A theorem on the existence of equilibrium of Γ is in order.

Theorem 3.2. *Γ has an equilibrium if for each $i \in N$, X_i is convex and f_i is D_i-quasi-convex in x_i, θ_i is convex-valued.*

Proof. By Theorem 3.1, it is sufficient to prove the existence of equilibrium of $\Gamma(d)$ for a $d \in \hat{D}$. Take any $d = (d_1, \cdots, d_n) \in \hat{D}$ and define set-valued mappings $\mu_i: X \twoheadrightarrow X_i (i \in N)$ and $\mu: X \twoheadrightarrow X$ by

$$\mu_i(x_i, x_{-i}) = \{x'_i \in \theta_i(x_{-i}) | f_i(x'_i, x_{-i}; d_i) = \mathrm{Min}\, f_i(\theta_i(x_{-i}), x_{-i}; d_i)\}, \ \ \mu(x) = \prod_{i \in N} \mu_i(x).$$

It is easy to see that μ is a u.s.c set-valued mapping with convex compact value by Lemma 2.3 and Lemma 2.4. From Kakutani fixed point theorem [2], there is $x^* \in X$ such that $x^* \in \mu(x^*)$. Hence x^* is an equilibrium of $\Gamma(d)$. In particular, when $D_i = \Re^{p(i)}_+$ and $d_i = (d_{i1}, \cdots, d_{ip(i)})$, $a_i = (a_{i1}, \cdots, a_{ip(i)})$, the payoff function of player i in $\Gamma(d)$ is

$$f_i(x; d_i) = \mathrm{Max}\{d^{-1}_{ij}(f_{ij}(x) - a_{ij}) | j = 1, \cdots, p(i)\}.$$

§4. Pareto-Optimal Absolute Security Strategy

In this section we are concerned with conventional multiobjective game, i.e., $D_i = \Re^{p(i)}_+$ and $\theta_i(x_{-i}) \equiv X_i$ for each $i \in N$.

The following solution concept was introduced in [2].

Definition 4.1. *For $i \in N$, define $\bar{f}_i: X_i \to \Re^{p(i)}$ by $\bar{f}_i = (\bar{f}_{i1}, \cdots, \bar{f}_{ip(i)})$, where $\bar{f}_{ij}(x_i) = \mathrm{Max}\, f_{ij}(x_i, X_{-i})$. $\bar{f}_i(x_i)$ is called an absolute security payoff vector* (ASPV) *of x_i for player i, x^*_i is called a Pareto-optimal absolute security strategy* (POASS) *for player i, if $\bar{f}_i(x^*_i) \in \mathrm{WMin}(\bar{f}_i(X_i); D_i)$.*

If x^* is a POASS, then there is not $x_i \in X_i$ whose ASPV is better than that of x^* and independent of the actions of other players.

POASS can be scalarized as follows. If $a = (a_1, \cdots, a_n) \in \Re^N$ such that $f_i(X) \subset a_i + \mathrm{Int}\Re^{p(i)}_+$, take $d = (d_1, \cdots, d_n) \in \hat{D}$ and define $\tilde{f}_i(\cdot; d_i): X_i \times (X_{-i})^{p(i)} \to \Re$ and $g_i(\cdot; d_i): X_i \to \Re$ by $\tilde{f}_i(x_i, \bar{x}_{-i}; d_i) = \mathrm{Max}\{d^{-1}_{ij}(f_{ij}(x_i, x^j_{-i}) - a_{ij}) | j = 1, \cdots, p(i)\}$

$$g_i(x_i; d_i) = \mathrm{Max}\, \tilde{f}_i(x_i, (X_{-i})^{p(i)}; d_i),$$

where $\bar{x}_{-i} = (x^1_{-i}, \cdots, x^{p(i)}_{-i}) \in (X_{-i})^{p(i)}$.

Definition 4.2. *$x^*_i \in X_i$ is called a Minimax strategy for player i w.r.t. d_i if $g_i(x^*_i; d_i) = \mathrm{Min}\, g_i(X_i; d_i)$.*

The name "Minimax strategy" comes from a two person zero-sum game with $\tilde{f}_i(\cdot; d_i)$, X_i, $(X_{-i})^{p(i)}$ as payoff and strategy sets respectively.

Theorem 4.1. *If x^* is a Minimax strategy for player i w.r.t. d_i, then x^* is a* POASS.

Proof. Suppose to the contrary, then there is $x'_i \in X_i$ such that $\bar{f}_{ij}(x'_i) < \bar{f}_{ij}(x^*_i)$ for each $j = 1, \cdots, p(i)$.

Observing that

$$g_i(x_i;d_i) = \mathrm{Max}\,\tilde{f}_i(x_i,(X_{-i})^{p(i)};d_i) = \underset{\bar{x}_{-i}}{\mathrm{Max}}\,\underset{j}{\mathrm{Max}}\{d_{ij}^{-1}(f_{ij}(x_i,x_{-i}^j)-a_{ij})\}$$
$$= \underset{j}{\mathrm{Max}}\,\underset{x_{-i}\in X_{-i}}{\mathrm{Max}}\{d_{ij}^{-1}(f_{ij}(x_i,x_{-i})-a_{ij})\} = \underset{j}{\mathrm{Max}}\{d_{ij}^{-1}(\bar{f}_{ij}(x_i)-a_{ij})\},$$

we have $g_i(x_i';d_i) < g_i(x_i^*;d_i)$, contradicting the assumption that x_i^* is a Minimax strategy.

Theorem 4.2. *If x_i^* is a* POASS*, then there is $d_i\in \mathrm{Int}\Re_+^{p(i)}$ such that x_i^* is a Minimax strategy w.r.t. d_i.*

Proof. Let $d_i = (\bar{f}_i(x_i^*)-a_i)/\|\bar{f}_i(x_i^*)-a_i\| \in \mathrm{Int}\Re_+^{p(i)}$, $h(x_i) = \mathrm{Max}\{d_{ij}^{-1}(\bar{f}_{ij}(x_i)-a_{ij})|j=1,\cdots,p(i)\}$. Then x_i^* is an optimal solution of the following programming: $\underset{x_i\in X_i}{\mathrm{Min}}\, h(x_i)$ by Lemma 2.2.

Since $h(x_i) = g_i(x_i;d_i)$ from the proof of Theorem 4.1 we have $g_i(x_i^*;d_i) = \mathrm{Min}\, g_i(X_i;d_i)$, i.e, x_i^* is a Minimax strategy w.r.t. d_i.

Theorem 4.3. *For each $i\in N$ there is a* POASS *of player i.*

Proof. It is immediate from Theorem 4.1, the continuity of $\tilde{f}_i(\cdot;d_i)$, $g_i(\cdot;d_i)$ and the compactness of X_i.

By the way, we obtain a theorem on the structure of set of POASS under the convexity condition.

Theorem 4.4. *If X_i is convex and f_i is quasi-convex in x_i, then the set of* POASS *is nonempty, closed and connected.*

Proof. Since $\bar{f}_{ij}(x_i) = \mathrm{Max}\, f_{ij}(x_i,X_{-i})$ is quasi-convex, the conclusion follows from Theorems 1.1 and 4.6 (ch.6) in [3].

§5. Pareto-Optimal Weakly Efficient Security Strategy

For a multiobjective game $\Gamma = \{N,X_i,f_i,D_i\}$, another concept about security strategy can be introduced.

Definition 5.1. *For* MOG Γ, *define set-valued mapping $\mathring{f}_i\colon X_i \twoheadrightarrow \Re^{p(i)}$ as*

$$\mathring{f}_i(x_i) = \mathrm{WMax}(f_i(x_i,X_{-i});D_i) \triangleq \mathrm{WMin}(f_i(x_i,X_{-i});-D_i),$$

x_i^* is called a Pareto-optimal weakly efficient security strategy (POWESS) *of player i, if $\mathring{f}_i(x_i^*)\cap \mathrm{WMin}(\mathring{f}_i(X_i);D_i) \neq \emptyset$, $y_i^* \in \mathring{f}_i(x_i^*)\cap \mathrm{WMin}(\mathring{f}_i(X_i);D_i)$ is called a* POWESS's *payoff vector of player i for x_i^*.*

In order to prove the existence of POWESS, the following lemma is needed.

Lemma 5.1. *Suppose that $f\colon X\times Y\to\Re^p$ is continuous, Y is compact and $D\subset\Re^p$ is a closed pointed convex cone with nonempty interior. Define $\mu\colon X\twoheadrightarrow Y$ and $F\colon X\twoheadrightarrow \Re^p$ by*

$$\mu(x) = \{y\in Y|f(x,y)\in \mathrm{WMax}(f(x,Y);D)\},\quad F(x) = f(x,\mu(x)).$$

Then μ and F are u.s.c set-valued mappings with compact value.

Proof. Let $x^k, x^0\in X$, $x^k\to x^0$, $z^k\in\mu(x^k)$, $z^k\to z^0$. Then there are $y^k\in Y$ such that $z^k = f(x^k,y^k)\in \mathrm{WMax}(f(x^k,Y);D)$. Without loss of generality, we can assume that $y^k\to y^0\in Y$. So $z^0 = \lim z^k = \lim f(x^k,y^k) = f(x^0,y^0)$.

If $z^0\notin\mu(x^0)$, then there is $\bar{y}\in Y$ such that $f(x^0,y^0) < f(x^0,\bar{y})$.

Since Int$D$ is open and f is continuous, we have $f(x^k,y^k) < f(x^k,\bar{y})$ for k sufficiently large. It contradicts $z^k\in\mu(x^k)$.

In particular, taking $x^k \equiv x$, and $z^k \in \mu(x)$, we see that $\mu(x)$ is closed. Hence μ is u.s.c with compact value.

Now, it is obvious that F has the same properties.

Theorem 5.1. *For each $i \in N$, there exists a* POWESS *of player i.*

Proof. Since $\overset{\circ}{f}_i\colon X_i \twoheadrightarrow \Re^{p(i)}$ is u.s.c with compact value by Lemma 5.2, the conclusion follows from Lemma 2.4.

When $D_i = \Re_+^{p(i)}$, for each $i \in N$, there are two different concepts of security strategy: POASS and POWESS. The following relation between them holds.

Theorem 5.2. *For* MOG $\Gamma = \{N, X_i, f_i, \Re_+^{p(i)}\}$*, if x_i^* is a* POWESS *and $\hat{x}_i$ is a* POASS *of player i, $y_i^* \in \overset{\circ}{f}_i(x_i^*) \cap \mathrm{WMin}(\overset{\circ}{f}_i(X_i); \Re_+^{p(i)})$, $\quad \hat{y}_i = \bar{f}_i(\hat{x}_i)$. Then $y_i^* \not> \hat{y}_i$.*

Proof. By the definition of POASS, $\overset{\circ}{y}_i = \bar{f}_i(\overset{\circ}{x}_i) \geqslant y_i$ for any $y_i \in \overset{\circ}{f}_i(\hat{x}_i)$. If $y_i^* > \hat{y}_i$, then $y_i^* > \hat{y}_i \geqslant y_i$ for any $y_i \in \overset{\circ}{f}_i(\hat{x}_i)$. It contradicts the assumption that x_i^* is a POWESS.

Theorem 5.2 shows that the payoff of POWESS is not worse than that of POASS.

At last, we discuss the relationship between the POWESS's payoff and the payoff of equilibrium in a zero-sum MOG of two players.

Theorem 5.3. *Let $\Gamma = \{\{1,2\}, X_i, f_i, D_i\}$ be a zero-sum* MOG *of two players, where $f_2 = -f_1$ and $D_1 = D_2$. If (x_1^*, x_2^*) is an equilibrium of Γ, then there are* POWESS $\hat{x}_1, \hat{x}_2$ *and* POWESS*'s payoffs $\hat{y}_1, \hat{y}_2$ of player* 1 *and* 2 *respectively such that*

$$\hat{y}_1 \leqslant f_1(x_1^*, x_2^*), \quad \hat{y}_2 \leqslant f_2(x_1^*, x_2^*).$$

Proof. From Theorem 4.3 in [6], since (x_1^*, x_2^*) is a saddle point of f_1,

$$f_1(x_1^*, x_2^*) \in \underset{x_1 \in X_1}{\mathrm{WMin}}(\underset{x_2 \in X_2}{\mathrm{WMax}}(f_1(x_1, x_2); D_1); D_1) + D_1 = \underset{x_1 \in X_1}{\mathrm{WMin}}(\overset{\circ}{f}_1(x_1); D_1) + D_1.$$

So, there is $\hat{x}_1 \in X_1$ and $\hat{y}_1 \in \overset{\circ}{f}_1(\hat{x}_1) \cap \mathrm{WMin}(\overset{\circ}{f}_1(X_1); D_1)$ such that $\hat{y}_1 \leqslant f_1(x_1^*, x_2^*)$.

On the other hand,

$$\begin{aligned} f_1(x_1^*, x_2^*) &\in \underset{x_2 \in X_2}{\mathrm{WMax}}(\underset{x_1 \in X_1}{\mathrm{WMin}}(f_1(x_1, x_2); D_1); D_1) - D_1 \\ &= -\underset{x_2 \in X_2}{\mathrm{WMin}}(\underset{x_1 \in X_1}{\mathrm{WMax}}(f_2(x_1, x_2); D_2); D_2) - D_2. \end{aligned}$$

So, there is $\overset{\circ}{x}_2 \in X_2$ and $\hat{y}_2 \in \overset{\circ}{f}_2(\hat{x}_2) \cap \mathrm{WMin}(\overset{\circ}{f}_2(X_2); D_2)$ such that $\hat{y}_2 \leqslant f_2(x_1^*, x_2^*)$.

Obviously, $\hat{x}_1, \hat{x}_2$ are POWESS.

Theorem 5.3 shows that, loosely speaking, the POWESS's payoff is better than the payoff of equilibrium.

For the classical game with scalar payoff, the POASS and the POWESS are coincident and their payoff is equal to the payoff of equilibruim.

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