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ABELIAN 3-FOLDS IN PRODUCTS OF PROJECTIVE SPACES

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Abstract

This paper deals with the existential problem of abelian 3-folds in products of projective spaces–$P_1 \times P_4$ and $P_2 \times P_3$. The answer to this problem is negative.

Keywords Abelian 3-fold, Embedding, Projective space.
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§0. Introduction

It is known that there exists no abelian 3-folds in P_5. To prove the existence of abelian varieties in projective spaces is a much harder problem. In this paper we shall investigate the existence of abelian 3-folds in $P_2 \times P_3$ and $P_1 \times P_4$. Again this problem falls into two parts. Here we shall show that there exists no abelian 3-folds in the two spaces in both cases. This follows mostly from the self-intersection formula, but not entirely.

§1. Preliminaries

If $Z = X \times Y$ is a product, we denote the canonical projections by p and q respectively:

$$\begin{array}{ccc} & Z = X \times Y & \\ & p \swarrow \quad \searrow q & \\ X & & Y \end{array}$$

If L and μ are line bundles on X and Y respectively, we set

$$L \otimes \mu := p^* L \otimes q^* \mu.$$

In particular, if $X = P_k$ and $Y = P_n$, we set

$$O_Z(a, b) := O_{P_k}(a) \otimes O_{P_n}(b).$$

We denote the class of $O_Z(1, 0)$ (resp. $O_Z(0, 1)$) in $H^2(Z, I)$ by h_1 (resp. h_2), where I denotes the ring of integral numbers.

Lemma 1.1. *Let X be a non-simple abelian 3-fold. Then there is no rational curves on X.*

Proof. We want to deduce a contradiction by the assumption that there exists a rational curve C on X. By the Poincaré's complete reducibility theorem (Theorem 1[4]), there is an abelian subvariety Z such that $E \cap Z$ is finite and $E + Z = X$, where E is an elliptic curve on X. In other words X is isogenous to $E \times Z$. Let $f : E \times Z \longrightarrow X$ be the isogenous

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map. We use G to denote the kernal of f. Then $E \times Z/G \cong X$. Thus C is a rational curve on $E \times Z/G$. We use p_1 and p_2 to denote the first and the second projections of $E \times Z/G$ respectively. Since E/G is an elliptic curve, $p_1(C)$ is just a point of E/G. Therefore we may regard C as a rational curve on Z/G. But Z/G is an abelian surface, and a abelian surface does not contain curves with negative self-intersection. Thus we reach a contradiction.

Proposition 1.1. *Let C be a curve. Then the products $C \times P_3$ does not contain an abelian 3-fold.*

Proof. First assume $g(C) \geq 2$. Then the assertion is obvious since the projection $X \to C$ must be surjective and this would imply the existence of a non-constant 1-form on X.

Case 1. $g(C) = 0$.

Then $C = P_1$ and

$$O_{P_1 \times P_3}(X) = O(a, b)$$

for some $a, b > 0$. By the adjunction formula

$$\omega_X = O_X(a-2, b-4).$$

Since X is abelian, $\omega_X = O_X$. This implies

$$(ah_1 + bh_2)((a-2)h_1 + (b-4)h_2)h_i = 0, \qquad (i = 1, 2)$$

i.e.,

$$b(b-4) = 0, \qquad (i = 1)$$

$$a(b-4) + b(a-2) = 0. \qquad (i = 2)$$

It follows that $b = 4$ and $a = 2$. On the other hand we get from

$$0 \to O(-2, -4) \to O \to O_X \to 0$$

an exact sequence

$$H^1(O) \to H^1(O_X) \to H^2(O(-2, -4)).$$

By the Serre-duality theorem and the Künneth formula

$$h^2(O(-2, -4)) = h^1(O) = 0.$$

Hence $h^1(O_X) = 0$, which is a contradiction.

Case 2. $g(C) = 1$.

Here our argument is very similar. By ([2, p.292])

$$\mathrm{Pic}(C \times P_3) = \mathrm{Pic}C \times \mathrm{Pic}P_3,$$

i.e., we can write

$$O_{C \times P_3}(X) = L \otimes O(b)$$

for some $b > 0$ and $L \in \mathrm{Pic}C$. Since

$$\omega_{C \times P_3} = O_C \otimes O(-4),$$

the adjuction formula gives

$$\omega_X = L \otimes O(b-4)|_X.$$

Let a =deg L. Then arguing as before we find

$$b(b-4)=o, \quad ab+a(b-4)=0,$$

i.e., $b=4, a=0$.

Since

$$H^0(L\otimes O(4))=H^0(L)\otimes H^0(O(4)),$$

it follows that $L=O_C$ and $X=C\times S$, where S is a quartic surface in P_3, i.e., S is a $K3$ surface. On the other hand, C is an abelian subvariety of the abelian variety X since C is an elliptic curve. So $S\cong X/_C$ is an abelian surface. Then we reach a contradiction that S is both an abelian surface and a $K3$ surface.

Finally we recall the self-intersection formula from [1, p.103]. For any regular embedding $i: X\to Z$ of codimension d with normal bundle $N_{X/Z}$,

$$i^*i_*[\alpha]=c_d(N_{X/Z})\cap[\alpha]$$

for all $\alpha\in A_*(X)$.

In particular, if X is a three-fold in a 5-manifold Z, then

$$[X]^2=c_2(N_{X/Z}).[X].$$

§2. Existence of Abelian 3-Folds in P_5

(see [2, Ex. 6.10, p.437])

In this section we shall prove

Proposition 2.1. *There exists no abelian 3-folds in P_5.*

Remark. Though the result is well known, we still prove it here in order to keep our question as a whole.

Proof. We shall deduce a contradiction by the assumption that there is an abelian 3-fold X in P_5.

Let h be the class of $O_{P_5}(1)$ in $H^2(P_5,\mathbb{Z})$. The class of X is of the form $[X]=ah^2$ with integers $a>0$. From the normal bundle sequence

$$o\to T_X\to T_{P_5}|_X\to N_{X/P_5}\to 0$$

and the fact that T_X is trivial, one finds

$$c(N_{X/P_5})=c(T_{P_5}|_X)=(1+6h+15h^2+20h^3+15h^4+6h^5).[X]$$

$$c_2(N_{X/P_5})=15ah^4.$$

Since $[X]^2=a^2h^4$, the self-intersection formula implies $15a=a^2$. So $a=15$, i.e., the degree of X in P_5 is 15. Let $L=O_X(1)$, and H be a hyperplane section of X. Then by the Riemman-Roth theorem of abelian varieties,

$$h^0(L)=\frac{H^3}{6}=\frac{15}{6}=\frac{5}{2},$$

which is a contradiction.

§3. Existence of Abelian 3-Folds in $P_2\times P_3$

In this section, we shall prove the following result.

Proposition 3.1. *There exists no abelian 3-folds in $P_2 \times P_3$.*

Proof. Assume that X is an abelian 3-fold in $P_2 \times P_3$. We shall deduce a contradiction. Let the class of X be

$$[X] = \alpha h_1^2 + \beta h_2^2 + \gamma h_1 h_2$$

with non-negative integers α, β and γ. As before we want to make use of the self-intersection formula. From

$$c(N_{X/(P_2 \times P_3)}) = c(T_{P_2} \times T_{P_3}|_X) = (1 + 3h_1 + 3h_1^2)(1 + 4h_2 + 6h_2^2 + 4h_2^3).[X]$$

we get

$$c_2(N_{X/(P_2 \times P_3)}) = (6\gamma + 12\beta)h_1 h_2^3 + (6\alpha + 12\gamma + 3\beta)h_1^2 h_2^2.$$

Since

$$[X]^2 = 2\beta\gamma h_1 h_2^3 + (\gamma^2 + 2\alpha\beta)h_1^2 h_2^2,$$

the self-intersection formula implies

$$3\gamma + 6\beta = \beta\gamma, \tag{3.1}$$

$$6\alpha + 3\beta + 12\gamma = \gamma^2 + 2\alpha\beta. \tag{3.2}$$

By (3.1), we have

$$\gamma = \frac{6\beta}{\beta - 3}. \tag{3.3}$$

Since $\alpha, \beta, \gamma \geq 0$, we have $\beta > 3$ or $\beta = 0$.

Case 1. $\beta > 3$.

By (3.2) and (3.3) we have

$$2(\beta - 3)\alpha = \frac{3\beta}{(\beta - 3)^2}[\beta^2 + 6\beta - 63]. \tag{3.4}$$

Since α, β and $\beta - 3$ are non-negative, we have $\beta^2 + 6\beta - 63 \geq 0$. Thus $\beta \geq 5$.

Similarly, by (3.1) we have

$$\beta = \frac{3\gamma}{\gamma - 6}. \tag{3.5}$$

By (3.2) and (3.3) we have

$$2\alpha(\beta - 3) = 3\beta + 12\gamma - \gamma^2 = \frac{9\gamma}{\gamma - 6} + 12\gamma - \gamma^2 = -\frac{\gamma}{\gamma - 6}(\gamma^2 - 18\gamma + 63).$$

With the same reason as before, we have $\gamma^2 - 18\gamma + 63 \leq 0$. Thus $7 \leq \gamma \leq 13$. But $\gamma = 13$ is impossible by (3.5). After trivial discussion, we have only one possibility for $\alpha = 3, \beta = 6$ and $\gamma = 12$.

Next we will deduce a contradiction for this one possibility. The projection onto the second factor gives a surjective map $q|_X : X \to P_3$ of degree α. On the other hand, the projection onto the first factor gives a surjective map $p|_X : X \to P_2$, whose fibres are curves. Then by the proposition (see [4, p.88]), we know that X is not a simple abelian varieties. Thus by Lemma 1.1, X does not contain rational curves. It follows that $q|_X$ is finite. Hence the Nakai-Moishezon criterion implies that

$$O_X(0, 1) = (q|_X)^* O_{P_3}(1)$$

is ample. By the Kodaira vanishing theorem

$$h^1(O_X(0,1)) = h^2(O_X(0,1)) = h^3(O_X(0,1)) = 0.$$

Therefore, the Riemman-Roth theorem gives

$$h^0(O_X(0,1)) = \frac{3}{6} = \frac{1}{2},$$

which leads to a contradiction. Thus Case 1 can not occure.

Case 2. α, β and γ are all 0.

In this case the projection onto the second factor gives a map $X \to D \subset P_2$, where D is a (possibly singular) curve. Let $v : \widetilde{D} \to D$ be the normalization map. Let D_0 be the smooth part of D and let X_0 be the open set of X which lies over D_0. Since $v \times id$ is an isomorphism away from the singularities of D, we can consider X_0 to be a subset of $\widetilde{D} \times P_3$. Let $\widetilde{X}$ be its Zariski-closure. Then we have a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{D} \times P_3 \\ g \downarrow & & \downarrow v \times id \\ X & \longrightarrow & D \times P_3 \end{array}$$

By constructions g is finite and birational. Since X is smooth, it follows from [5, Theorem 5, p.115] that g is an isomorphism. By Proposition 1.1 it leads to a contradiction. Therefore, Case 2 can not occure either.

§4. Existence of Abelian 3-Folds in $P_1 \times P_4$

Here we prove

Proposition 4.1. *There exists no abelian 3-folds in* $P_1 \times P_4$.

Proof. We will deduce a contradiction by the assumption that there is an abelian 3-fold X in $P_1 \times P_4$.

Let the class of X be $[X] = \alpha h_1 h_2 + \beta h_2^2$. Then $\alpha, \beta \geq 0$. As before we want to make use of the self-intersection formula. From

$$c(N_{X/(P_1 \times P_4)}) = c(T_{P_1} \times T_{P_4}|_X) = (1 + 2h_1)(1 + 5h_2 + 10h_2^2 + 10h_2^3 + 5h_2^4).[X],$$

we get

$$c_2(N_{X/(P_1 \times P_4)}) = (10h_2^2 + 10h_1 h_2).[X] = (10\alpha + 10\beta)h_1 h_2^3 + 10\beta h_2^4.$$

Since $[X]^2 = 2\alpha\beta h_1 h_2^3 + \beta^2 h_2^4$, we find

$$10\alpha + 10\beta = 2\alpha\beta, \quad 10\beta = \beta^2.$$

If $\beta = 0$, then X can be embedded into P_4 , which is impossible by Proposition 2.1. Thus $\alpha = \beta = 10$. Now the second projection q induces a surjective map

$$\bar{q} : X \to \overline{X} \subset P_4.$$

On the other hand, the projection onto the first factor gives a surjective map $p|_X : X \to P_1$. By proposition (see [4, p.88]), we know that X is not simple. Then Lemma 1.1 says that X does not contain rational curves. Thus the projection $\bar{q}$ is a finite map. By Proposition 1.2, $\overline{X}$ can not be a hyperplane. Hence $\overline{X}$ spans P_4. Since $\bar{q}$ is finite, the line bundle $O_X(0,1) = \bar{q}^* O_{P_4}(1)$ is ample. By the Kodaira vanishing theorem and Riemman-Roth theorem, this shows

$$h^0(O_X(0,1)) = \frac{1}{6}(O_X(0,1)^3) = \frac{1}{6}h_2^3.[X] = \frac{10}{6},$$

which is a contradiction.

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