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ON THE BOUNDED AND UNBOUNDED SOLUTIONS OF ONE DIMENSIONAL NONLINEAR REACTION-DIFFUSION PROBLEM

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Abstract

The existence of bounded and unbounded solutions to nonlinear reaction-diffusion problem $u_t = \Delta\Phi(u) + F(u, x, t)$ with initial or initial-boundary conditions is discussed when $u = u(x, t), x \in R$. Simple criteria are given.

Keywords Reaction-Diffusion equation, Bounded solution, Cauchy problem.
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§1. Introduction

A lot of chemical reactions taken place in a nonlinear medium result in nonlinear reaction-diffusion equations. Some of their solutions become unbounded in a finite or infinite period while some remain bounded for ever[1−5]. As a special case, the Cauchy problem

$$\begin{cases} u_t = \Delta u^m + u^n, \\ u(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

where $\varphi(x) \geq 0$ is bounded, was studied in detail in [4]. It is well known that when $1 < n < m + 2$, all solutions of Equation (1.1) blow up, i.e., they become unbounded in a finite period provided that $\varphi \in C^0(R, [0, \infty))$ and $\varphi \not\equiv 0$. But the situation changes completely when a boundary condition is added to Equation (1.1). It will become even more complicated when u^m and u^n in Equation (1.1) are replaced by two generalized functions. Our purpose in this paper is to study the boundedness and unboundedness of solutions for generalized reaction-diffusion equations.

§2. Bounded Solutions

We consider the initial-boundary problem

$$\begin{cases} u_t = \Delta\Phi(u) + F(u, x, t), \\ u(x, 0) = \varphi(x), \qquad x \in (-a, a), \\ u(\pm a, t) = h_\pm(t), \qquad t \in (0, \infty), \end{cases} \tag{2.1}$$

where $a > 0$ and

1) $\Phi \in C^1(R^+, R^+)$, indecreasing and $\Phi(0) = 0$;

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2) $F : R^+ \times R \times R^+ \to R$ locally Lipschitzian continuous in u and $F(0, x, t) = 0$;

3) $\varphi \in C^0([-a, a], R^+)$;

4) $h_\pm(t) \in L_{loc}(R^+)$, bounded.

Denote by Ω and Ω_T the sets $(-a, a)$ and $(-a, a) \times (0, T)$ respectively.

Definition 2.1. *A function $u \in C^0([0,T], L^1(\Omega)) \cap L^\infty(Q_T)$ is a solution to Equation (2.1) on $[0,T]$ if it satisfies*

$$\int_\Omega u(t)\psi(t) - \iint_{Q_t} (u\psi_t + \Phi(u)\Delta\psi) + \int_0^t [\Phi(h_+(s))\psi_x(a,s) - \Phi(h_-(s))\psi_x(-a,s)]ds$$
$$= \int_\Omega \varphi\psi(0) + \iint_{Q_t} F\psi$$

for all $t \in (0,T]$ and $\psi \in C^2(\bar{Q}_T, R)$ such that $\psi \geq 0$ and $\psi = 0$ when $(x,t) \in \{-a, a\} \times [0,T]$. A solution on $[0,\infty)$ means a solution on each $[0,T]$ for any $T > 0$. A supersolution (subsolution) is defined by (2.2) with equality replaced by $\geq$ $(\leq)$ and with $\bar{h}_\pm(t) \geq h_\pm(t)$ $(\underline{h}_\pm(t) \leq h_\pm(t))$.

Lemma 2.1.[1, Theorem 12] *Let $\bar{u}$ be a supersolution and $\underline{u}$ a subsolution of (2.1) when $F(t,x,u) = F(u)$ with initial data $\bar{u}_0$ and $\underline{u}_0$ respectively. If $\underline{u}_0 \leq \bar{u}_0$, then*

$$\underline{u} \leq \bar{u}.$$

Obviously Lemma 2.1 ensures the uniqueness of solution to Equation (2.1). Denote by Φ^{-1} the inverse function of $\Phi : R^+ \to R^+$.

Theorem 2.1. *Suppose $h_\pm(t) = 0, F(u) > 0$ for $u > 0$, $\lim_{u\to\infty} \Phi(u) = \infty$ and*

$$\lim_{u\to 0} \frac{F(u)}{\Phi(u)} = \alpha > \frac{\pi^2}{4a^2}, \qquad \lim_{u\to\infty} \frac{F(u)}{\Phi(u)} = \beta < \frac{\pi^2}{4a^2}.$$

Then

$$\begin{cases} \Delta\phi(u) + F(u) = 0, \\ u(\pm a) = 0 \end{cases} \tag{2.2}$$

has a unique nontrivial solution $\tilde{u}(x)$ which satisfies $u(x) > 0$ for $|x| < a$ and if $F(u)$ satisfies

$$F(\Phi^{-1}(sv)) < sF(\Phi^{-1}(v)) \tag{2.3}$$

for any $s > 1$ and $v > 0$, then every solution of (2.1) is bounded and approaches $\tilde{u}(x)$ as $t \to \infty$ provided $\varphi(x) \not\equiv 0$.

Proof. Let $v = \Phi(u)$. Then (2.2) is equivalent to

$$\begin{cases} \Delta v + F(\Phi^{-1}(v)) = 0, \\ v(\pm a) = 0. \end{cases} \tag{2.4}$$

For $\Delta v + F(\Phi^{-1}(v)) = 0$, its trajectories are determined by

$$\frac{1}{2}\dot{v}^2 + \int_0^v F(\Phi^{-1}(s))ds = C. \tag{2.5}$$

Extend the domains for F and Φ as $\Phi(-u) = -\Phi(u)$ and $F(-u) = -F(u)$ for $u < 0$ although we have interest only in the case $u \geq 0$. Then curve (2.5) is simply closed with period, say, $4T$. By use of the method of qualitative analysis in ordinary differential equations we can easily prove that $T > a$ as $C \to \infty$ and $T < a$ as $C \to 0$. Therefore there exists at least one $C_0 > 0$ such that, $T = a$. Then the equation

$$\frac{d^2 v}{dx^2} + F(\Phi^{-1}(v)) = 0$$

has a solution $v(x)$ such that $v'(0) = 0$ and

$$\int_0^{v(0)} F(\Phi^{-1}(s))ds = C_0.$$

Hence $v(\pm T) = v(\pm a)$. Obviously $\tilde{u}(x) = \Phi^{-1}(v(x))$ is a solution to Equation (2.2). The uniqueness is ensured by Lemma 2.1.

For the boundedness of solutions of Equation (2.1), it suffices to prove that $\|u\|_{L^\infty}$ is bounded since the solution exists. It follows from the condition

$$\lim_{u\to\infty} \frac{F(u)}{\Phi(u)} = \beta < \frac{\pi^2}{4a^2}$$

that we can choose M_0 large enough such that for a constant $e \in (a, \pi/2\sqrt{\beta})$

$$\frac{F(u)}{\Phi(u)} \le \frac{\pi^2}{4e^2}$$

holds for $u \ge M_0$. Let $M_1 = \max\limits_{0\le u\le M_0} F(u)$. Then $F(u) < M_1 + \frac{\pi^2}{4e^2}\Phi(u)$. Set

$$\widetilde{M} =\ge \sec\frac{\pi a}{2e} \cdot \left[\frac{4M_1 e^2}{\pi^2} + \max\{\Phi(u), \|\varphi\|_{L^\infty}\}\right]$$

and

$$\bar{u}(x,t) := \Phi^{-1}\Big(\widetilde{M}\cos\frac{\pi x}{2e} - \frac{4M_1 e^2}{\pi^2}\Big).$$

Then

$$\begin{aligned}\bar{u}_t - \Delta\Phi(\bar{u}) = -\Delta\Phi(\bar{u}) &= \frac{\widetilde{M}\pi^2}{4e^2}\cos\frac{\pi x}{2e} \\ &= \frac{\pi^2}{4e^2}\Big[\widetilde{M} cos\frac{\pi x}{2e} - \frac{4M_1e^2}{\pi^2}\Big] + M_1 > F(u)\end{aligned}$$

and

$$\begin{aligned}\bar{u}(x,0) &= \Phi^{-1}\Big(\widetilde{M}\cos\frac{\pi x}{2e} - \frac{4M_1e^2}{\pi^2}\Big) \\ &\ge \Phi^{-1}\Big(\widetilde{M}\cos\frac{\pi a}{2e} - \frac{4M_1e^2}{\pi^2}) \\ &= \Phi^{-1}(\max\{\Phi(M_0), \|\varphi\|_{L^\infty}\}) \\ &\ge \|\varphi\|_{L^\infty},\end{aligned}$$

i.e., $\bar{u}(t,x)$ is a supersolution of (2.1). Since $\underline{u} \equiv 0$ is a subsolution, Lemma 2.1 implies the conclusion.

In order to prove the remaining part of the theorem we need the following lemma.

Lemma 2.2. *Suppose that $u_1(x)$ and $u_2(x)$, with $u_1'(0) = u_2'(0) = 0$, are in their own support two arbitrary solutions to the stationary equation*

$$\frac{d^2\Phi(u)}{dx^2} + F(u) = 0, \tag{2.6}$$

where Φ and F satisfy the requirements of Theorem 2.1 *and for some $\epsilon > 0$*

$$\text{supp}u_1 \subset (-a, a) \subset (-a-\epsilon, a+\epsilon) \subset \text{supp}u_2.$$

If

$$\varphi(x)=\begin{cases}u_1(x), & x\in \mathrm{supp}u_1\\ 0, & x\in[-a,a]\backslash\mathrm{supp}u_1\end{cases}\qquad [\varphi(x)=u_2(x),|x|\le a],$$

then the solution $u(x,t)$ *to Equation* (2.1) *with* $h_\pm(t)=0$ *satisfies*

$$u_t(x,t)\ge 0\ [\le 0]\quad for\quad (x,t)\in[-a,a]\times(0,\infty).$$

Proof. Since $\lim\limits_{t\to 0}u(x,0)=u_1(x)$ for $x\in\mathrm{supp}u_1$, it is obvious that

$$\frac{\partial u(x,0)}{\partial t}=\frac{d^2(\Phi(u(x,0))}{dx^2}+F(u(x,0))=0$$

for $x\in[-a,a]\backslash\partial\{\mathrm{supp}u_1\}$. It follows from $u_1(x)$ satisfying (2.6) that $\Phi(u_1(x))$ satisfies (2.5) with $C=C_0>0$. Let $d\Phi(u_1(x))/dx=w_1(x)$. Then

$$\lim_{x\to b^+[c^-]}\frac{d\Phi(u_1(x))}{dx}=\lim_{x\to b^+[c^-]}w_1(x)=\sqrt{2C_0}\ \ [-\sqrt{2C_0}]$$

since $u_1(b)=u_1(c)=0$. But

$$\lim_{x\to b^-[c^+]}\frac{d\Phi(u_1(x))}{dx}=0.$$

Therefore $d\Phi(u_(x))/dt$ has discontinuous points at $x=b,c$ with right limits greater than left ones. In solving Equation (2.1), both $d^2\Phi(u_1(b))/dx^2$ and $d^2\Phi(u_1(c))/dx^2$ serve as $+\infty$. Then the inequality

$$\frac{\partial u(x,0)}{\partial t}=\frac{\partial^2\Phi(u(x,0))}{\partial x^2}+F(u(x,0))\ge 0$$

implies that there exists $\delta>0$ such that $u(x,\Delta t)\ge u(x,0)$ for $(x,\Delta t)\in[-a,a]\times(0,\delta)$. Consider $u(x,\Delta t)$ and $u(c,0)$ as two initial functions. By applying Lemma 2.1, we have $u(x,t+\Delta t)\ge u(x,t)$ and hence $u(x,t)\ge 0$.

The proof for the conclusion about $u_2(x)$ is the same as above.

We now continue to prove Theorem 2.1.

Suppose that $u_1(x)$ and $u_2(x)$ satisfy the requirements of Lemma 2.2 and $u_1(x)<u_2(x)$ for $|x|\le a$. Denote by $u_i(x,t)$ the solutions of (2.1) with $\varphi(x)=u_i(x), i=1,2$. As $u_1(x,t)$ is monotone and bounded, there is $u_l(x)\ge 0$ $(>0, x\in(-a,a))$ such that $\lim\limits_{t\to\infty}u_1(x,t)=u_l(x)$. But $u_l(x)$ must satisfy $u'=0$ and therefore (2.2) holds. This means $u_l(x)=\tilde u(x)$. Similarly $\lim\limits_{t\to\infty}u_2(x,t)=\tilde u(x)$.

If $\varphi(x)\ge 0, |x|\le a$ is an arbitrary continuous function with $\varphi(x)\not\equiv 0$, then the conditions

$$\lim_{u\to 0}F(u)/\Phi(u)=\alpha>\frac{\pi^2}{4a^2}\quad\text{and}\quad\lim_{u\to\infty}F(u)/\Phi(u)=\beta<\frac{\pi^2}{4a^2}$$

ensure that there exist $u_1(x)$ and $u_2(x)$ such that $u_1(x)\le\varphi(x)\le u_2(x)$. Then $u_1(x,t)\le u_2(x,t)\le u_2(x)$ and

$$\lim_{t\to\infty}u(x,t)=\tilde u(x),\quad |x|<a. \tag{2.7}$$

The proof is now completed.

Before giving another theorem we consider at first a special equation when $F(u,x,t)=\Phi(u)$, i.e.,

$$\Delta\Phi(u)+\Phi(u)=0. \tag{2.8}$$

Lemma 2.3. *Suppose that $\Phi(u)$ satisfies the requirements given above. If $u(x)$ is a solution of Equation (2.8) together with the initial conditions*

$$u(0) = u_0 > 0, \quad u'(0) = 0, \tag{2.9}$$

then $u(t)$ satisfies $u(\pm\pi/2) = 0$ and

$$xu'(x) < 0, \ a.e., \qquad for\ \ 0 < |x| < \frac{\pi}{2}. \tag{2.10}$$

Proof. Let $v = \Phi(u)$. Then $v(x) = \Phi(u(x))$ is the solution of initial problem

$$\begin{cases} v'' + v = 0, \\ v(0) = \Phi(u_0), \quad v'(0) = 0. \end{cases} \tag{2.11}$$

Solve (2.11) and we have

$$v(x) = \Phi(u_0)\cos x. \tag{2.12}$$

Then $u(x) = \Phi^{-1}(v(x)) = \Phi^{-1}(\Phi(u_0)\cos x)$.

The fact that Φ is an increasing function implies $\Phi' \geq 0$, $\text{meas}\{u|\Phi'(u) = 0\} = 0$ and hence

$$u'(x) = -\frac{\phi(U_0)\sin x}{\Phi'(\Phi(u_0)\cos x)}, \quad a.e., \quad \text{for } |x| < \frac{\pi}{2}. \tag{2.13}$$

The truth of Lemma 2.3 is now obvious.

Theorem 2.2. *Suppose $F(u,x,t) \leq A\Phi(u)$, $A > 0$ a constant. If $u < \pi/2\sqrt{A}$, then the solution $u(x,t)$ to the initial-boundary problem (1.2) with $h_\pm(t) = 0$ is bounded and tends to the trivial solution $u = 0$.*

Proof. Without loss of generality we suppose that $A = 1$.

Let $u_0(x)$ be the solution to Equation (2.8) with the initial conditions $u(0) = c, u'(0) = 0$. Clearly $B = \{u_c(x)|c \geq 0\}$ is a strip in the x,u-plane: $(-\pi/2, \pi/2) \times R^+$. For any $a \in (0,\pi/2), M > 0$ there exists a $c > 0$ such that $u_c(x) > M$, $|x| \leq a$. Fix $M = \max\limits_{|x|\leq a}|\varphi(x)|$ and let $\tilde{u}_0(x) = u_c(x)$, $|x| \leq a$. Denote by $\tilde{u}(x,t)$ the solution of the initial-boundary problem (1.1) with $u(x,0) = \tilde{u}_0(x), h_\pm(t) = 0$. Then according to the comparison theorem we have $u(x,t) \leq \tilde{u}(x,t)$ in their common existence interval $t \in (0,T)$.

To prove Theorem 2.2 it suffices to prove that $\tilde{u}(x,t)$ exists for $t < T$, where T is any positive constant, and $\lim\limits_{t\to\infty}\tilde{u}(x,t) = 0$.

Since $\tilde{u}_0(x)$ is a solution to Equation (2.8) and

$$\lim_{t\to 0}\tilde{u}(x,t) = \tilde{u}_0(x) > 0, \quad |x| < a,$$

we have $\frac{\partial}{\partial t}u(x,0) = 0$, $|x| < a$. At the same time, it follows from $\tilde{u}(-a) = \tilde{u}(a) > M > 0$ that

$$\lim_{t\to 0}\Delta\tilde{u}(a,t) = \lim_{t\to 0}\Delta\tilde{u}(-a,t) = -\infty.$$

Then

$$\lim_{t\to 0}\frac{\partial\tilde{u}(x,t)}{\partial t} \leq 0, \quad x \in [-a,a],$$

i.e., $\tilde{u}(x,t)$ does not increase as t increases from $t = 0$ to $t = \delta$, where $\delta > 0$ is small enough. Then based on the comparison theorem it is easy to prove that $\tilde{u}_t(x,t) \leq 0$. This, together with the fact $\tilde{u}(x,t) \geq 0$ which follows from $\tilde{u}_0(x) > 0$, implies that $\tilde{u}(x,t)$ exists for ever and $\lim_{t\to\infty} \tilde{u}(x,t) = \bar{u}(x)$, $|x| \leq a$. Obviously $\bar{u}(x)$ must satisfy Equation (2.8). But when $a < \pi/2$, Lemma 2.1 implies that Equation (2.8) with the boundary condition $u(\pm a) = 0$ has only the trivial solution $u(x) \equiv 0$. Therefore $\bar{u}(x) \equiv 0$ and hence $\lim_{t\to\infty} u(x,t) = 0$, $|x| \leq a$, since $0 \leq u(x,t) \leq \tilde{u}(x,t)$.

§3. Unbounded Solutions

Consider the Cauchy problem

$$\begin{cases} u_t = \Delta\Phi(u) + F(u,x,t), \\ u(x,0) = \varphi(x), \end{cases} \tag{3.1}$$

where $\varphi \in C^0(R,R^+), \varphi \geq 0,$ suppφ bounded and $\Phi(0) = F(0,x,t) = 0$.

Theorem 3.1. *Suppose that $F \in C^0(R^+,R), \Phi^{1/m} \in C^1(R^+,R), [\Phi^{1/m}(u)]'_u \geq 1$ for some $m > 1$. If there are two constants $A > 0$ and $n \in (1,m)$ such that $F(u,x,t) > A\Phi^{n/m}(u)$, then the solution $u(x,t)$ to (3.1) will be unbounded in a finite period provided that $\varphi \not\equiv 0$.*

Such a solution is called a blow-up solution of (3.1).

Proof. It follows from the condition $[\Phi^{1/m}(u)]'_u \geq 1$ that $\Phi^{1/m}(u)$ and hence $\Phi(u)$ are strictly increasing on $(0,\infty)$. Then Φ^{-1} exists. Without loss of generality we suppose $A = 1$ and $0 \in \text{int}\{\text{supp}\varphi\}$.

Take in account the problem

$$\begin{cases} v_t = \Delta v^m + v^n, \\ v(x,0) = \phi(x), \end{cases} \tag{3.2}$$

where $m > n > 1, \phi \in C^0(R,R^+)$ and $\phi \geq 0[\not\equiv 0]$ with the support bounded.

Let $y = x/\sqrt{m}$. Then Equation (3.2) is equivalent to

$$\begin{cases} v_t = \dfrac{\partial}{\partial y}(v^{m-1}\dfrac{\partial v}{\partial y}) + v^n, \\ v(y,0) = \phi(\sqrt{m}y). \end{cases} \tag{3.3}$$

It is well known (see [4]) that the first equation in (3.3) has self similarity solutions

$$\bar{v}(y,t) = \frac{1}{(T-t)^{1/(n-1)}}\theta\left(|y|(T-t)^{\frac{m-n}{2(n-1)}}\right)_+,$$

where $(f(x))_+$ means $\max\{f(x),0\}$ and $\theta(r) \geq 0$ is the compactly and connectedly supported solution to

$$\begin{cases} \dfrac{d}{dr}\left(\theta^{m-1}\dfrac{d\theta}{dr}\right) + \dfrac{m-n}{2(n-1)}r\dfrac{d\theta}{dr} - \dfrac{1}{n-1}\theta + \theta^n = 0, \\ \dfrac{d\theta(0)}{dt} = 0,\ \theta(+\infty) = 0,\ \theta(0) > 0. \end{cases} \tag{3.4}$$

Besides, θ has the property that $\theta'(r) > 0$, where $r, \theta > 0$. Then $\bar{v}(.,t)$ is compactly supported for any $t \in [0,T)$ and so is the self similarity solutions of (3.2)

$$v(x,t) = \frac{1}{(T-t)^{1/(n-1)}}\theta(|x|(T-t)^{(m-n)/2(n-1)}/\sqrt{m})_+.$$

It is easy to see that $v(x,0) \geq 0$ for $x \in R$ and $\Delta u^m + u^n \geq 0$ since $v(x,t)$ is a solution of $v_t = \Delta v^m + v^n$ on $R \times [0,t)$.

Let $v^m = \Phi(u)$. Then

$$w(x,t) = \Phi^{-1}\left[\frac{1}{(T-t)^{m/(n-1)}}\theta^m(|x|(T-t)^{(m-n)/2(n-1)}/\sqrt{m})_+\right] \tag{3.5}$$

is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{[\Phi^{1/m}(u)]'_u}[\Delta\Phi(u) + \Phi^{n/m}(u)], \\ u(x,0) = w(x,0). \end{cases} \tag{3.6}$$

For a given t in $[0,T]$ denote by X^t the set $\{(x,t)|w(x,t)>0\}$ and $X = \{X^t|t\in[0,T]\}$. Obviously $w_t(x,t) > 0$ for $(x,t)\in X$.

Choose $T>0$ so large that $w(x,0)\le\phi(x),\quad (x,0)\in X^0$. Let $u(x,t)$ be the solution of Equation (3.1). Then there exists $\delta>0$ such that

$$w(x,t) < u(x,t), \qquad \text{for} \quad (x,t)\in\{X^t|t\in(0,\delta\}. \tag{3.7}$$

Suppose that $u(x,t)$ does not blow up before $t=T$. It suffices to prove that Equation (3.7) holds for all $t\in(0,T)$. Otherwise we suppose that there exists a point $(x_0,t_0), t_0<T$, such that

$$w(x_0,t_0) = u(x_0,t_0), \quad w(x,t) < u(x,t) \text{ for } \ (x,t)\in X^t, \quad t\in(0,t_0).$$

Then

$$\begin{aligned} \frac{\partial u(x_0,t_0)}{\partial t} &= \Delta\Phi(u) + F(u,x_0,t_0) > \Delta\Phi(u) + \Phi^{n/m}(u) \\ &\ge \frac{1}{[\Phi^{1/m}(u)]^0{\prime}_u}[\Delta\Phi(u)+\Phi^{n/m}(u)] \\ &= \frac{\partial w(x_0,t_0)}{\partial t} \end{aligned}$$

since $w_t > 0$. Here u stands for $u(x_0,t_0)$. Thus there is $\delta>0$ such that $u(x_0,t) < w(x_0,t)$ for $t\in(t_0-\delta,t_0)$, a contradiction.

So (3.7) holds for $t\in(0,T)$ and this means that $u(x,t)$ must blow up in $(0,T)$.

Consider the problem

$$\begin{cases} u_t = \Delta u^m + u^m, & m>1, \\ u(\pm a,t) = 0, & t\in R^+. \end{cases} \tag{3.8}$$

Lemma 3.1.[4,p.1266] *For any $a>\pi/2$, (3.8) has a solution $\bar u(x,t)$ which blows up in a finite period and satisfies $\bar u_t(x,t)>0$ when $\bar u(x,t)>0$.*

Theorem 3.2. *Suppose there are constants $m>1$ and $A>0$ such that $\Phi^{1/m}\in C^1(R^+,R), [\Phi^{1/m}(u)]'_u \ge 1$ and $F(u,x,t) > A\Phi(u)$. If $a>\pi/2\sqrt{A}$, then the solution $u(x,t)$ to the initial-boundary problem (2.1) with $h_\pm(t)=0$ will blow up in a finite period provided $\varphi(x)\not\equiv 0$.*

Proof. Without loss of generality we assume $A=1$.

Suppose that the theorem is false. Then for some $a>\pi/2$ there is $\varphi(x)\ge 0$ continuous and $\varphi(x)\not\equiv 0$ such that for any $T>0$ the solution $u(x,t)$ to (2.1) remains bounded when $t\in(0,T]$.

Assume that $\varphi(x_0)>0$ for some $x_0\in(-a,a)$. Then for the given $\varphi(x)$ there exist $T>0$

large enough and $\gamma > 0$ small enough such that

$$\phi(x) = \left[\frac{(m-1)\gamma^2}{2(m+1)}\right] T^{-\frac{1}{m+1}} \left[1 - \left(\frac{x - x_0}{m\gamma T^{\frac{1}{m+1}}}\right)^2\right]_+^{\frac{1}{m-1}}$$

satisfies supp$\phi \subset (-a, a)$ and $\phi(x) \le \varphi(x), \quad |x| < a$. Consider

$$\begin{cases} u_t = \Delta\Phi(u), \\ u(x,0) = \phi(x), \\ u(\pm a, t) = 0. \end{cases} \tag{3.9}$$

Its solution is

$$v(x,t) = \left[\frac{(m-1)\gamma^2}{2(m+1)}\right] (T+t)^{-\frac{1}{m+1}} \left[1 - \left(\frac{x - x_0}{m\gamma (T+t)^{\frac{1}{m+1}}}\right)^2\right]_+^{\frac{1}{m-1}}$$

when

$$t \le t_1 = \min\Big\{\Big(\frac{a - x_0}{m\gamma}\Big)^{m+1} - T, \Big(\frac{a + x_0}{m\gamma}\Big)^{m+1} - T\Big\}.$$

Clearly $t_1 > 0$ when γ is small enough.

Since $v(x,t)$ is the solution of diffusion problem (3.9), it approaches a constant $d > 0$ for $x \in (-a, a)$ as $t \to \infty$ and then there is $T_1 > t_1$ such that

$$v(x,t) > \frac{d}{2} > 0, \qquad |x| < a, \ t \ge T_1. \tag{3.10}$$

It is not difficult to prove that

$$\frac{\partial}{\partial x} v(a,t) < -b < 0, \qquad \frac{\partial}{\partial x} v(-a,t) > b > 0, \quad \text{for } t > T_1. \tag{3.11}$$

By applying the comparison theorem, it follows that

$$u(x,t) > \frac{d}{2} > 0 \quad \text{for } |x| < a, t \ge T_1$$

and hence

$$\frac{\partial}{\partial x} u(a,t) < -b, \qquad \frac{\partial}{\partial x} u(-a,t) > b, \quad \text{for } \ t > T_1. \tag{3.12}$$

Let $w(x,t) = \Phi^{1/m}(u(x,t))$. Then $w(x,t))$ is the solution of the initial-boundary problem

$$\begin{cases} w_t = [\Phi^{1/m}]'_u [\Delta w^m + F(\Phi^{-1}(w^m), x, t)], \\ w(x,0) = \Phi^{1/m}(\varphi(x)), \\ w(\pm a, t) = 0. \end{cases} \tag{3.13}$$

Furthermore, let $\tilde{w}(x,t) = w(x, t + T_1)$. Then $w(x,t)$ satisfies the first equation of (3.13) and the initial condition $\tilde{w}(x,0) = \Phi^{1/m}(u(x, T_1))$.

For the solution $y(t)$ of the equation

$$y'' - \frac{1}{m-1} y^{1/m} + y = 0, \quad y \ge 0,$$

with $y(\pm a) = 0$, it follows from

$$\tilde{w}_x(\pm a, 0) = \frac{1}{m} \lim_{x \to \pm a} \Phi^{\frac{1-m}{m}}(u) u'(x, T_1) = \mp\infty$$

that we can choose $T > T_1 > 0$ large enough such that

$$\frac{1}{T^{\frac{1}{m-1}}} y^{\frac{1}{m}}(x) < \tilde{w}(x,0), \quad |x| < a.$$

We now prove that for $\tilde{v}(x,t) = (T-t)^{-\frac{1}{m-1}} y^{\frac{1}{m}}(x)$ it holds that

$$\tilde{v}(x,t) < \tilde{w}(x,t), \qquad |x| < a, t \in (0,T). \tag{3.14}$$

If inequality (3.14) is not true, then there exist $t_0 \in (0,T)$ and $x_0 \in (-a,a)$ such that

$$\tilde{v}(x,t) < \tilde{w}(x,t), \qquad (x,t) \in (-a,a) \times (0,t_0) \tag{3.15}$$

and $\tilde{v}(x_0,t_0) = \tilde{w}(x_0,t_0)$. This implies

$$\frac{\partial \tilde{v}(x_0,t_0)}{\partial x} = \frac{\partial \tilde{w}(x_0,t_0)}{\partial x}, \quad \Delta \tilde{v}(x_0,t_0) \le \Delta \tilde{w}(x_0,t_0).$$

At the same time we have at (x_0,t_0)

$$\frac{\partial \tilde{v}(x_0,t_0)}{\partial t} = \frac{1}{(m-1)(T-t_0)^{\frac{m}{m-1}}} y^{\frac{1}{m}}(x_0) > 0,$$

i.e.,

$$\Delta \tilde{v}^m(x_0,t_0) + \tilde{v}^m(x_0,t_0) > 0.$$

Therefore

$$\begin{aligned} \frac{\partial \tilde{v}(x_0,t_0)}{\partial t} &= \Delta \tilde{v}^m(x_0,t_0) + \tilde{v}^m(x_0,t_0) \\ &\le [\Phi^{1/m}(u)]'_u [\Delta \tilde{v}^m(x_0,t_0) + \tilde{v}^m(x_0,t_0)] \\ &\le [\Phi^{1/m}(u)]'_u [\Delta \tilde{w}^m(x_0,t_0) + \tilde{w}^m(x_0,t_0)] \\ &< [\Phi^{1/m}(u)]'_u [\Delta \tilde{w}^m(x_0,t_0) + F(\tilde{w}(x_0,t_0),x_0,t_0)] \\ &= \frac{\partial \tilde{w}(x_0,t_0)}{\partial t}, \end{aligned}$$

that is to say, there is $\bar{t} \in (0,t_0)$ such that

$$\tilde{v}(x_0,\bar{t}) > \tilde{w}(x_0,\bar{t}),$$

a contradiction to (3.15). Then the fact that $v(x,t)$ blows up at $t = T$ implies that $\tilde{w}(x,t)$ must blow up before $t = T$. Obviously $w(x,t)$ blows up before $t = T_1 + T$. It follows from $w^m = \Phi(u)$ that $u(x,t)$ blows up before $t = T_1 + T$. The proof is now completed.

Example. Consider

$$\begin{cases} u_t = \Delta(u^3 + u^2) + (3u^3 + 2u^2 + |x|u), |x| \le a, \\ u(\pm a,t) = 0, & t > 0, \\ u(x,0) = \varphi(x), & |x| < 0, \end{cases} \tag{3.16}$$

where $\varphi \in C^0([-a,a], R^+), \varphi(x) \not\equiv 0$.

Here $\Phi(x) = u^3 + u^2$. Therefore

$$f(u) = [\Phi^{\frac{1}{3}}(x)]'_u = \frac{3u^2 + 2u}{3(u^3 + u^2)^{\frac{2}{3}}}.$$

It is easy to show that $\lim\limits_{u \to \infty} f(u) = 1$ and $f(u)$ decreases on R^+ . Therefore $f(u) > 1$. It follows from

$$F(u,x,t) = 3u^3 + 2u^2 + |x|u > 2\Phi(u)$$

that the solution to (3.16) will blow up in a finite period provided $a > \pi/2\sqrt{2}$.

Remark. The comparison theorem can only be applied for two solutions in a common existence interval in t. Consider

$$\begin{cases} u_t = \Delta u^m + f(u) & \text{in } S = R^N \times R^+, \\ u(x,0) = u_0(x), & x \in R^N, \end{cases} \tag{3.17}$$

where $u_0 \geq 0$ continuous with bounded support, $f(0) = 0, f \in C^1(R^+, R^+)$.

By comparing (3.17) to the special case $f(x) \equiv 0$, paper [2] gave a conclusion that if $f(s) \geq 0$ when $s \geq 0$, then for any $x \in R^N$ there exists $T(x) \geq 0$ such that $u(x,t) > 0$ for any $t > T$.

But it is false. For example, when $f(u) = u^m$ and

$$u_0(x) = \begin{cases} \left[\dfrac{2m}{(m^2-1)T}\right]^{\frac{1}{m-1}} \cos^{\frac{2}{m-1}} \left(\dfrac{m-1}{2m}\right) x, & |x| \leq \dfrac{m\pi}{m_1}, \\ 0, & |x| > \dfrac{m\pi}{m-1}, \end{cases} \tag{3.18}$$

Equation (3.17) has the solution

$$u(x,t) = \begin{cases} \left[\dfrac{2m}{(m^2-1)(T-t)}\right]^{\frac{1}{m-1}} \cos^{\frac{2}{m-1}} \left(\dfrac{m-1}{2m}\right) x, & |x| \leq \dfrac{m\pi}{m-1}, \\ 0, & |x| > \dfrac{m\pi}{m-1}. \end{cases} \tag{3.19}$$

Obviously $u(x,t)$ remains 0 when $|x| \geq m\pi/(m-1)$. That is because $u(x,t)$ will blow up at $t = T$. After then there is no reason to compare it with other solutions.

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