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POINTED REPRESENTATIONS OF INFINITE DIMENSIONAL LIE ALGEBRAS

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Abstract

A contravariant bilinear pairing K on every $M(\rho) \times M(\rho\theta)$ is determined and it is proved that $M(\rho)$ is irreducible if and only if K is left nondegenerate. It is also proved that every cyclic pointed module is a quotient of some Verma-like pointed module; moreover if it is irreducible then it is a quotient of the Verma-like pointed module by the left kernel of some bilinear pairing K. In case the mass function is symmetric, there exists a bilinear form on $M(\rho)$. It is proved that unitary pointed modules are integrable. In addition, a characterization of the mass functions of Kac-Moody algebras is given, which is a generalization of the finite dimensional Lie algebras case.

Keywords Pointed representation, Primitive cycle, Mass function, Bilinear pairing.
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§1. Introduction

In literature, pointed representations are weight representations which admit a one-dimensional weight space. This class of representations is a natural generalization of the highest weight representations. In the complicated theory of representations, pointed representations are more feasible to access next to the highest weight representations. For a simple Lie algebra L, if $U(L)$ is the universal enveloping algebra of L, we denote by $C(\mathfrak{h})$ the centralizer of the Cartan subalgebra $\mathfrak{h}$ in $U(L)$. Suppose that $\lambda : \mathfrak{h} \to \mathbb{C}$ is a weight of an L-module V such that $\dim V_\lambda = 1$, then we get a map $\rho : C(\mathfrak{h}) \to \mathbb{C}$ defined by $\rho(c)v = cv$ for $c \in C(\mathfrak{h})$. In fact ρ is an algebra homomorphism and we call it a mass function of V. Clearly ρ restricted to $\mathfrak{h}$ is equal to λ. Conversely, given any algebra homomorphism $\rho : C(\mathfrak{h}) \to \mathbb{C}$, one can construct a unique irreducible pointed module V, which admits ρ as a mass function.

The pointed representations of finite dimensional simple Lie algebras have been studied by D. J. Britten, F. W. Lemire, I. Z. Bouwer, etc. In 1987, Britten and Lemire classified all pointed L modules for arbitrary simple Lie algebra of finite dimension[1]. But for infinite dimensional Lie algebras, there are no literature by now except for the torsion free pointed representations of affine algebras[2].

In this paper we devote ourselves to the study of pointed representations of infinite dimensional Lie algebras. We give some theorems and propositions, and give a generalization of a result of finite dimensional case to the infinite dimensional case.

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§2. Definition

Let $A=(a_{ij})$ be an $n\times n$ generalized Cartan matrix, $g(A)$ or g the associated Kac-Moody algebra over $\mathbb{C}$ with Chevalley generators $e_i,\ f_i,\ h_i\ (i=1,\dots,n)$.

Definition 2.1. *A representation $\pi: g(A)\to gl(V)$ is called pointed, if V can be decomposed into a direct sum of weight spaces such that there are at least one weight space which is of dimension* 1.

Let $U(g)$ denote the universal enveloping algebra of $g(A)$ or g. Since g is a direct sum of root spaces and every root space is of finite dimension, we can give an order to one basis which consists of root vectors of g. With this order one can give a PBW basis of the universal enveloping algebra $U(g)$.

Let $g=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ be the triangular decomposition of g. Then $u(g)=U(\mathfrak{n}^-)\otimes U(\mathfrak{n}^+)\otimes U(\mathfrak{h})$. In the given order of one basis of $U(g)$, we usually suppose that the elements of $U(\mathfrak{n}^-)$ are in the left, and the elements of $U(\mathfrak{h})$ are in the right. Thus

$$U(g)=\langle Y_k^{t_k}\cdots Y_1^{t_1}X_1^{j_1}\cdots X_k^{j_k}h_1^{l_1}\cdots h_n^{l_n}\ \mid\ t_i,\ l_i,\ j_i\geq 0,\ k\geq 0\rangle. \qquad (*)$$

As in the triangular decomposition, $\mathfrak{h}$ is the Cartan subalgebra of g. We denote by $C(\mathfrak{h})$ the centralizer of $\mathfrak{h}$ in $U(g)$. Suppose that X_i is of root β_i and Y_i is of root $-\beta_i$. Then

$$C(\mathfrak{h})=\langle Y_k^{t_k}\cdots Y_1^{t_1}X_1^{j_1}\cdots X_k^{j_k}h_1^{l_1}\cdots h_n^{l_n}\Big|\sum_{i=1}^{k}(t_i-j_i)\beta_i=0\rangle.$$

The elements of $C(\mathfrak{h})$ are called cycles. A homomorphism $\rho: C(\mathfrak{h})\to\mathbb{C}$ is called a mass function of g.

Let $Q=\sum\limits_{i=1}^{n}\mathbb{Z}\alpha_i$ denote the root lattice of g, where $\alpha_i\ (i=1,\dots,n)$ are simple roots. $Q^+=\sum\limits_{i=1}^{n}\mathbb{Z}^+\alpha_i,\ Q^-=\sum\limits_{i=1}^{n}\mathbb{Z}^-\alpha_i$. Then $U(g)=\bigoplus\limits_{\alpha\in Q}U(g)_\alpha$.

For any basis element u in (*), we denote by $u(i)$ the number of times that the factors $X_i\ (i>0)$ or $Y_i\ (i<0)$ is contained in u; $u^j(0)$ the number of times that h_j appears in u. For any such element u, we associated it with a set of ordered numbers:

$$[u]=(\cdots,u(-k),\cdots,u(-1),u^1(0),\cdots,u^n(0),u(1),\cdots,u(k),\cdots).$$

Now , $[u]\geq[v]$ if and only if $u(k)\geq v(k)$ for $k\in\mathbb{Z}$, and $u^i(0)\geq v^i(0),\ i=1,\dots,n$; while $[u]>[v]$ if $[u]\geq[v]$ and $[u]\neq[v]$.

Definition 2.2. *For any $u,\ u'\in(*)$, u is said to contain (properly contain) u', if $[u]\geq[u']$ ($[u]>[u']$). An element $u\neq 1$ of $(*)$ is called primitive if it does not properly contain any non-trivial cycle.*

§3. Pointed Representations of Kac-Moody Algebras

With $A,\ g,\ C(\mathfrak{h})$, etc. as in Section 2, the Cartan decomposition of g induces a gradation of $U(g)$

$$U(g)=U(g)_0\oplus\bigoplus_{\alpha\neq 0}U(g)_\alpha.$$

It is clear that $U(g)_0=C(\mathfrak{h})$. Any mass function is an algebra homomorphism from $U(g)_0$ to $\mathbb{C}$.

Definition 3.1. *A pointed module is called cyclic if it is generated by its one dimensional weight space vectors.*

Let θ be an antilinear involution of the Lie algebra g such that

$$\theta(e_i) = f_i, \quad \theta(f_i) = e_i, \quad \theta(h_i) = h_i,$$
$$\theta[e_i\ f_i] = [\theta(f_i)\ \theta(e_i)], \quad i, j = 1, \cdots, n.$$

Such an involution does exist and can be extended to an antilinear involution of the associative algebra $U(g)$, and we also denote the extended involution by θ (cf. [3]).

Let $\rho : C(\mathfrak{h}) \to \mathbb{C}$ be any mass function. Then $\ker(\rho)$ is a maximal ideal of $C(\mathfrak{h})$ and it generates a proper left ideal $I(\rho)$ of $U(g)$, and $U(g)/I(\rho)$ is a g module.

Definition 3.2. *For any mass function $\rho : C(\mathfrak{h}) \to \mathbb{C}$, the g module $M(\rho) = U(g)/I(\rho)$ is called Verma-like pointed module.*

Theorem 3.1. *Any cyclic pointed module is a quotient of some Verma-like pointed module.*

Proof. Let $V = \underset{\lambda}{\oplus} V_\lambda$ be a pointed module with $\dim V_{\lambda_0} = 1$, and the corresponding mass function be $\rho : C(\mathfrak{h}) \to \mathbb{C}$.

Since V is cyclic, $V = U(g)v$ for some nonzero $v \in V_{\lambda_0}$. We define a homomorphism $\phi : U(g) \to V$ by $\phi(u) = uv$. Then ϕ is surjective. It is obvious that $I(\rho) \subset \ker(\phi)$. This shows that ϕ induces a homomorphism $\tilde{\phi} : M(\rho) \to V$, and V is a quotient of $M(\rho)$.

For the decomposition $U(g) = U(g)_0 \oplus \underset{\alpha\neq 0}{\oplus} U(g)_\alpha$, let $P : U(g) \to U(g)_0$ be the projection of $U(g)$ onto $U(g)_0$ parallel to the space $\underset{\alpha\neq 0}{\oplus} U(g)_\alpha$. If $u,\ v \in U(g)$, let $K(u,\ v) = \rho(P(\theta(v)u))$. Then K is a contravariant bilinear form on $U(g)$. We define

$$\ker_L(K) = \{\ u \in U(g)\ \mid\ K(u,\ v) = 0 \text{ for any } v \in U(g)\ \},$$

$$\ker_R(K) = \{\ u \in U(g)\ \mid\ K(u,\ v) = 0 \text{ for any } v \in U(g)\ \}.$$

Since $\mathbb{C}$ is a commutative algebra, we see that if ρ is a mass function, then $\rho\theta$ is also a mass function. Now a moment's consideration shows that $\ker(\rho) \subset \ker_L(K)$, $\ker(\rho\theta) \subset \ker_R(K)$. Thus we get a contravariant bilinear pairing K on $M(\rho) \times M(\rho\theta)$. By the definition of K, $U(g)$ decomposes into a direct sum of mutually orthogonal spaces which coincide with the gradation of $U(g)$ by root lattice.

Lemma 3.1. *Let $\mathfrak{h}$ be a commutative Lie algebra, V a diagonalizable $\mathfrak{h}$ module, i.e., $V = \underset{\lambda\in\mathfrak{h}^*}{\oplus} V_\lambda$, where $V_\lambda = \{\ v \in V\ \mid\ hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$. Then any submodule U of V is graded with respect to the above gradation.*

Proof. See [3].

Now we have the following

Theorem 3.2. *Let g be a Kac-Moody algebra, $\rho : C(\mathfrak{h}) \to \mathbb{C}$ a mass function of g, K the contravariant bilinear pairing on $M(\rho) \times M(\rho\theta)$. Then $\ker_L(K)$ ($\ker_R(K)$) is a maximal submodule of $M(\rho)$ ($M(\rho\theta)$), and $M(\rho)/\ker_L(K)$ ($M(\rho\theta)/\ker_R(K)$) is an irreducible pointed module with ρ ($\rho\theta$) as a mass function.*

Proof. Consider K on $U(g)$. Then $\ker_L(K)$ is a left ideal of $U(g)$, which infers that $\ker_L(K)$ is a submodule of $M(\rho)$.

Let $L(\rho) = M(\rho)/\ker_L(K)$. Since $M(\rho)$ is graded, by Lemma 3.1, $\ker_L(K)$ is graded and $L(\rho)$ is also graded. Similar result is true for $\rho\theta$.

Suppose that V is a proper submodule of $L(\rho)$. Let $v \in V$ be a weight vector of V. We shall show that $v \in \ker_L(K)$. In fact, $L(\rho)$ and $L(\rho\theta)$ are decomposed into direct sum of weight spaces, and under K the weight spaces $L(\rho)_{\rho|_\mathfrak{h}+\alpha}$ are orthogonal to $L(\rho\theta)_{\rho\theta|_\mathfrak{h}+\beta}$ if $\alpha \neq \beta$. Assume that v is of weight $\rho|_\mathfrak{h} + \alpha$. Than to show $v \in \ker_L(K)$, we need only to show that $K(v,\ v') = 0$ for any $v' \in L(\rho\theta)_{\rho\theta|_\mathfrak{h}+\alpha}$. As $L(\rho\theta)$ is cyclic, there exists a $u \in U(g)$ such that $uv_0 = v'$, where $v_0 \in L(\rho\theta)_{\rho\theta|_\mathfrak{h}}$. So $K(v,\ v') = K(\theta(u)v,\ v_0) = 0$, since $\theta(u)v \in V_{\rho|_\mathfrak{h}}$ and $V_{\rho|_\mathfrak{h}} = 0$ for V is a proper submodule of $L(\rho)$. Thus $v \in \ker_L(K)$ and $V = 0$, which implies that $L(\rho)$ is irreducible.

Remark 3.1. $L(\rho) = M(\rho)/\ker_L(K)$ is the unique irreducible quotient module of $M(\rho)$.

Theorem 3.3. *Let ρ be any mass function of a Kac-Moody algebra g. Then on $L(\rho) \times L(\rho\theta)$ there exists a unique, up to constant factors, nondegenerate contravariant bilinear pairing K, and with respect to K, $L(\rho)_{\rho|_\mathfrak{h}+\alpha}$ is orthogonal to $L(\rho\theta)_{\rho\theta|_\mathfrak{h}+\beta}$ if $\alpha \neq \beta$.*

Proof. The existence of such a contravariant bilinear pairing is given as above, it is clearly nondegenerate.

Now, suppose that K_1 and K_2 are two nondegenerate contravariant bilinear pairing on $L(\rho) \times L(\rho\theta)$. Let $v \in L(\rho)_{\rho|_\mathfrak{h}},\ v' \in L(\rho\theta)_{\rho\theta|_\mathfrak{h}}$ be two vectors. Then $K_1(v,\ v') \neq 0$ and $K_2(v,\ v') \neq 0$ for K_1 and K_2 are nondegenerate. Let $c = K_1(v,\ v')/K_2(v,\ v')$. We claim that $K_1 = cK_2$. In fact, since $L(\rho)$ and $L(\rho\theta)$ are decomposed into direct sum of weight spaces and $L(\rho)_{\rho|_\mathfrak{h}+\alpha}$ is orthogonal to $L(\rho\theta)_{\rho\theta|_\mathfrak{h}+\beta}$ when $\alpha \neq \beta$, we need only to show that for any two weight vectors v_1 and v_2 with weights $\rho|_\mathfrak{h} + \alpha$ and $\rho\theta|_\mathfrak{h} + \alpha$ respectively, one has $K_1(v_1,\ v_2) = cK_2(v_1,\ v_2)$. Suppose $v_1 = u_1 v,\ v_2 = u_2 v$ where $u_1,\ u_2 \in U(g)_\alpha$. Then $\theta(u_2)u_1 \in U(g)_0$. Let $c' = \rho(\theta(u_2)u_1) \in \mathbb{C}$, which is a constant. Then

$$\begin{aligned} K_1(v_1,\ v_2) &= K_1(\theta(u_2)u_1 v,\ v') = c'K_1(v,\ v') = c'cK_2(v,\ v') \\ &= cK_2(c'v,\ v') = cK_2(v_1,\ v_2). \end{aligned}$$

This shows that $K_1 = cK_2$.

Definition 3.3. *Let θ be an antilinear involution of g. A mass function ρ is called symmetric, if $\rho(\theta(c)) = \rho(c)$ for all $c \in C(\mathfrak{h})$.*

Example 3.1. Let A be a generalized Cartan matrix not of finite type, g the Kac-Moody algebra associated with A. View g as a g module via the adjoint action. Then g is a pointed module (the real root spaces are of dimension 1). If A is not symmetrizable, then for a fixed nonzero real root vector the corresponding mass function is not symmetric. Conversely, if A is symmetrizable, then for any fixed nonzero real root vector, the corresponding mass function is symmetric.

Corollary 3.1. *Let $\rho : C(\mathfrak{h}) \to \mathbb{C}$ be a symmetric mass function, then there exists a unique nondegenerate symmetric contravariant bilinear form K on $L(\rho)$, and with respect to K, $L(\rho)$ is decomposed into a direct sum of mutually orthogonal weight spaces.*

Let ω be a compact antilinear involution of g, such that $\omega(e_i) = f_i,\ \omega(f_i) = e_i,\ \omega(h_i) = h_i$ and $\omega(au) = \bar{a}\omega(u)$ for $a \in \mathbb{C},\ u \in g$, where $\bar{a}$ is the complex conjugate of a. If $\rho_R : C(\mathfrak{h}_R) \to \mathbb{R}$ is a mass function of the compact form g_R of g, then g_R can be extended uniquely to a mass function of g.

Corollary 3.2. *Let $\rho_R : C(\mathfrak{h}_R) \to \mathbb{R}$ be a symmetric mass function (with respect to the compact involution ω) of g_R. Then there exists a unique nondegenerate contravariant Hermitian form K on $L(\rho)$, such that $L(\rho)$ is a direct sum of mutually orthogonal weight spaces.*

Definition 3.4. *Let ρ be a mass function. If there exists a positive definite contravariant Hermitian form on $L(\rho)$, then $L(\rho)$ is called a unitary pointed representation of g.*

Definition 3.5. An $\mathfrak{h}$-diagonalizable module V over a Kac-Moody algebra g is called integrable if all e_i and f_i, $(i = 1, \ldots, n)$ are locally nilpotent on V.

The following lemma can be seen in [3].

Lemma 3.2. (1) *Let $v_1,\ v_2,\ \ldots,$ be a system of generators of a g-module V, and let $x \in g$ be such that adx is locally nilpotent on g and $x^{N_i}(v_i) = 0$ for some positive integers $N_i,\ i = 1, \ldots$. Then x is locally nilpotent on V.*

(2) *For Kac-Moody algebra $g(A)$, ade_i and adf_i are locally nilpotent on $g(A)$.*

Theorem 3.3. *Let $L(\rho)$ be a unitary cyclic pointed representation of a Kac-Moody algebra $g(A)$, then $L(\rho)$ is integrable.*

Proof. Let v be a weight vector of $L(\rho)$ with weight $\rho|_{\mathfrak{h}}$. Since v generates $L(\rho)$, by Lemma 3.2 we need only to prove that there exist integers N_i and N_i' $(i = 1, \ldots, n)$ such that $f_i^{N_i} v = e_i^{N_i'} v = 0$. Suppose that $\rho(f_i e_i) = c_i,\ \rho(e_i f_i) = d_i$. Then $\rho(h_i) = d_i - c_i$. As $K(f_i v,\ f_i v) = \rho(e_i f_i)$, $K(e_i v,\ e_i v) = \rho(f_i e_i)$, it is from the unitary of K that $c_i \geq 0,\ d_i \geq 0$. For v as above, we establish two formulas:

$$e_i f_i^k = (k\rho(h_i) - k(k-1) + c_i) f_i^{k-1} v, \tag{3.1}$$

$$f_i e_i^k v = (-k\rho(h_i) - k(k-1) + d_i) e_i^{k-1} v. \tag{3.2}$$

Indeed

$$\begin{aligned} e_i f_i^k v &= (f_i e_i + h_i) f_i^{k-1} v \\ &= (\rho(h_i) - 2(k-1)) f_i^{k-1} v + f_i e_i f_i^{k-1} v \\ &= (k\rho(h_i) - k(k-1) + c_i) f_i^{k-1} v. \end{aligned}$$

Similarly

$$\begin{aligned} f_i e_i^k v &= (e_i f_i - h_i) e_i^{k-1} v \\ &= -(\rho(h_i) + 2(k-1)) e_i^{k-1} v + e_i f_i e_i^{k-1} v \\ &= (-k\rho(h_i) - k(k-1) + d_i) e_i^{k-1} v. \end{aligned}$$

Now if $f_i^k v \neq 0$ for all $k \in \mathbb{Z}^+$, then $K(f_i^k v,\ f_i^k v) > 0$ by the assumption of unitarity. As $\lim\limits_{k\to\infty} (k(\rho(h_i) - k(k-1) + c) = -\infty$, choose k_0 such that $k_0\rho(h_i) - k_0(k_0 - 1) + c_i < 0$ and $k(\rho(h_i) - k(k-1) + c_i) \geq 0$, for $0 < k < k_0$. Then

$$\begin{aligned} K(f_i^{k_0} v,\ f_i^{k_0} v) &= K(e_i f_i^{k_0} v,\ f_i^{k_0 - 1}) \\ &= (k_0\rho(h_i) - k_0(k_0 - 1) + c_i) K(f_i^{k_0 - 1} v,\ f_i^{k_0 - 1} v). \end{aligned} \tag{3.3}$$

Since $K(f_i^{k_0-1} v,\ f_i^{k_0-1} v) > 0$ and $K(f_i^{k_0} v,\ f_i^{k_0} v) > 0$, (3.3) is a contradiction. So there exist some positive integers N_i $(i = 1, \ldots, n)$ such that $f_i^{N_i} v = 0$. Similarly, there exist some positive integers N_i' $(i = 1, \ldots, n)$ such that $e_i^{N_i'} v = 0$. Thus we complete the proof of Theorem 3.3.

As a particular case of the theorem we have the following

Corollary 3.3. *Let $L(\Lambda)$ be an irreducible highest weight representation of Kac-Moody algebra $g(A)$, where $\Lambda \in \mathfrak{h}^*$ is the highest weight. If $L(\Lambda)$ is unitary then $\Lambda \in P^+$.*

Proof. $L(\Lambda)$ is unitary, and then it is integrable by Theorem 3.3. But $L(\Lambda)$ is integrable if and only if $\Lambda \in P^+$ (the set of dominant integral weights)[3].

Proposition 3.1. *Let g be a Kac-Moody algebra and $L(\rho)$ an integrable cyclic pointed module. Then there exist positive integers k_i and k_i' $(i = 1, \ldots, n)$ such that $\rho(h_i) = k_i - k_i'$, $\rho(f_i e_i) = k_i(k_i' - 1)$ which are integers.*

Proof. $L(\rho)$ is integrable. By definition, there exist positive integers k_i $(i = 1, \ldots, n)$ such that $f_i^{k_i} v = 0$ and $f_i^k v \neq 0$ for $k < k_i$, where v is the vector of weight $\rho|_{\mathfrak{h}}$. As

$$e_i f_i^{k_i} v = (k_i \rho(h_i) - k_i(k_i - 1) + \rho(f_i e_i)) f_i^{k_i - 1} v, \text{ and } f_i^{k_i - 1} v \neq 0,$$

this implies that

$$k_i \rho(h_i) - k_i(k_i - 1) + \rho(f_i e_i) = 0. \tag{3.4}$$

Similarly, there exist positive integers k_i' $(i = 1, \ldots, n)$ such that $e_i^{k_i'} v = 0$ and $e_i^k v \neq 0$ for $k < k_i'$, and $-k_i' \rho(h_i) - k_i'(k_i' - 1) + \rho(e_i f_i) = 0$. As $\rho(h_i) = \rho(e_i f_i) - \rho(f_i e_i)$, we have

$$(1 - k_i') \rho(h_i) - k_i'(k_i' - 1) + \rho(f_i e_i) = 0. \tag{3.5}$$

Combining (3.4) and (3.5), we have $\rho(f_i e_i) = k_i(k_i' - 1), \quad \rho(h_i) = k_i - k_i'$.

Remark 3.2. This proposition does not completely determine a concrete mass function of an integrable cyclic pointed module, as the primitive cycles of generating set are much more than what we have listed (h_i, $f_i e_i$, $i = 1, \ldots, n$, are only part of them) and in general, the set of primitive cycles is very complicated.

For any Kac-Moody algebra g, its primitive cycles are countable. Thus we can fix an order to the set π of all primitive cycles. Now we have the following theorem which is a generalization of Theorem 4.4 of [4].

Theorem 3.4. *Let π be the set of all primitive cycles with a fixed order. Suppose that c_i, $c_j \in \pi$ are two primitive cycles. Then $c_i c_j$ can be decomposed into a sum of products of primitive cycles, i.e., $c_i c_j = \sum \prod_{c \in \pi} c^{n(c)}$ (in $\prod_{c \in \pi} c^{n(c)}$ all c are ordered). Moreover, if $\rho : \pi \to \mathbb{C}$ is a map such that $\rho(c_i)\rho(c_j) = \sum \prod_{c \in \pi} \rho(c)^{n(c)}$, then ρ defines a mass function.*

Proof. It is analogous to the proof of Theorem 4.4 of [4].

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