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BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF NONLINEAR SECOND ORDER DIFFERENTIAL DFFERENCE EQUATIONS

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Abstract

The author studies the boundary value problems for systems of nonlinear second order differential difference equations and adopts a new-type Nagumo condition, in which the control function is a vector-valued function of several variables and which can guarantee simultaneously and easily finding a priori bounds of each component of the derivatives of the solutions. Under this new-type Nagumo condition the existence results of solution are proved by means of differential inequality technique.

Keywords Boundary value problem, Nonlinear differential difference system, New-type Nagumo condition, Existence of solution, Differential inequality.
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§1. Introduction

In study of some practical problems, for example, population problems, optimal control problems and some problems in biomathematics, one must consider the time delay in order that the true prosesses can be described more precisely. So the established mathematical models are often certain differential difference equations (or systems). This motivates mathematicians to study the initial or boundary value problems for differential difference equations, for example, L. J. Grimm and K. Schmitt[1,2], K. Schmitt[3,4], Miao Shumei and Zhou Qinde[5] and Miao Shumei[6]. They discuss mainly the following boundary value problems for nonlinear second order differential difference equations

$$x'' = f(t, x(t-\tau_1), \cdots, x(t-\tau_m), x, x'),$$
$$x(t) = \phi(t), \quad -\tau \le t \le 0, \quad x(1) = \alpha,$$

where $0 < \tau_i < 1,\ i = 1, 2, \cdots, m,\ \phi(t) \in C([-\tau, 0], R),\ \tau = \max\{\tau_1, \tau_2, \cdots, \tau_m\}$, by means of differential inequality technique. However it is very difficult to apply this technique to the boundary value problems for systems of nonlinear second order differential difference equations, because the control functions in the classical Nagumo conditions are all functions of one variable (see [7–11]). Hence the related works are rare.

In this paper we study the boundary value problems for systems of nonlinear second order differential difference equations and adopt a new-type Nagumo condition, in which the control function is a vector-valued function of several variables and which can guarantee

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simultaneously and easily finding a priori bounds of each component of the derivatives of the solutions. Under this new-type Nagumo condition we prove the existence results of solution for the boundary value problems of nonlinear differential difference systems by means of differential inequality technique. This is the contents in Section 3. Finally in Section 4 we exhibit an example as an application of the results obtained in Section 3.

§2. Notations and Definitions

Consider the following boundary value problems for the systems of nonlinear second order differential difference equations

$$x'' = f(t, x(t-\tau), x, x'), 0 \le t \le 1, \tag{2.1}$$

$$x(t) = \phi(t), -\tau \le t \le 0, x(1) = A, \tag{2.2}$$

where $x, f \in R^n, \phi(t) \in C([-\tau, 0], R^n), A \in R^n$ is a constant vector.

For simplification of writing we will adopt following notations: For $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ and $f(x) \in C(R^n, R^n)$,

$$x \ge y \text{ means } x_i \ge y_i, i = 1, 2, \cdots, n,$$

$$|x| = (|x_1|, |x_2|, \cdots, |x_n|), \quad \|x\| = \max\{|x_1|, |x_2|, \cdots, |x_n|\},$$

$$x[y_i]_i = (x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n),$$

$$x[0]_i = (x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n),$$

$$f(x[y]) = (f_1(x[y_1]_1), f_2(x[y_2]_2), \cdots, f_n(x[y_n]_n)),$$

$$f(x[0]) = (f_1(x[0]_1), f_2(x[0]_2), \cdots, f_n(x[0]_n)).$$

For $N \in R$, let $\vec{N} = (N, N, \cdots, N) \in R^n$. And for $g(t) \in C([a, b], R^n)$, let

$$\|g(t)\|_{[a,b]} = \max_{1 \le i \le n} \{\max_{a \le t \le b} |g_i(t)|\}.$$

Definition 2.1. *A function $x(t) \in C([-\tau, 1], R^n) \cap C^2([0, 1], R^n)$ is said to be a solution of the boundary value problem* (2.1), (2.2), *if $x(t)$ satisfies* (2.1), (2.2).

Definition 2.2. *Two functions $\bar{\omega}(t), \underline{\omega}(t) \in C([-\tau, 1], R^n) \cap C^2([0, 1], R^n)$ are said to be upper and lower solutions of the boundary value problem* (2.1),(2.2) *respectively, if*

$$\underline{\omega}(t) \le \bar{\omega}(t), \quad -\tau \le t \le 1,$$

$$\underline{\omega}(t) \le \phi(t) \le \bar{\omega}(t), \quad -\tau \le t \le 0,$$

$$\underline{\omega}(1) \le A \le \bar{\omega}(1)$$

and for any function $g(t) \in B[\underline{\omega}, \bar{\omega}]$

$$\underline{\omega}''(t) \ge f(t, g(t-\tau), g[\underline{\omega}], g'[\underline{\omega}']), \quad 0 \le t \le 1,$$

$$\bar{\omega}''(t) \le f(t, g(t-\tau), g[\bar{\omega}], g'[\bar{\omega}']), \quad 0 \le t \le 1,$$

where

$$B[\underline{\omega}, \bar{\omega}] = \{g(t) : g(t) \in C([-\tau, 1], R^n) \cap C^2([0, 1], R^n), \\ \underline{\omega}(t) \le g(t) \le \bar{\omega}(t), \quad -\tau \le t \le 1\}.$$

Definition 2.3. *If for any real number $r > 0$, there exists a function $H(\xi) \in C([0, \infty)^n, (0, \infty)^n)$, which is nondecreasing in every ξ_i, such that $|f(t, x, y, z)| \le H(|z|)$ for $0 \le t \le$*

1, $\|x\|, \|y\| \le r$ and there exists a real number $N_0 > 0$ such that

$$\int_{2r}^{N} [\xi_i/h_i(\vec{N}[\xi_i]_i)]d\xi_i > 2r, \quad i = 1, 2, \cdots, n$$

for any $N > N_0$, where h_i is the i-th component of H. Then we say that the function $f(t, x, y, z)$ satisfies Nagumo condition with respect to z.

§3. Existence Theorems of Solution

For the boundary value problem (2.1), (2. 2) our essential hypotheses are as follows:

(H$_1$) $f(t, x, y, z) \in C([0, 1] \times R^{3n}, R^n)$ satisfies Nagumo condition with respect to z.

(H$_2$) $f_i(t, x, y, z)$ is strictly increasing in y_i as any other variables are fixed, $i = 1, 2, \cdots, n$.

(H$_3$) Boundary value problem (2.1) (2.2) has upper and lower solutions $\bar{\omega}(t)$ and $\underline{\omega}(t)$.

The following theorems are our main results.

Theorem 3.1. *Assume that* (H$_1$), (H$_2$) *and* (H$_3$) *hold. Then boundary value problem* (2.1), (2.2) *has a solution $x(t)$ satisfying the inequality*

$$\underline{\omega}(t) \le x(t) \le \bar{\omega}(t), \quad 0 \le t \le 1. \tag{3.1}$$

Proof. Let $r = \max\{\|\bar{\omega}(t)\|_{[-\tau,1]}, \|\underline{\omega}(t)\|_{[-\tau,1]}\}$. From (H_1) there exists a function $H(\xi) \in C([0, \infty)^n, (0, \infty)^n)$, which is nondecreasing in every ξ_i, such that $|f(t, x, y, z)| \le H(|z|)$ for $0 \le t \le 1$, $\|x\|, \|y\| \le r$ and there exists a real number $N_0 > 0$ such that

$$\int_{2r}^{N} [\xi_i/h_i(\vec{N}[\xi_i]_i)]d\xi_i > 2r, \quad i = 1, 2, \cdots, n$$

for any $N > N_0$. For such an $N > N_0$ we define the set of functions as follows:

$$\overline{B} = \{g(t) : g(t) \in B[\underline{\omega}, \bar{\omega}],\ g(t) \equiv \phi(t),\ -\tau \le t \le 0,\ g(1) = A,\ \|g'(t)\|_{[0.1]} \le N\}.$$

Obviously $\overline{B}$ is a bounded closed convex subset of Banach space $C([-\tau, 1], R^n)$ with the norm $\|\cdot\|_{[-\tau,1]}$.

Next we divide the proof into three steps.

1) We prove that for each $g(t) \in \bar{B}$ the corresponding boundary value problem

$$x'' = f(t, g(t-\tau), g[x], g'[x']), \tag{3.2}$$

$$x(0) = \phi(0), x(1) = A \tag{3.3}$$

has a unique solution $x_g(t)$ satisfying the inequality

$$\underline{\omega}(t) \le x_g(t) \le \overline{\omega}(t), \quad 0 \le t \le 1. \tag{3.4}$$

Since (3.2) consists of n equations independent of each other, from Theorem 7.3 in [11] we immediately conclude that (3.2), (3.3) has a solution $x_g(t)$ satisfying (3.4). In addition, from (H_2) it is clear that $x_g(t)$ is the unique solution of (3.2),(3.3).

2) We prove that for all functions $g(t) \in \overline{B}$, the solutions $x_g(t)$ of the corresponding boundary value problems (3.2), (3.3) all satisfy the inequality

$$\|x_g'(t)\|_{[0,1]} \le N. \tag{3.5}$$

Assume that (3.5) is not true. Then there exist a $g_0(t) \in \bar{B}$, an $i_0 (1 \le i_0 \le n)$ and a $t_0 \in [0, 1]$ such that $|x'_{g_0,i_0}(t_0)| > N$. From Lagrange mean value theorem there exists a $t_1 \in (0, 1)$ such that $|x'_{g_0,i_0}(t_1)| = |x_{g_0,i_0}(1) - x_{g_0,i_0}(0)| \le 2r$. Owing to the continuity of $x'_{g_0,i_0}(t)$

there exist $t_2, t_3 \in [0,1]$ with $|x'_{g_0,i_0}(t_2)| = 2r, |x'_{g_0,i_0}(t_3)| = N$ and $2r < |x'_{g_0,i_0}(t)| < N$ for $t_2 < t < t_3$ (or $t_3 < t < t_2$). Thus taking note of

$$|f_{i_0}(t, g_0(t-\tau), g_0[x_{i_0}]_{i_0}, g'_0[x'_{i0}]_{i_0})| \le h_{i_0}(|g'_0[x'_{i_0}]_{i_0}|) \le h_{i_0}(\vec{N}[|x'_{i_0}|]_{i_0}),$$

we have

$$2r < \int_{2r}^{N} [\xi_{i_0}/h_{i_0}(\vec{N}[\xi_{i_0}]_{i_0})] d\xi_{i_0} \le \Big| \int_{t_2}^{t_3} \frac{x'_{g_0,i_0}(t)|x''_{g_0,i_0}(t)|}{h_{i_0}(\vec{N}[|x'_{g_0,i_0}(t)|]_{i_0})} dt \Big|$$

$$\le \Big| \int_{t_2}^{t_3} x'_{g_0,i_0}(t) dt \Big| = |x_{g_0,i_0}(t_3) - x_{g_0,i_0}(t_2)| \le 2r.$$

This contradiction shows that (3.5) holds.

3) We prove that boundary value problem (2.1),(2.2) has a solution $x(t)$ satisfying (3.1).

For 1) and 2) we know that for each $g(t) \in \overline{B}$ there exists a unique function $x_g(t)$, which is the unique solution of (3.2), (3.3) and satisfies (3.4), (3.5). Define

$$x(t) = \begin{cases} \phi(t), & -\tau \le t \le 0, \\ x_g(t), & 0 \le t \le 1, \end{cases} \tag{3.6}$$

so that $x(t) \in \overline{B}$. This defines a mapping $T: \overline{B} \to \overline{B}$ as follows: $T: g(t) \to x(t)$, where $g(t) \in \overline{B}, x(t)$ is given by (3.6). It is clear that T is a continuous mapping on $\overline{B}$. Furthermore we will prove that T is a completely continuous mapping. Assume that $\{x_k(t)\} \subset T(\overline{B})$. Then

$$\|x_k(t)\|_{[-\tau,1]} \le r, \quad \|x'_k(t)\|_{[0,1]} \le N \text{ and } \|x''_k(t)\|_{[0,1]} \le M,$$

where M is the maximum of $\|f(t,x,y,z)\|$ on $0 \le t \le 1, \|x\|, \|y\| \le r, \|z\| \le N$. This shows that $\{x_k(t)\}$ is a uniformly bounded and equicontinuous sequence of functions on $[-\tau, 1]$ because $x_k(t) \equiv \phi(t), -\tau \le t \le 0$ and $\{x'_k(t)\}$ is also such on [0,1]. Consequently, from Ascoli-Arzela Theorem there exist subsequences $\{x_{k_j}(t)\}$ and $\{x'_{k_j}(t)\}$ converging uniformly on $[-\tau, 1]$ and [0,1] respectively, such that

$$\lim_{j\to\infty} x_{k_j}(t) = \hat{x}(t), \quad -\tau \le t \le 1, \quad \lim_{j\to\infty} x'_{k_j}(t) = \hat{x}'(t), \quad 0 \le t \le 1$$

and

$$\|\hat{x}(t)\|_{[-\tau,1]} \le r, \quad \|\hat{x}'(t)\|_{[0,1]} \le N, \quad \hat{x}(t) \equiv \phi(t), \quad -\tau \le t \le 0, \quad \hat{x}(1) = A.$$

Hence $\hat{x}(t) \in \overline{B}$. This shows T is a completely continuous mapping on $\overline{B}$. Thus by Schauder fixed point theorem T has a fixed point $x^*(t)$ in $\overline{B}$. This $x^*(t)$ is a solution of (2.1),(2.2) and satisfies (3.1). The proof of Theorem 3.1 is completed.

Theorem 3.2. *Assume that* $f(t,x,y,z) \in C([0,1] \times R^{3n}, R^n), f_i(t,x,y,z)$ *satisfies Lipschitz condition with respect to* $z, i = 1, 2, \cdots, n$, *and conditions* (H_2), (H_3) *hold. Then boundary value problem* (2.1), (2.2) *has a solution* $x(t)$ *satisfying* (3.1).

Proof. We only need to prove that $f(t,x,y,z)$ satisfies Nagumo condition with respect to z.

For any real number $r > 0$, let

$$M_i = \max_{\substack{0\le t\le 1 \\ \|x\|,\|y\|\le r}} |f_i(t,x,y,0)|, \quad i = 1, 2, \cdots, n.$$

Then we have

$$|f_i(t,x,y,z)| \le |f_i(t,x,y,z) - f_i(t,x,y,0)| + |f_i(t,x,y,0)|$$
$$\le L_i \sum_{j=1}^{n} |z_j| + M_i, \quad i = 1,2,\cdots,n,$$

for $0 \le t \le 1, \|x\|, \|y\| \le r$, where L_i is the Lipschitz constant. We define $H(\xi) = (h_1(\xi), h_2(\xi), \ldots, h_n(\xi)) \in C([0,\infty)^n, (0,\infty)^n)$ by

$$h_i(\xi) = L_i \sum_{j=1}^{n} \xi_j + M_i, \quad i = 1,2,\cdots,n.$$

Since for $i = 1,2,\cdots,n$,

$$\int_{2r}^{N} \frac{\xi_i}{L_i\xi_i + (n-1)NL_i + M_i} d\xi_i$$
$$= \frac{1}{L_i}\left[N - 2r - \Big((n-1)N + \frac{M_i}{L_i}\Big) \ln \frac{nNL_i + M_i}{(n-1)NL_i + M_i + 2rL_i}\right],$$

let us consider the function $f(\theta) \in C([2r,\infty), R)$:

$$f(\theta) = \theta - 2r - \left[(n-1)\theta + \frac{M_i}{L_i}\right] \ln \frac{n\theta L_i + M_i}{(n-1)\theta L_i + M_i + 2rL_i}.$$

It is easy to see that

$$\lim_{\theta\to\infty} \frac{f(\theta)}{\theta} = 1 - (n-1)\ln\frac{n}{n-1} \overset{\text{def.}}{=} k > 0,$$
$$\lim_{\theta\to\infty} (f(\theta) - k\theta) = -2r - \frac{M_i}{L_i}\ln\frac{n}{n-1} + \frac{2rnL_i + M_i}{nL_i} \overset{\text{def}}{=} b.$$

So $w = k\theta + b$ is the asymptotic line of $f(\theta)$ for $\theta \to \infty$. Hence $f(\theta) \to \infty (\theta \to \infty)$. Consequently there exists a real number $N_0 > 0$ such that $f(\theta) > 2rL_i$ for $\theta > N_0$. This shows that

$$\int_{2r}^{N} \frac{\xi_i}{L_i\xi_i + (n-1)NL_i + M_i} d\xi_i > 2r$$

for any $N > N_0$. Thus we conclude that $f(t,x,y,z)$ satisfies Nagumo condition. The proof of Theorem 3.2 is completed.

Theorem 3.3. *Assume that* $f(t,x,y,z) \in C^1([0,1] \times R^{3n}, R^n)$,

$$\left|\frac{\partial f_i}{\partial z_j}\right| \le m, \quad i,j = 1,2,\cdots,n, \frac{\partial f_i}{\partial y_i} \ge l_i > 0, \quad i = 1,2,\cdots,n$$

and $|f(t,x,y[0],z[0])| \le M$, *where* $M = (M_1,\ldots,M_n)$. *Then the boundary value problem* (2.1),(2.2) *has at least a solution.*

Proof. $\left|\frac{\partial f_i}{\partial z_j}\right| \le m, i,j = 1,2,\cdots,n$, implies that $f_i(t,x,y,z)$ satisfies Lipschitz condition, $i = 1,2,\cdots,n$. Hence similarly to the proof of Theorem 3.2 we conclude that $f(t,x,y,z)$ satisfies Nagumo condition with respect to z. Furthermore the condition (H_2) holds because $\frac{\partial f_i}{\partial y_i} \ge l_i > 0, i = 1,2,\cdots,n$. Finally let

$$\overline{\omega}_i(t) = \begin{cases} |\phi_i(t)| + |\alpha_i| + \frac{M_i}{l_i}, & -\tau \le t \le 0, \\ (|\phi_i(0)| + |\alpha_i|)e^{\lambda_i t} + \frac{M_i}{l_i}, & 0 \le t \le 1, \end{cases}$$
$$\underline{\omega}_i(t) \equiv -\overline{\omega}_i(t), \quad -\tau \le t \le 1,$$

where $\lambda_i = (-m + \sqrt{m^2 + 4l_i})/2, \phi_i, \alpha_i$ are i-th components of ϕ, A respectively. We define

$$\overline{\omega}(t) = (\overline{\omega}_1(t), \overline{\omega}_2(t), \cdots, \overline{\omega}_n(t)), \underline{\omega}(t) = (\underline{\omega}_1(t), \underline{\omega}_2(t), \cdots, \underline{\omega}_n(t)).$$

Then it is easy to prove that $\overline{\omega}(t), \underline{\omega}(t)$ are upper and lower solutions of (2.1),(2.2), respectively. Thus from Theorem 3.1, (2.1),(2.2) has at least a solution. The proof is completed.

§4. An Example

Consider the following boundary value problem

$$x'' = (2 + \sin[x(t-\tau) + y(t-\tau)])\, x' + \frac{y'}{1 + (y')^2} + x + \text{ arctg } y + e^t, \tag{4.1}$$

$$y'' = |x'|^{1/2} \cdot y' + y\text{ch}x + \exp(-[x^2(t-\tau) + y^2(t-\tau)] + \ln(1+t), \tag{4.2}$$

$$x(t) = y(t) = 0, \quad -\tau \le t \le 0, \quad x(1) = y(1) = 0. \tag{4.3}$$

Obviously it is impossible or very difficult to apply the classical Nagumo conditions to this example. However it is easy to verify that the functions on the right hand sides of (4.1), (4.2) satisfy Nagumo condition.

Next let $\overline{\omega}(t) = (\overline{\omega}_1(t), \overline{\omega}_2(t)), \underline{\omega}(t) = (\underline{\omega}_1(t), \underline{\omega}_2(t))$, where

$$\overline{\omega}_1(t) = \begin{cases} \frac{\pi}{2} + \frac{1}{2}, & -\tau \le t \le 0, \\ \exp\left(\frac{1+\sqrt{5}}{2}t\right) + \left(\frac{\pi}{2} - \frac{1}{2}\right), & 0 \le t \le 1, \end{cases}$$

$$\underline{\omega}_1(t) \equiv -\overline{\omega}_1(t), \quad -\tau \le t \le 1,$$

$$\overline{\omega}_2(t) = \begin{cases} 1, & -\tau \le t \le 0, \\ e^t, & 0 \le t \le 1, \end{cases}$$

$$\underline{\omega}_2(t) \equiv -\overline{\omega}_2(t), \quad -\tau \le t \le 1.$$

We can prove that $\overline{\omega}(t)$ and $\underline{\omega}(t)$ are upper and lower solutions of (4.1)-(4.3) respectively. Consequently, from Theorem 3.1 boundary value problem (4.1)-(4.3) has at least a solution.

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