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TENSOR PRODUCT OF SEMIGROUPS AND THE EQUATION $AC-CB=Q$***

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Abstract

Properties for tensor products of semigroups are considered and the solutions of the equation $AC-CB=Q$ are discussed. Results obtained in this paper considerably generalize those obtained in [9].

Keywords Tensor product, Semigroup of operator, Complex Banach space.
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§1. Introduction

Let X, Z be complex Banach spaces and let $\underline{X}=B(Z,X)$ be the space of all bounded linear operators from Z into X. $\{H(t):t>0\}$, $\{G(t):t>0\}$ are semigroups of operators on X, Z, respectively. The family $\{T(t):t>0\}$ of operators on $\underline{X}$, defined by $T(t)C=H(t)CG(t)$, with $C\in X$, is a semigroup of operators on $\underline{X}$ and will be referred to as the tensor product of $H(\,\cdot\,)$ and $G(\,\cdot\,)$.

For a linear operator E, $R(E), N(E)$ and $D(E)$ denote the range, the null space and the domain of E.

Assume that the generators A and $-B$ of $H(\,\cdot\,)$ and $G(\,\cdot\,)$ exist, respectively, in a sense that will be made clear in §4. Our objective in this paper is to study the existence and uniqueness of the operator equation

$$AC-CB=Q, \tag{1.1}$$

with Q in $\underline{X}$. By a solution of equation (1.1), we mean an element $C\in\underline{X}$, satisfying the following conditions:

$$CD(B)\subset D(A),\quad ACz-CBz=Qz\ \text{ for all } z\in D(B).$$

Let Δ be the operator defined in $\underline{X}$ whose domain $D(\Delta)$ consists of all C's in $\underline{X}$, such that $CD(B)\subset D(A), AC-CB$ is bounded on $B(D)$, and which sends each C to the (unique) closure of $AC-CB$. Here, we assume that $D(B)$ is norm-dense in Z. Then, an equivalent formulation of (1.1) is to find $C\in X$ such that $\Delta C=Q$.

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M. Rosenblum[7] and J. A. Goldstein[2] considered equation (1.1) for the case in which A and B were selfadjoint operators in a separable Hilbert space $X = Z = H$. J. M. Freeman[1] studied the case in which A and B were generators of C_0-semigroups on a reflexive Banach space $X = Z$. S. Y. Shaw et al.[9] examined a more general case and obtained a new criterion for the solvability of (1.1), with A and B still generators of C_0-semigroups. For other papers on this subject, see the bibliographies in [1,2,7,9].

Our starting point is a more general one than the above mentioned [1,2,7,9] and we shall give a criterion for the solvability of (1.1) based on the characterization of Abel-ergodic properties established in [10]. Our result, compared to [1,2,7], is new and it is an essential generalization of [9].

Since we shall identify Δ to the generator of the tensor product $T(\,\cdot\,)$ of $H(\,\cdot\,)$ and $G(\,\cdot\,)$, under certain addition conditions, it will be convenient to consider first the general operator equation

$$Ax = q, \tag{1.2}$$

where A is the generator of a semigroup.

§2. The Operator Equation $Ax=q$

For the complex Banach space X, assume that Y is a norm-closed subspace of X^*, the dual of X, so that X and Y are reciprocal, that is,

$$\|x\| = \sup\{|\langle x, y\rangle|/\|y\| : y \in Y,\ y \neq 0\},$$

for all $x \in X$. Let $T(\,\cdot\,)$ be a semigroup of operators on X, satisfying the following conditions (see [9]):

(W1) Y is invariant under $T(t)^*$, for each $t > 0$;

(W2) $T(\,\cdot\,)x$ is $\sigma(X, Y)$-continuous on $(0, \infty)$, for each $x \in X$;

(W3) (a) for each $x \in X$ and $y \in Y$, $\langle T(t)x, Y\rangle$ as a function of t is L-integrable on $[0, 1]$;

(b) $\int_0^1 \langle T(t)x, y\rangle dt$ is $\sigma(Y, X)$-continuous with respect to $y \in Y$, for each fixed $x \in X$;

(W4) $X_0 = \cup\{\underline{R}(T(\eta)) : \eta > 0\}$ is $\sigma(X, Y)$-dense in X and $\cap\{N(T(\eta)) : \eta > 0\} = \{0\}$.

Such a $T(\,\cdot\,)$ is referred to as a weakly Y-integrable semigroup. As mentioned in [10], the semigroups of translations on spaces of Hölder-continuous functions are not strongly continuous. Actually they are weakly Y-integrable for some suitably chosen Y. Also, it was reported in [11] that the tensor product of two strongly continuous semigroups (even of C_0-class) may no longer be strongly continuous. A simple example of a strongly discontinuous semigroup is $T(\,\cdot\,)$ defined on $L^\infty(-\infty, \infty)$ by

$$[T(t)x](s) = x(s + t), \quad x \in L^\infty(-\infty, \infty).$$

A simple calculation shows that $T(\,\cdot\,)$ is weakly Y-integrable, with $Y = L(-\infty, \infty)$.

If $T(\,\cdot\,)$ satisfies (W1), (W2), (W3) and the following

(W4)$'$ $\cap\{N(T(\eta)) : \eta > 0\} = \{0\}$,

then $T(\,\cdot\,)$ is referred to as a pre-weakly Y-integrable semigroup.

If $T(\,\cdot\,)$ only satisfies (W1), (W2), and (W3), then $T(\,\cdot\,)$ is referred to as a quasi-weakly Y-integrable semigroup. In [10], we studied the Abel-ergodic properties for a quasi-weakly Y-integrable semigroup, which will be used in this paper (see Theorem 2.1).

Now assume that $T(\,\cdot\,)$ is pre-weakly Y-integrable. It is easily seen that Theorem 3.5 and Proposition 4.2 of [11] are applicable to $T(\,\cdot\,)$, under consideration. Hence the resolvent $R(\lambda)$ of $T(\,\cdot\,)$ exists for λ with $\mathrm{Re}\lambda > \omega_0$, where ω_0 is the type of $T(\,\cdot\,)$. Further, following [10], we may define the Laplace transform $R_Y(\,\cdot\,)$ of $T(\,\cdot\,)$ in a weaker sense. Let $R_Y(\lambda, t)$ be the operator defined by the equality

$$\langle R_Y(\lambda,t)x, y\rangle = \int_0^t e^{-\lambda u}\langle T(u)x, y\rangle du.$$

It follows from [10] that $R_Y(\lambda, t)$ is a bounded linear operator on X, for each $t > 0$ and $\lambda \in \mathbf{C}$. Consider those λ's for which $\lim\limits_{t\to\infty}\langle R_Y(\lambda,t)x, y\rangle$ exists for all $x \in X$ and $y \in Y$, and it defines a bounded linear operator $R_Y(\lambda) \in B(X)$ such that

$$\langle R_Y(\lambda)x, y\rangle = \lim_{t\to\infty}\langle R_Y(\lambda,t)x, y\rangle = \lim_{t\to\infty}\int_0^t e^{-\lambda u}\langle T(u)x, y\rangle du.$$

It has been proved in [10, Proposition 7] that if for a complex number $\lambda_0, R_Y(\lambda_0)$ is a bounded linear operator on X, then so is $R_Y(\lambda)$ for all λ with $\mathrm{Re}\lambda > \mathrm{Re}\lambda_0$. This enables us to define the number

$$\sigma_a := \inf\{u \in (-\infty,\infty) : R_Y(\lambda) \text{ is analytic for } \lambda \text{ with } \mathrm{Re}\lambda > u\}. \tag{2.1}$$

The following example[3,8] shows that the following strict inequality may occur:

$$-\infty = \sigma_a < 0 < \omega_0. \tag{2.2}$$

Example 2.1. Let $1 \le p \le q < \infty$, and let X be the set of all L-measurable functions on $(0,\infty)$ such that

$$\|f\| := \Big(\int_0^\infty e^{ps^2}|f(s)|^p ds\Big)^{1/p} + \Big(\int_0^\infty |f(s)|^q ds\Big)^{1/q} < \infty \quad \text{for} \quad f \in X.$$

Then $(X, \|\,\cdot\,\|)$ is a Banach lattice. For $\alpha \ge 0$, let $T_\alpha(\,\cdot\,)$ be the semigroup defined by

$$(T_\alpha(t)f)(s) = e^{\alpha t}f(t+s), \quad f \in X;\ s,t \ge 0.$$

Then $T_a(t) = e^{\alpha t}T_0(t)$. It was shown in [3] that $\|T_0(t)\| = 1$ for all $t \ge 0$ and for $T_0(\,\cdot\,)$, $\sigma_a = -\infty$. Further, the type of $T_\alpha(\,\cdot\,)$ is clearly equal to α. If $\alpha > 0$, then (2.2) holds for $T_\alpha(\,\cdot\,)$.

We return to discuss the pre-weakly Y-integrable semigroup $T(\,\cdot\,)$. From the definition, it is easily seen that $R_Y(\lambda) = R(\lambda)$ for λ with $\mathrm{Re}\lambda > \omega_0$. To simplify notation, we shall denote $R_Y(\lambda)$ by $R(\lambda)$, for all λ with $\mathrm{Re}\lambda > \sigma_a$. The generator of $T(\,\cdot\,)$ is denoted to be the following operator A:

$$\begin{cases} D(A) = \underline{R}(R(\lambda)), \\ (\lambda - A)^{-1} = R(\lambda), \quad \lambda \in \mathbf{C},\ \mathrm{Re}\lambda > \sigma_a. \end{cases} \tag{2.3}$$

Clearly, A is closed and hence $N(A)$ is closed.

The following theorem is a special case of [10, Corollary 3], that will serve our purpose.

Theorem 2.1. *Let $R(\cdot)$ be the resolvent of the pre-weakly Y-integrable semigroup $T(\cdot)$. If $\sigma_a \le 0$ and*

$$\overline{\lim_{\lambda\to 0}}\,\|\lambda R(\lambda)\| < \infty, \tag{2.4}$$

then the operator P_S defined by $P_S x := s - \lim_{\lambda\to 0} \lambda R(\lambda)x$, provided that the limit exists, has the following properties:

(i) *P_S is a bounded projection with its domain $D(P_S)$ closed;*

(ii) *$N(P_S) = \overline{\underline{R}(R(1)-I)} = \overline{\underline{R}(A)}$, $\underline{R}(P_S) = \overline{N(R(1)-I)} = N(A)$ and hence*

$$D(P_S) = \overline{\underline{R}(A)} \oplus N(A). \tag{2.5}$$

In terms of Theorem 2.1, one can prove the following

Theorem 2.2. *Let $T(\cdot)$ be a pre-weakly Y-integrable semigroup with generator A. If condition* (2.4) *holds, then the following statements are equivalent:*

(i) *$q \in A[D(A) \cap \overline{\underline{R}(A)}]$;*

(ii) *$x = s - \lim_{\lambda\to 0}[-R(\lambda)q]$ exists in X;*

(iii) *there exists a sequence $\{\lambda_n\}$ converging to zero such that $x = s - \lim_{n\to\infty}[-R(\lambda_n)q]$ exists in X;*

(iv) *there exists a sequence $\{\lambda_n\}$ converging to zero such that $x = w - \lim_{n\to\infty}[-R(\lambda_n)q]$ exists in X.*

If any of conditions (i)–(iv) *holds, then x is the unique solution of equation* (1.2) *in $\overline{\underline{R}(A)}$.*

Proof. (i)⇒(ii): Assuming that $q \in A(D(A) \cap \overline{\underline{R}(A)})$, we see that there exists $x \in D(A) \cap \overline{\underline{R}(A)}$ such that $Ax = q$ and hence

$$-R(\lambda)q = -R(\lambda)Ax = x - \lambda R(\lambda)x, \text{ or } x = -R(\lambda)q + \lambda R(\lambda)x.$$

Since $x \in \overline{\underline{R}(A)} = N(P_S)$, one has

$$x = s - \lim_{\lambda\to 0}[-R(\lambda)q + \lambda R(\lambda)x] = s - \lim_{\lambda\to 0}[-R(\lambda)q].$$

Implications (ii)⇒(iii) and (iii)⇒(iv) are clear.

(iv)⇒(i): The existence of $x = w - \lim_{n\to\infty}[-R(\lambda_n)q]$ implies

$$s - \lim_{n\to\infty}[\lambda_n R(\lambda_n)q] = 0.$$

Thus, it follows that

$$A[-R(\lambda_n)q] = q - \lambda_n R(\lambda_n)q \to q$$

in the norm topology. Since A is closed in the norm (and hence in the weak) topology, $x \in D(A)$ and $Ax = q$. Thus x is a solution of (1.2).

The equalities

$$x = w - \lim_{n\to\infty}[-R(\lambda_n)q] = w - \lim_{n\to\infty}[-R(\lambda_n)Ax] = w - \lim_{n\to\infty}[x - \lambda_n R(\lambda_n)x]$$

imply that $x \in \overline{\underline{R}(R(1)-I)} = \overline{\underline{R}(A)}$. Therefore $q \in A[D(A) \cap \overline{\underline{R}(A)}]$ and hene (i) is proved.

Finally, assume that one of conditions (i)–(iv) holds. Since $N(A) \cap \overline{\underline{R}(A)} = \{0\}$ by (2.5), x is evidently the unique solution of (1.2) in $\overline{\underline{R}(A)}$.

It has been proved in [10] that the strong and weak Abel-ergodicity for pseudo-resolvents (hence for pre-weakly Y-integrable semigroups) are equivalent, and when this property holds, one has

$$X = \overline{\underline{R}(A)} \oplus N(A) = N(P_S) \oplus \underline{R}(P_S).$$

Thus

$$\underline{R}(A) = A(D(A)) = A[D(A) \cap (\overline{\underline{R}(A)} \oplus N(A))] = A(D(A) \cap \overline{\underline{R}(A)}).$$

A straightforward consequence is the following

Corollary 2.1. *Assume that $T(\,\cdot\,)$ is strongly (hence weakly) Abel-ergodic. The following statements are equivalent:*

(i)$'$ $q \in \underline{R}(A)$;

(ii) $x = s - \lim\limits_{\lambda \to 0} [-R(\lambda)q]$;

(iii) *there exists a sequence* $\{\lambda_n\} \to 0$, *as* $n \to \infty$, *such that*

$$x = s - \lim_{n \to \infty} [-R(\lambda_n)q]$$

exists in X;

(iv) *there exists a sequence* $\{\lambda_n\} \to 0$, *as* $n \to \infty$, *such that*

$$x = w - \lim_{n \to \infty} [-R(\lambda_n)q]$$

exists in X.

Proof. Implications (i)$'$ $\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$ (iv) are clear. We claim that (iv)$\Rightarrow$(i)$'$ also holds. Indeed, it follows from $A[-R(\lambda_n)q] = q - \lambda_n R(\lambda_n)q$ that one has $Ax = q$ and hence $q \in \underline{R}(A)$.

We can compare the above Theorem 2.2 to [8, Theorem 2.3]. For this we need the following

Proposition 2.1. *If the pre-weakly Y-integrable semigroup $T(\,\cdot\,)$ satisfies condition*

$$\overline{\lim_{t \to \infty}}\, t^{-1} \|S(t)\| < \infty, \tag{2.6}$$

then $\sigma_a \le 0$ and (2.4) *holds, where $S(T) \in B(X)$ is defined by*

$$\langle S(t)x, y\rangle = \int_0^t \langle T(\tau)x, y\rangle d\tau.$$

If, in addition to (2.6), *we assume that the following limit exists*

$$x = s - \lim_{t \to \infty} [t^{-1} F(t)q], \tag{2.7}$$

then q satisfies (i) *of Theorem* 2.2. *In* (2.7), *$F(t) \in B(X)$ is defined by*

$$\langle F(t)x, y\rangle = \int_0^t \langle S(\tau)x, y\rangle d\tau.$$

Remark 2.1. The existence and the properties of $S(\,\cdot\,)$ have been studied in [12], those for $F(\,\cdot\,)$ can be deduced from $S(\,\cdot\,)$.

Proof of Proposition 2.1. By condition (2.6), there exists a number $M > 0$ so that

$t^{-1}\|S(t)\| \le M$ for $t \le 1$. Then, for $x \in X,\ y \in Y$ one has

$$\Big|\int_{t_0}^{t} e^{\lambda u}\langle T(u)x, y\rangle du\Big| \le \Big|[e^{-\lambda u}\langle S(u)x, y\rangle]_{t_0}^{t} + |\lambda| \int_{t_0}^{t} e^{-\lambda u}\langle S(u)x, y\rangle du\Big|$$
$$\le \Big[te^{(-\mathrm{Re}\lambda)t} + t_0 e^{(-\mathrm{Re}\lambda)t_0} + |\lambda| \int_{t_0}^{t} e^{(-\mathrm{Re}\lambda)u} udu\Big] M\|x\|\,\|y\|.$$

Hence, for any given $\varepsilon > 0$, the inequality

$$\Big|\int_{t_0}^{t} e^{-\lambda u}\langle T(u)x, y\rangle du\Big| \le \varepsilon\|x\|\,\|y\| \tag{2.8}$$

holds uniformly for λ in every compact subset of the half plane $\{\lambda : \mathrm{Re}\lambda > 0\}$, whenever $t \ge t_0 (\ge 1)$ are sufficiently large. Consequently, $R_Y(\lambda) = \lim\limits_{t\to\infty} R_Y(\lambda, t)$ exists uniformly for λ in every compact subset of the upper half plane. $R_Y(\lambda, t)$ is analytic in λ on $\{\lambda : \mathrm{Re}\lambda > 0\}$ and so is $R_Y(\lambda)$. Hence $\sigma_a \le 0$. Letting $t_0 = 0$ and $t = \infty$ in (2.8) one gets for $\lambda > 0$,

$$|\langle \lambda R(\lambda)x, y\rangle| \le \Big(\lambda^2 \int_0^{\infty} e^{-\lambda u} udu\Big) M\|x\|\,\|y\| = M\|x\|\,\|y\|$$

and hence (2.4) holds.

Finally, we assume the additional condition (2.7). From [11, Theorem 3.5] and by integration by parts, one has

$$|\langle R(\lambda)q + x, y\rangle| = \lambda^2 \Big|\int_0^{t} e^{-\lambda t} t\langle t^{-1}F(t)q + x, y\rangle dt\Big|$$
$$\le \lambda^2 \Big(\int_N^{\infty} e^{-\lambda t} tdt\Big) \sup_{t\ge N} \|t^{-1}F(t)q + x\|\,\|y\|$$
$$+ \lambda^2 \int_0^{N} e^{-\lambda t} tdt \sup_{0\le t\le N} \|F(t)q\|\,\|y\| + \lambda^2 \int_0^{N} e^{-\lambda t} dt \|x\|\,\|y\|.$$

In view of (2.7) and the boundedness of $F(\,\cdot\,)$ on every closed interval $[a, b] \subset [0, \infty)$, it is easy to see that $\lim\limits_{\lambda\to 0} \|R(\lambda)q + x\| = 0$. Thus q satisfies Theorem 2.2 (ii) and hence (i).

Remark 2.2. A result similar to Theorem 2.2 was obtained in [9, Theorem 2.3]. In the latter there were assumed (2.7),

$$\|T(t)x\| = o(t) \quad \text{as} \quad t \to \infty, \text{ for each } x \in D(A) \tag{2.9}$$

and conditions on $T(\,\cdot\,)$, much stronger than (W1), (W2) and (W3) (see [8] for details). Proposition 2.1 and the following example show that Theorem 2.2 is an essential extension of [9, Theorem 2.3].

Example 2.2. Let $X = L_2(0, 1)$ and define

$$(J^{\zeta}x)(t) = \frac{1}{\Gamma(\zeta)} \int_0^{t} (t-u)^{\zeta-1} f(u)du.$$

In [4, pp. 664-665], the following were proved:

(i) the type ω_0 of $\{J^{\xi} : \xi = \mathrm{Re}\zeta > 0\}$ satisfies

$$\omega_0 = \lim_{\xi\to\infty} \xi^{-1} \log \|J^{\xi}\|_2 = -\infty$$

and hence the spectrum of the infinitesimal generator A of J^{ξ} is empty;

(ii) $\{J^{i\eta} : \eta = \mathrm{Im}\zeta \in (-\infty, \infty)\}$ is a strongly continuous group of operators on $L^2(0,1)$ with infinitesimal generator iA, so $\sigma(iA) = \emptyset$.

In view of [4, Theorem 23.16.1], it is easily seen that the function x, with $x(t) = 1$ on $[0,1]$ is in $D(A)$. By a few computations, one obtains

$$\|J^{i\eta}x\|_2 \geq |\langle J^{i\eta}x, x\rangle| = \frac{1}{|\Gamma(i\eta+2)|}.$$

It follows from [6, p.550] that

$$\log \frac{1}{|\Gamma(i\eta+2)|} = 1 - \frac{3}{4}\log(4+\eta^2) + \eta\arg(i\eta+2) + C(i\eta+2), \tag{2.10}$$

where $C(i\eta+2)$ is such that $|C(i\eta+2)| \leq \frac{\pi}{8}$. Relation (2.10) implies that

$$\lim_{\eta\to\infty}\left|\frac{1}{\Gamma(i\eta+2)}\right| = \infty.$$

Therefore, $J^{i\eta}$ does not satisfy (2.9) and hence [9, Theorem 2.3] does not apply to $J^{i\eta}$.

On the other hand, $\sigma(iA) = \emptyset$ shows that $\lim\limits_{\lambda\to 0}\|\lambda R(\lambda, iA)\| = 0$. Consequently $J^{i\eta}$ satisfies (2.4) and hence our Theorem 2.2 is applicable.

§3. Tensor Products of Semigroups

Let $H(\,\cdot\,), G(\,\cdot\,)$ be the semigroups given in §1. In order to reach the target of this paper, some additional conditions on $H(\,\cdot\,), G(\,\cdot\,)$ will be needed. Assume that $H(\,\cdot\,)$ is a pre-weakly Y-integrable semigroup, $G(\,\cdot\,)$ is a C_0-semigroup. The tensor product $T(\,\cdot\,)$ of $H(\,\cdot\,)$ and $G(\,\cdot\,)$ (see §1) is defined to be the family $\{T(t) : t > 0\}$ of operators on X satisfying

$$T(t)C = H(t)CG(t), \quad t > 0, \quad C \in X.$$

For each $z \in Z,\ y \in Y$, let $f_{z\otimes y}$ be the linear functional on X, defined by $\langle C, f_{z\otimes y}\rangle = \langle Cz, y\rangle$. Then $f_{z\otimes y}$ is bounded and $\|f_{z\otimes y}\| = \|z\|\,\|y\|$ (see [9]). Let $\underline{Y} \subset \underline{X}^*$ be the norm-closed linear span of all $f_{z\otimes y}$, with $z \in Z, y \in Y$. It has also been shown in [9] that $\underline{X}$ and $\underline{Y}$ are reciprocal.

Lemma 3.1. *The tensor product has the following property*

$$\|T(t)\| = \|H(t)\|\,\|G(t)\|, \quad t > 0. \tag{3.1}$$

Proof. Let $\varepsilon > 0$ and let $t > 0$ be fixed. Choose $x_0 \in X$ with $\|x_0\| = 1$ and $z_0 \in Z$ with $\|z_0\| = 1$ such that

$$\|H(t)x_0\| \geq \|H(t)\| - \varepsilon, \quad \|G(t)x_0\| \geq \|G(t)\| - \varepsilon.$$

Let $z_0^* \in Z$ satisfy conditions $\langle G(t)z_0/\|G(t)z_0\|, z_0^*\rangle = 1$ and $\|z_0^*\| = 1$. Define $C \in \underline{X}$ by

$$Cz = \langle z, z_0^*\rangle x_0, \quad \text{for all} \quad z \in Z.$$

Then $\|C\| = 1$ and

$$\begin{aligned}\|T(t)\| &\geq \|[T(t)C]z_0\| = \|H(t)CG(t)z_0\| \\ &= \|H(t)x_0\|\,\|G(t)\|z_0\| \geq (\|H(t)\| - \varepsilon)(\|G(t)\| - \varepsilon).\end{aligned}$$

Since ε is arbitrary, one obtains $\|T(t)\| \geq \|H(t)\| \, \|G(t)\|$. This, together with the evident opposite inequality, yields (3.1).

Corollary 3.1. *Let $\omega, \omega_0, \omega_1$ be the type of $T(\,\cdot\,), H(\,\cdot\,), G(\,\cdot\,)$, respectively. Then* $\omega = \omega_0 + \omega_1$.

Proposition 3.1. *If $H(\,\cdot\,)$ is pre-weakly $Y-$integrable, $G(\,\cdot\,)$ is of C_0-class, then $T(\,\cdot\,)$ is $\sigma(\underline{X}, \underline{Y})$-continuous on* $(0, \infty)$.

Proof. For each $z \in Z,\ y \in Y$ and $t > 0$, one has

$$\begin{aligned}
&|\langle T(t+\Delta t)C, f_{z\otimes y}\rangle - \langle T(t)C, f_{z\otimes y}\rangle| \\
&= |\langle CG(t+\Delta t)z, H(t+\Delta t)'y\rangle - \langle CG(t)z, H(t)'y\rangle| \\
&\leq |\langle CG(t+\Delta t)z, CG(t)z, H(t+\Delta t)y\rangle| + |\langle CG(t)z, H(t+\Delta t)'y - H(t)'y\rangle| \\
&\leq \|C\| \, \|[G(t+\Delta t) - G(t)z]\| \, \|H(t+\Delta t)'y\| \\
&\quad + |\langle CG(t)z, [H(t+\Delta t)' - H(t)']y\rangle| \to 0, \quad \text{as } \Delta t \to 0, \qquad (3.2)
\end{aligned}$$

where $H(t)' = H(t)^*|Y$. By Lemma 3.1, $T(\,\cdot\,)$ is bounded on every closed subinterval $[a, b]$ of $(0, \infty)$. This, together with (3.2), asserts that

$$\langle T(t+\Delta t)C, f\rangle \to \langle T(t)C, f\rangle \quad \text{as} \quad \Delta t \to 0, \quad \text{for all } f \in \underline{Y}.$$

Proposition 3.2. *$\underline{Y}$ is invariant under $T(t)^*$, for each $t > 0$.*

Proof. Let $z \in Z,\ y \in Y$. For every $C \in \underline{X}$, one obtains successively:

$$\langle C, T(t)^* f_{z\otimes y}\rangle = \langle T(t)C, f_{z\otimes y}\rangle = \langle CG(t)z, H(t)'y\rangle = \langle C, f_{G(t)z\otimes H(t)'y}\rangle.$$

Thus $T(t)^* f_{z\otimes y} = f_{G(t)z\otimes H(t)'y} \in \underline{Y}$ and hence $\underline{Y}$ is invariant under $T(t)^*$.

Propositions 3.1, 3.2 assert that $T(\,\cdot\,)$ satisfies conditions (W1), (W2), respectively. In the sequel, we shall denote $T(\,\cdot\,)' = T(\,\cdot\,)^*|\underline{Y}$. The following proposition gives a sufficient condition for $T(\,\cdot\,)$ to satisfy condition (W3).

Proposition 3.3. *Suppose that $G(\,\cdot\,)$ is a C_0-semigroup, $H(\,\cdot\,)$ satisfies properties* (W1), (W2), (W4)$'$ *and*

$$\|H(t)\| \leq \psi(t), \quad \text{a.e.} \quad t \in (0, \infty), \qquad (3.3)$$

where $\psi(\,\cdot\,)$ is a non-negative L-integrable function on $[0, \infty)$. Then $T(\,\cdot\,)$ satisfies (W3).

Remark 3.1. Condition (3.3) implies (W3) by [11, Proposition 3.5], therefore $H(\,\cdot\,)$ is a pre-weakly Y-integrable semigroup.

Proof of Proposition 3.3. The inequality

$$\|T(t)\| = \|H(t)\| \, \|G(t)\| \leq M\psi(t),$$

where $M > 0$ is a constant, and [11, Proposition 3.5] imply that $T(\,\cdot\,)$ satisfies (W3).

Proposition 3.4. *With the condition of Proposition 3.3, $T(\,\cdot\,)$ has the following property:*

$$\bigcap\{N(T(\eta)) : \eta > 0\} = \{0\}. \qquad (3.4)$$

Proof. Let $C \in \underline{X}$ be such that $T(\eta)C = 0$ for all $\eta > 0$. Then, for each $z \in Z,\ y \in Y$ and all $\eta > 0$, we have

$$\langle CG(\eta)z, H(\eta)'y\rangle = \langle T(\eta)C, f_{z\otimes y}\rangle = 0. \qquad (3.5)$$

Let $u \in \cup\{C\underline{R}(G(\eta)) : \eta > 0\}$ and $v \in \cup\{\underline{R}(H(\eta)') : \eta > 0\}$. There exist $\eta_1, \eta_2 > 0$ such that $u \in C\underline{R}(G(\eta_1))$, $v \in \underline{R}(H(\eta_2)')$. Set $\eta = \min\{\eta_1, \eta_2\}$. Then, clearly $u \in C\underline{R}(G(\eta))$, $v \in \underline{R}(H(\eta)')$. Choose z and y such that $u = CG(\eta)z$ and $v = H(\eta)'y$. By (3.5), $\langle u, v\rangle = 0$ and hence

$$\cup\{C\underline{R}(G(\eta)) : \eta > 0\} \perp \cup \{\underline{R}(H(\eta)') : \eta > 0\}.$$

(W4)$'$ applied to $H(\,\cdot\,)$ gives

$$\cap\{N(H(\eta)) : \eta > 0\} = \{0\}$$

and hence $\cup\{R(H(\eta)') : \eta > 0\}$ is $\sigma(Y, X)$-dense in Y. Consequently,

$$\cup\{C\underline{R}(G(\eta)) : \eta > 0\} = \{0\},$$

or equivalently,

$$C(\cup\{\underline{R}(G(\eta)) : \eta > 0\}) = \{0\}.$$

Since $\cup\{\underline{R}(G(\eta)) : \eta > 0\}$ is norm-dense in Z, one has $C = 0$, and hence (3.4) holds.

Corollary 3.2. *The resolvent $R_T(\lambda)$ of $T(\,\cdot\,)$ is injective on $\underline{X}$.*

Proof. The statement of the corollary follows from [11, Proposition 4.2] and (3.4).

So far we do not know whether $R_T(\lambda)$ has a $\sigma(\underline{X}, \underline{Y})$-dense range in $\underline{X}$. We can prove the following weaker result. Let $\underline{Y}_0$ be the linear span of $f_{z\otimes y}$, with $z \in Z,\ y \in Y$.

Proposition 3.5. *Assume that $G(\,\cdot\,)$ is a C_0-semigroup, $H(\,\cdot\,)$ satisfies* (W1),(W2), (W4), *and* (3.3). *Then*

$$\underline{X}_0 := \cup\{\underline{R}(T(\eta)) : \eta > 0\} \tag{3.6}$$

is $\sigma(\underline{X}, \underline{Y})$-dense in $\underline{X}$.

Proof. To prove that $\underline{X}_0$ is $\sigma(\underline{X}, \underline{Y})$-dense in $\underline{X}$, it suffices to show that

$$\cap\{N(T(\eta)'|\underline{Y}_0) : \eta > 0\} = \{0\}.$$

Now assume that $T(\eta)'f_{z\otimes y} = 0$ for all $\eta > 0$, and for some $z \in Z,\ y \in Y$. For $C \in \underline{X}$, we have

$$\begin{aligned} 0 &= \langle C, T(\eta)'f_{z\otimes y}\rangle = \langle T(\eta)C, f_{z\otimes y}\rangle \\ &= \langle H(\eta)CG(\eta)z, y\rangle = \langle CG(\eta)z, H(\eta)'y\rangle \\ &= \langle C, f_{G(\eta)z\otimes H(\eta)'y}\rangle. \end{aligned} \tag{3.7}$$

Thus, $f_{G(\eta)z\otimes H(\eta)'y} = 0$, or equivalently,

$$G(\eta)z \otimes H(\eta)'y = 0, \quad \text{for all } \eta > 0. \tag{3.8}$$

There are only two possible cases implied by (3.8):

(a) $G(\eta)z = 0$ for all $\eta > 0$. In this case $z = 0$, because $z = s - \lim\limits_{\eta\to 0+} G(\eta)z$.

(b) $G(\eta_0) \neq 0$ for some $\eta_0 > 0$. In this case $G(\eta)z \neq 0$ for $0 \le \eta \le \eta_0$, hence $H(\eta)'y = 0$ for $0 < \eta \le \eta_0$. $H(\,\cdot\,)'$ being a weakly X-integrable semigroup on Y by [12, Theorem 2.1], one has $y = 0$. Thus either of cases (a) and (b) implies $f_{z\otimes y} = 0$.

Next, assume that for some $f = f_{w_n}$, where $w_n = \sum\limits_{j=1}^{n} z_j \otimes y_j$, one has

$$T(\eta)'f = T(\eta)'f_{w_n} = 0.$$

A calculation similar to that of (3.7) produces the following analogue of (3.8):

$$\sum_{j=1}^{n} G(\eta)z_j \otimes H(\eta)'y_j = 0 \quad \text{for all} \quad \eta > 0. \tag{3.9}$$

We may assume that one of $\{z_j\}_{j=1}^n$, $\{y_j\}_{j=1}^n$, say the latter, is linearly independent. We shall assert that the system $\{H(\eta)'y_j\}_{j=1}^n$ is linearly independent for a sufficiently small $\eta > 0$. Assuming the contrary, there exists, at least, one decreasing sequence $\{\eta_m\}$ that converges to zero such that $\{H(\eta_m)'y_j\}_{j=1}^n$ is linearly dependent. Hence, for each m, there exists a system of numbers $\{b_j^{(m)}\}_{j=1}^n$ satisfying the relations

$$H(\eta_m)'\Big(\sum_{j=1}^{n} b_j^{(m)}y_j\Big) = \sum_{j=1}^{m} b_j^{(m)}H(\eta_m)'y_j = 0, \text{ for all } m;$$

$$\sum_{j=1}^{n} |b_j^{(m)}| = 1. \tag{3.10}$$

Clearly, we may assume that, for each j, $b_j^{(m)} \to b_j$, as $m \to \infty$. Then (3.10) implies

$$\sum_{j=1}^{n} |b_j| = 1. \tag{3.11}$$

Let m_0 be fixed. Then, for $m > m_0$,

$$H(\eta_{m_0})'\Big[\sum_{j=1}^{n} b_j^{(m)}y_j\Big] = H(\eta_{m_0} - \eta_m)'\Big[\sum_{j=1}^{m} b_j^{(m)}H(\eta_m)'y_j\Big] = 0.$$

Letting $m \to \infty$, one obtains

$$H(\eta_{m_0})'\Big[\sum_{j=1}^{n} b_jy_j\Big] = 0$$

for each m_0. Hence

$$\sum_{j=1}^{n} b_jy_j \in \cap\{N(H(\eta)') : \eta > 0\},$$

and

$$\sum_{j=1}^{n} b_jy_j = 0.$$

The latter implies that $b_j = 0$ $(j = 1, 2, \cdots, n)$, contradicting (3.11). Therefore, the system $\{H(\eta)'y_j\}_{j=1}^n$ is linearly independent for sufficiently small $\eta > 0$. It follows from (3.9) that $G(\eta)z_j = 0$ for each $j = 1, 2, \cdots, n$ and sufficiently small $\eta > 0$. Thus $z_j \in \cap\{N(G(\eta)) : \eta > 0\}$ and hence $z_j = 0$ for $j = 1, 2, \cdots, n$. One has $f = f_{w_n} = 0$.

$\underline{Y}_0$ being a linear span of all $f_{z\otimes y}$ with $z \in Z,\ y \in Y$, the previous argument asserts that

$$\cap\{N(T(\eta)'|\underline{Y}_0) : \eta > 0\} = \{0\}.$$

Thus $\cup\{\underline{R}(T(\eta)) : \eta > 0\}$ is $\sigma(\underline{X}, \underline{Y}_0)$-dense in $\underline{X}$.

§4. The Solution of ΔC=Q

Throughout this section we shall assume that $H(\,\cdot\,)$ satisfies conditions (W1), (W2), (W4) and (3.3), $G(\cdot)$ is a C_0-semigroup, A and $-B$ are the generators of $H(\cdot)$ and $G(\cdot)$, respectively. It has been shown that A is $\sigma(X,Y)$-closed and densely defined, the dual A' on Y is $\sigma(X,Y)$-closed and densely defined, A is the dual of A' in X[12]. For the C_0-semigroup $G(\cdot)$, B is norm-closed and densely defined.

Lemma 4.1. *Δ is $\sigma(\underline{X},\underline{Y})$-closed, where Δ is defined in §1.*

Proof. Assume that $\{C_\alpha\} \subset D(\Delta)$ converges to C and $\{\Delta C_\alpha\}$ converges to $\underline{C}$ in the $\sigma(\underline{X},\underline{Y})$-topology. For each $z \in D(B),\ y \in D(A')$, we have

$$\langle A, C_\alpha z, y\rangle - \langle C_\alpha Bz, y\rangle = \langle \Delta C_\alpha z, y\rangle,$$

or equivalently

$$\langle C_\alpha, f_{z\otimes A'y}\rangle - \langle C_\alpha, f_{Bz\otimes y}\rangle = \langle \Delta C_\alpha, f_{z\otimes y}\rangle. \tag{4.1}$$

Going to the limit in (4.1), one obtains

$$\langle C, f_{z\otimes A'y}\rangle - \langle C, f_{Bz\otimes y}\rangle = \langle \underline{C}, f_{z\otimes y}\rangle,$$

that is,

$$\langle Cz, A'y\rangle - \langle CBz, y\rangle = \langle \underline{C}z, y\rangle. \tag{4.2}$$

Since $\langle CBz, y\rangle,\ \langle \underline{C}z, y\rangle$ are $\sigma(X,Y)$-continuous linear functionals on $D(A')$, so is $\langle Cz, A'y\rangle$. Consequently, $Cz \in D(A)$ and $\langle ACz, y\rangle = \langle Cz, A'y\rangle$. (4.2) implies

$$(AC - CB)z = \underline{C}z \ \text{ for all } \ z \in D(B).$$

Thus $C \in D(\Delta),\ \Delta C = \underline{C}$ and hence Δ is $\sigma(\underline{X},\underline{Y})$-closed.

Under the conditions set in this section on $H(\cdot)$ and $GF(\cdot)$, Propositions 3.2-3.5 assert that $T(\cdot)$ is a pre-weakly $\underline{Y}$-integrable semigroup on $\underline{X}$. Furthermore, the generator Δ_1 of $T(\cdot)$ is defined by (see §2.):

$$D(\Delta_1) = \underline{R}(R_T(\lambda)); \quad (\lambda - \Delta_1)^{-1} = R_T(\lambda), \ \text{ for all } \lambda \ \text{ with } \ \mathrm{Re}\lambda > \omega.$$

Furthermore, if $\mathrm{Re}\lambda > \omega$, we have

$$\begin{aligned}\langle R_T(\lambda)Cz, y\rangle &= \langle\ R_T(\lambda)C, f_{z\otimes y}\rangle \\ &= \int_0^\infty e^{-\lambda t}\langle T(t)C, f_{z\otimes y}\rangle dt \\ &= \int_0^\infty e^{-\lambda t}\langle T(t)Cz, y\rangle dy\end{aligned} \tag{4.3}$$

for all $C \in \underline{X},\ z \in Z,\ y \in Y$.

Theorem 4.1. $\Delta = \Delta_1$, *that is, Δ is the generator of $T(\cdot)$.*

Proof. Let $z \in D(B),\ y \in D(A')$. Then for each $C \in D(\Delta)$, (4.3) implies

$$\begin{aligned}
&\langle R_T(\lambda)\Delta C z, y\rangle \\
&= \int_0^\infty e^{-\lambda t}\langle [T(t)\Delta C, y]\rangle dt = \int_0^\infty e^{-\lambda t}\langle [T(t)(AC - CB)]z, y\rangle dt \\
&= \int_0^\infty e^{-\lambda t}\langle H(t)(AC - CB)G(t)z, y\rangle dt \\
&= \int_0^\infty e^{-\lambda t}\langle AH(t)CG(t)z, y\rangle dt - \int_0^\infty e^{-\lambda t}\langle H(t)CG(t)Bz, y\rangle dt \\
&= \int_0^\infty e^{-\lambda t}\langle CG(t)z, H(t)'A'y\rangle dt - \int_0^\infty e^{-\lambda t}\langle CG(t)Bz, H(t)'y\rangle dt \\
&= \int_0^\infty e^{-\lambda t}\frac{d}{dt}\langle CG(t)z, H(t)'y\rangle dt = \int_0^\infty e^{-\lambda t}\frac{d}{dt}\langle H(t)CG(t)z, y\rangle dt \\
&= \langle \lambda R_T(\lambda)Cz, y\rangle - \langle Cz, y\rangle, \quad \text{for all } \ \lambda \text{ with } \mathrm{Re}\lambda > \omega. \qquad (4.4)
\end{aligned}$$

In the equalities of (4.4), we used integration by parts and relations

$$\begin{aligned}
&\frac{d}{dt}G(t)z = -G(t)Bz \quad \text{for all } z \in D(B), \\
&\frac{d}{dt}H(t)'y = H(t)'A'y \quad \text{for all } \ y \in D(A'). \qquad (4.5)
\end{aligned}$$

The first equality of (4.5) is an easy consequence of C_0-semigroups and the second one has been verified in [12]. Since $D(A')$ is $\sigma(Y,X)$-dense in Y and $D(B)$ is norm dense in X, one obtains

$$R_T(\lambda)\Delta C = \lambda R_T(\lambda)C - C. \qquad (4.6)$$

Hence $C \in \underline{R}(R_T(\lambda)) = D(\Delta_1)$, and $\Delta \subset \Delta_1$. To claim the opposite inclusion, we still assume $z \in D(B),\ y \in D(A')$. Then, for each $C \in \underline{X}$, by a similar argument of (4.4), one obtains

$$\langle [R_T(\lambda)C]z, A'y\rangle = \langle \lambda[R_T(\lambda)C]z, y\rangle - \langle Cz, y\rangle + \langle [R_T(\lambda)C]Bz, y\rangle. \qquad (4.7)$$

Thus $\langle [R_T(\lambda)C]z, A'y\rangle$ is a $\sigma(Y,X)$-continuous linear functional on $D(A')$ because so is the right-hand side of (4.7). Therefore, $[R_T(\lambda)C]z \in D(A)$ and

$$\langle A[R_T(\lambda)C]z, y\rangle - \langle [R_T(\lambda)C]Bz, y\rangle = \langle \lambda[R_T(\lambda)C]z, y\rangle - \langle Cz, y\rangle.$$

Thus

$$A[R_T(\lambda)C] - [R_T(\lambda)C]B = \lambda R_T(\lambda)C - C, \quad R_T(\lambda)C \in D(\Delta)$$

and hence $D(\Delta_1) \subset D(\Delta)$. This, together with the inclusion $\Delta \subset \Delta_1$ yields $\Delta = \Delta_1$.

The following theorem is a direct consequence of Theorem 2.2.

Theorem 4.2. *Suppose that*

$$\overline{\lim_{\lambda\to 0}}\, \|\lambda R_T(\lambda)\| < \infty.$$

Then, the following statements are equivalent:

(i) $Q \in \Delta(D(\Delta) \cap \overline{R_T(\lambda)})$, *where* $\overline{R_T(\lambda)}$ *is the uniform closure of* $R_T(\lambda)$;

(ii) $\lim\limits_{\lambda\to 0} R_T(\lambda)Q$ *exists in the uniform operator topology;*

(iii) *there exists a sequence* $\{\lambda_n\}$ *converging to zero, as* $n \to \infty$, *such that* $\lim_{n\to\infty} R_T(\lambda_n)Q$ *exists in the uniform operator topology;*

(iv) *there exists a sequence* $\{\lambda_n\}$ *converging to zero, as* $n \to \infty$, *such that* $\lim_{n\to\infty} R_T(\lambda_n)Q$ *exists in the weak topology of* $\underline{X}$.

If one of (i)-(iv) *holds, then* $C = \lim_{n\to\infty}[-R_T(\lambda_n)Q]$ *is the unique solution of* (1.1) *in* $\underline{R}(\Delta)$.

Application of the results presented here will be the subject of a forthcoming paper.

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