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REGULARITY ESTIMATES FOR THE OBLIQUE DERIVATIVE PROBLEM ON NON-SMOOTH DOMAINS (I)**

Guan Pengfei* E. Sawyer*

Abstract

The authors consider the existence and regularity of the oblique derivative problem:

$$\begin{cases} Pu = f & \text{in } \Omega, \\ \vec{\ell}\, u = g & \text{on } \partial\Omega, \end{cases}$$

where P is a second order elliptic differential operator on R^n, Ω is a bounded domain in R^n and $\vec{\ell}$ is a unit vector field on the boundary of Ω (which may be tangential to the boundary). All above are assumed with limited smoothness. The authors show that solution u has an elliptic gain from f in Holder spaces (Theorem 1.1). The authors obtain L^p regualrity of solution in Theorem 1.3, which generalizes the results in [7] to the limited smooth case. Some of the application nonlinear problems are also discussed.

Keywords Oblique derivative, Degenerate boundary value problem, Existence, Regularity.
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§1. Introduction

The oblique derivative problem was first posed by Poincaré in 1910 (see [11]):

$$\begin{cases} Pu = f & \text{in } \Omega, \\ \vec{\ell}\, u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where P is an elliptic second order differential operator on R^n, Ω is a bounded domain in R^n and $\vec{\ell}$ is a unit vector field on R^n. The prototypical example is the Neumann problem where P is the Laplacian $\triangle$ and $\vec{\ell} = \vec{\mathbf{n}}$, the unit outward normal to $\partial\Omega$. In this case the existence, uniqueness and regularity for (1.1) are explicitly known under very weak smoothness requirements on the structures P, Ω and $\vec{\ell}$. Roughly speaking, a solution u exists if and only if $\int_{\partial\Omega} g = \int_\Omega f$, is unique up to constants, and is 2 degrees smoother than f and 1 degree smoother (plus $\frac{1}{p}$ from the extension to Ω) than g. If $\vec{\ell}$ is everywhere transversal to $\partial\Omega$, then (1.1) is an elliptic boundary value problem and the existence, uniqueness and regularity results are the same as those for the Neumann problem, except that the existence

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*Department of Mathematics and Statistics McMaster University, Hamilton, Ontario L8S 4K1, Canada.
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requires finitely many compatibility conditions, and the uniqueness holds only up to a finite dimensional space (see [1]).

When $\vec{\ell}$ is permitted to become tangent to $\partial\Omega$, the situation is significantly different. For example, as shown in [3], if $P = \triangle$, Ω is the unit ball in R^3 and $\vec{\ell} = \frac{\partial}{\partial x}$, then any harmonic function $u = u(x, y, z)$ that is independent of x satisfies (1.1) with $f = 0$ and $g = 0$, thus exhibiting an infinite dimensional null space. On the other hand, infinitely many compatibility conditons are required for the existence in the same problem but with Ω taken to be the complement of the unit ball. These examples suggest, as established in [9] and [5], that the values of u should be prescribed on the manifold Γ on which $\vec{\ell} = \vec{\mathbf{T}} + a\vec{\mathbf{n}}$ changes from pointing into Ω to pointing out of Ω (i.e. a changes sign from $-$ to $+$) as $\vec{\ell}$ crosses Γ in the direction of $\vec{\mathbf{T}}$, and that a jump discontinuity should be accepted on the manifold where a changes sign from $+$ to $-$. Thus in the event $\Gamma \neq \phi$, we consider the problem

$$\begin{cases} Pu = f & \text{in } \Omega, \\ \vec{\ell} u = g & \text{on } \partial\Omega, \\ u = h & \text{on } \Gamma. \end{cases} \tag{1.2}$$

In 1969, Egorov and Kondrat'ev[5] showed that in the case where the structures P, $\partial\Omega$, $\vec{\ell}$ and Γ are smooth, then u in (1.2) gains 1 derivative from f and 0 derivatives (plus $\frac{1}{p}$ from the extension to Ω) from g. Moreover, they showed that the gain of 0 from g was sharp, but left open the sharpness of the gain from f. Subsequently, the question of completely characterizing the gain from g was solved in the smooth case. In his book [4], Egorov showed that u gains $\frac{1}{k+1}$ derivatives (plus $\frac{1}{p}$ from the extension to Ω) from g in the L^p-Sobolev scale of spaces, if and only if $\vec{\ell}$ is of type k on $\partial\Omega$. As for the gain from f, Guan[6] and Smith[13] showed in 1990 that u actually has the elliptic gain of 2 derivatives from f in Hölder and L^p-Sobolev spaces respectively provided $\vec{\ell}$ is of finite type on $\partial\Omega$. More recently, the elliptic gain of 2 from f in Hölder spaces was extended to general smooth $\vec{\ell}$ by the first author (see below where this is proved for less regular structures). The gain from f was characterized completely in [7] in 1993. The result is that u gains 2-ϵ derivatives from f for every $\epsilon > 0$, and achieves the elliptic gain of 2 derivatives from f in the L^p Besov-Sobolev scale of spaces if and only if $\vec{\ell}$ is of finite type in $\partial\Omega \setminus \Gamma^*$ and satisfies the $\mathcal{A}_p^{\mp}$ condition on Γ (see subsection 1.1 and Section 2 of in [8] below for precise definitions).

This paper is a sequel to [7], in which we considered (1.2) for smooth P, $\partial\Omega$, $\vec{\ell}$ and Γ. However, in dealing with nonlinear versions of this problem, it is necessary to consider the case where P and $\vec{\ell}$ in (1.2) have limited smoothness. Thus the purpose of this paper is to consider the same problem, but with nonsmooth structure, namely where P has leading coefficients in $\mathcal{C}^\lambda$, Ω has $\mathcal{C}^{\lambda+3}$ boundary $\partial\Omega$, $\vec{\ell} = \vec{\mathbf{T}} + a\vec{\mathbf{n}}$ is a $\mathcal{C}^{\lambda+2}$ unit vector field, and Γ is a $\mathcal{C}^{\lambda+2}$ manifold. Of course, the methods used in [7] in the smooth case carry over for λ sufficiently large depending on the dimension n, but we will obtain optimal results for $\lambda > 0$ in any dimension. Here $\mathcal{C}^d$ denotes the usual Lipschitz space of continuous functions whose derivatives of order $[d]$ are bounded (when d is an integer), or satisfy a Hölder condition of order $d-[d]$ (when d is not an integer). See Theorems 1.4 and 1.5 below for applications to the regularity and existence of a nonlinear problem. More general nonlinear oblique derivative problems will be considered in a future paper.

In the case of limited smoothness, the techniques in [7] cannot be used for the reduction to the boundary, nor for the pseudodifferential calculus. For example, in the reduction to the boundary, we use here the solution to the Neumann problem to construct an approximate Poisson operator, and for the pseudodifferential calculus we use the method of symbol splitting. Several new features arise that were not present in our previous paper [7]. For example:

• The gain from f in the L^p Besov-Sobolev scale of spaces can be strictly less than 2, and this is characterized by fractional variations on the $\mathcal{A}_p^{\mp}$ condition.

• The gain from g in the L^p Besov-Sobolev scale of spaces takes the form $\nu+\frac{1}{p}$, where ν can be any number in $\left[0,\frac{1}{3}\right]\cup\left\{\frac{1}{2}\right\}$, and this is characterized by a generalized "type" condition on $\partial\Omega$.

• The gain from h in the L^p Besov-Sobolev scale of spaces, which had not been considered at all in [7], takes the form $\frac{\gamma}{p}+\frac{1}{p}$, where γ can be any number in $\left[0,\frac{1}{3}\right]\cup\left\{\frac{1}{2}\right\}$, and this is characterized by a generalized "type" condition near Γ.

• In addition to the L^p Besov-Sobolev scale of spaces, we obtain analogous results for the Hölder scale of spaces, $\Lambda^s(\Omega)$.

• In the elliptic setting, the solution u to (1.2) can be taken in $\Lambda^s(\Omega)$ for s all the way up to $\lambda+2$, and up to $\lambda+2-\epsilon$ in $H_p^s(\Omega)$ (recall the leading coefficients of P are only in $\mathcal{C}^\lambda$). This realization of the smoothness "cap" in the case of limited smoothness is achieved with the aid of symbol smoothing.

• In this paper, we need to construct a left parametrix (in [7] we used the left parametrix from [10]).

Our approach to solving (1.2) can be summed up as:

1. reduction of (1.2) to a pseudodifferential equation on $\partial\Omega$.

2. construction of a parametrix for the boundary problem and reduction to the operators $\mathcal{K}$, K and T:

$$\begin{aligned}
\mathcal{K}f(x,t)&=\int_{R^n}e^{ix\cdot\xi}e^{-\int_0^t a(x,\theta)Q(x,\theta,\xi)d\theta}\hat{f}(\xi)d\xi,\\
Kf(x,t)&=\int_{R^n}e^{ix\cdot\xi}\int_0^t e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta}f^{\sim}(\xi,t')dt'd\xi,\\
Tf(x,t)&=\int_{R^n}e^{ix\cdot\xi}\int_0^t a(x,t')Q(x,t',\xi)e^{-\int_{t'}^t a(x,\theta)Q(x,\theta,\xi)d\theta}f^{\sim}(\xi,t')dt'd\xi,
\end{aligned}\tag{1.3}$$

for $x\in R^n$ and $t\in(-1,1)$.

3. reduction of boundedness properties of T to weighted norm inequalities for the Hardy operator, and corresponding estimates for $\mathcal{K}$ and K.

The first two points will be taken up in the next section, while the third point is part of the subject matter of the adjoining paper [8], concerned with mapping properties of the special classes of pseudodifferential operators that arise in the oblique derivative problem.

1.1. Statement of Theorems

Let $H_p^s(R^{n+2})$, $B_p^{s,p}(R^{n+2})$ and $\Lambda^s\left(R^{n+2}\right)$ be the L_p - Sobolev, Besov and Hölder spaces respectively on R^{n+2}. Let Ω be a connected open set in R^{n+2} with $\mathcal{C}^{\lambda+3}$ boundary $\partial\,\Omega=\Sigma$. Let

$$P(x,D)=\sum_{i,j=1}^{n+2}a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\sum_{i=1}^{n}b_i(x)\frac{\partial}{\partial x_i}+c(x)$$

be an elliptic second order real differential operator on R^{n+2} with $\mathcal{C}^\lambda$ coefficients, i.e. there are C, $\delta>0$ such that

$$C|\xi|^2\geq\sum_{i,j=1}^{n+2}a_{ij}(x)\xi_i\xi_j\geq\delta|\xi|^2,\qquad x,\xi\;\in\;R^{n+2},$$

where a_{ij}, b_i and c are $\mathcal{C}^\lambda$ on R^{n+2}. Let $\vec{\ell}(x)$ be a real $\mathcal{C}^{\lambda+2}$ unit vector field in R^{n+2} with

$$\vec{\ell}|_\Sigma=\vec{\mathbf{T}}+a(x)\vec{\mathbf{n}},$$

where $\vec{\mathbf{n}}$ is the unit outward normal to Σ at x and $\vec{\mathbf{T}}$ is tangential to Σ. Finally let $\mathcal{N}=\{x\in\Sigma:a(x)=0\}\neq\phi$. We assume throughout the following restriction ($\mathcal{R}_{int}$) on $\vec{\mathbf{T}}$:

$$\text{No integral curve of } \vec{\mathbf{T}} \text{in } \Sigma \text{ lies in } \mathcal{N} \text{ for an infinite interval of time.}\tag{1.4}$$

Furthermore, we suppose that $\vec{\ell}$ satisfies one of the following two cases:

Case (I) $a(x)\geq0$ for all $x\in\Sigma$ or $a(x)\leq0$ for all $x\in\Sigma$.

Case (II) $a(x)$ takes on both positive and negative values and there is an n-dimensional $\mathcal{C}^{\lambda+2}$ submanifold Γ (possibly with several components) contained in $\mathcal{N}$ and open subsets Σ^+ and Σ^- of Σ such that Σ is a disjoint union of Σ^+, Σ^- and Γ and

(i) $a(x)\geq0$ for $x\in\Sigma^+$,

(ii) $a(x)\leq0$ for $x\in\Sigma^-$,

(iii) $\partial\Sigma^+=\partial\Sigma^-=\Gamma$,

(iv) $\vec{\ell}$ is transversal to Γ and points in the direction of Σ^+.

For case (**I**) we consider the problem

$$\begin{cases}Pu=f & \text{in }\Omega,\\ \vec{\ell}\,u=g & \text{on }\partial\Omega,\end{cases}\tag{1.5}$$

and for case (**II**) the problem is

$$\begin{cases}Pu=f & \text{in }\Omega,\\ \vec{\ell}\,u=g & \text{on }\partial\Omega,\\ \quad u=h & \text{on }\Gamma.\end{cases}\tag{1.6}$$

We always assume that either $u\in\Lambda^s(\Omega)$ for $s>1$, or that $u\in H_p^s(\Omega)$ for $s>1+\frac{1}{p}$, so that the traces of $\vec{\ell}\,u$ on $\partial\Omega$ and u on Γ exist.

To define the $\mathcal{A}_{p,\alpha}$ conditions, we will use the flow for $\vec{\mathbf{T}}$ on Σ. Given $x\in\Sigma$, let $\vec{\gamma}(x,s)$ denote the integral curve of $\vec{\mathbf{T}}$ through x, i.e.

$$\begin{cases}\dfrac{\partial}{\partial s}\vec{\gamma}(x,s)=\vec{\mathbf{T}}(\vec{\gamma}(x,s)), & s\text{ real},\\ \vec{\gamma}(x,0)=x.\end{cases}\tag{1.7}$$

We will also need some more notation. Let $y \in \mathcal{N}$. Define I_y to be the largest closed interval containing 0 (which may be $\{0\}$ itself) such that $\vec{\gamma}(y, I_y) \subset \mathcal{N}$, i.e. $a(\vec{\gamma}(y,s)) = 0$ for $s \in I_y$. Then $F_y = \vec{\gamma}(y, I_y) = \{\vec{\gamma}(y,s) : s \in I_y\}$ is the longest segment of integral curve of $\vec{\mathbf{T}}$ through y that lies in $\mathcal{N}$ and is thus bounded by restriction $(\mathcal{R}_{int})$ (see (1.4)). We refer to F_y as the fibre of $\mathcal{N}$ at y. Set $\Gamma^* = \bigcup_{y\in\Gamma} F_y$.

Definition 1.1. *The vector field $\vec{\ell}$ (or equivalently a) satisfies the $\mathcal{A}^{\mp}_{p,\alpha}$ condition at the fibre F_y, $y \in \Gamma$, if there are constants $r > 0$, $R^- < 0 < R^+$, such that $a(\vec{\gamma}(x,R^-)) \neq 0$ and $a(\vec{\gamma}(x,R^+)) \neq 0$ for $x \in \Gamma$, $|x-y| < r$ and both of the following conditions hold:*

$$\left[\frac{1}{\int_\sigma^\beta a(\vec{\gamma}(x,t))\,dt}\int_\sigma^\beta a(\vec{\gamma}(x,t))^{p'}\,dt\right]^{p-1} \leq C\frac{1}{\gamma-\beta}\left[\int_\beta^\gamma a(\vec{\gamma}(x,t))\,dt\right]^{1-p\alpha} \tag{1.8}$$

for all $x \in \Gamma$, $|x-y| < r$ and all $0 < \sigma < \beta < \gamma < R^+$ with $\int_\sigma^\beta a(\vec{\gamma}(x,t))\,dt = \int_\beta^\gamma a(\vec{\gamma}(x,t))\,dt$, and also

$$\left[\frac{1}{\int_\beta^\gamma |a(\vec{\gamma}(x,t))|\,dt}\int_\beta^\gamma |a(\vec{\gamma}(x,t))|^{p'}\,dt\right]^{p-1} \leq C\frac{1}{\beta-\sigma}\left[\int_\sigma^\beta |a(\vec{\gamma}(x,t))|\,dt\right]^{1-p\alpha} \tag{1.9}$$

for all $x \in \Gamma$, $|x-y| < r$ and all $R^- < \sigma < \beta < \gamma < 0$ with $\int_\sigma^\beta |a(\vec{\gamma}(x,t))|\,dt = \int_\beta^\gamma |a(\vec{\gamma}(x,t))|\,dt$. If the above holds for all $y \in \Gamma$, we say that $\vec{\ell}$ satisfies the $\mathcal{A}^{\mp}_{p,\alpha}$ condition on Γ.

Definition 1.2. *The vector field $\vec{\ell}$ (or equivalently a) satisfies the $\mathcal{A}^{=}_{p,\alpha}$ condition at the fibre F_y, $y \in (\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, if there are constants $r > 0$, $R^- < 0 < R^+$, such that $a(\vec{\gamma}(x,R^-)) \neq 0$ and $a(\vec{\gamma}(x,R^+)) \neq 0$ for $x \in (\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, $|x-y| < r$ and (1.8) holds for all $x \in (\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, $|x-y| < r$ and all $R^- < \sigma < \beta < \gamma < R^+$ with $\int_\sigma^\beta a(\vec{\gamma}(x,t))\,dt = \int_\beta^\gamma a(\vec{\gamma}(x,t))\,dt$. If the above holds for all $y \in (\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, we say that $\vec{\ell}$ satisfies the $\mathcal{A}^{=}_{p,\alpha}$ condition on $(\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$.*

Definition 1.3. *The vector field $\vec{\ell}$ (or equivalently a) satisfies the $\mathcal{A}^{\ddagger}_{p,\alpha}$ condition at the fibre F_y, $y \in (\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$, if there are constants $r > 0$, $R^- < 0 < R^+$, such that $a(\vec{\gamma}(x,R^-)) \neq 0$ and $a(\vec{\gamma}(x,R^+)) \neq 0$ for $x \in (\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$, $|x-y| < r$ and (1.9) holds for all $x \in (\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$, $|x-y| < r$ and all $R^- < \sigma < \beta < \gamma < R^+$ with $\int_\sigma^\beta |a(\vec{\gamma}(x,t))|\,dt = \int_\beta^\gamma |a(\vec{\gamma}(x,t))|\,dt$. If the above holds for all $y \in (\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$, we say that $\vec{\ell}$ satisfies the $\mathcal{A}^{\ddagger}_{p,\alpha}$ condition on $(\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$.*

Definition 1.4. *The vector field $\vec{\ell}$ (or equivalently a) satisfies the $(\mathcal{T}_\nu)$ condition on Σ if*

$$\beta - \alpha \leq C\left(\int_\alpha^\beta |a(\vec{\gamma}(x,t))|\,dt\right)^\nu \quad \text{for all } \alpha < \beta, \quad \text{and } x \in \Sigma.$$

Definition 1.5. *The vector field $\vec{\ell}$ (or equivalently a) satisfies the $(\mathcal{P}_\gamma)$ condition on Γ if*

$$|\beta| \leq C\left|\int_0^\beta a(\vec{\gamma}(x,t))\,dt\right|^\gamma \quad \text{for all } \beta \in R, \quad \text{and } x \in \Gamma.$$

Remark 1.1. We remark that for the purposes of the above five definitions, the family of integral curves $\vec{\gamma}(x,t)$ can be replaced by any smooth family $\vec{\gamma}(x,t)$ satisfying the following condition, weaker than (1.7):

$$\begin{cases} \left\| \dfrac{\partial}{\partial s} \vec{\gamma}\,(x,t) - \vec{\mathbf{T}}\,(\vec{\gamma}\,(x,t)) \right\| \leq C\,|a\,(x,t)|\,, \\ \vec{\gamma}\,(x,0) = x. \end{cases} \tag{1.10}$$

This is a simple consequence of Lemma 6.38 of [7], which required only that the curves be $\mathcal{C}^1$ and that the vector fields be Lipschitz.

We can now state our main theorems. Fix $0 < \mu_\infty < \min\left\{1-\delta, \frac{\lambda+2}{2(\lambda+3)}\right\}$ and set $\mu_p = \min\left\{1-\delta-\frac{1}{p(\lambda+3)}, \frac{\lambda+2}{2(\lambda+3)}\right\}$ for $1 < p < \infty$ where $\delta = \max\left\{\frac{1}{2}, \frac{1}{\lambda+1}\right\}$. We point out that the choice of δ is dictated by (2.13) below, and that μ_p arises in Lemma 1.7 of [8].

Theorem 1.1. *Suppose that P, Ω and $\vec{\ell}$ are as at the beginning of this subsection with $\lambda > 0$.*

(i) *If $\vec{\ell}$ satisfies case* (**I**) *and the $\mathcal{T}_\nu$ condition on Σ for some $\nu \geq 0$, then for all $1 < s \leq \lambda+2$, there is a subspace $\mathcal{F}$ of finite codimension in $\Lambda^{s-2}(\Omega)\times \Lambda^{s-\nu}(\partial\Omega)$ such that for $(f,g) \in \mathcal{F}$, there exists u satisfying* (1.5) *and $u \in \Lambda^s(\Omega)$. Moreover, if $u \in \Lambda^{s'}(\Omega)$ for some $s' > 1$ satisfies* (1.5) *with f, g as above, then $u \in \Lambda^s(\Omega)$, and there is C_s such that*

$$\|u\|_{\Lambda^s(\Omega)} \leq C_s \left(\|f\|_{\Lambda^{s-2}(\Omega)} + \|g\|_{\Lambda^{s-\nu}(\partial\Omega)} + \|u\|_{\Lambda^{s-\mu_\infty}(\Omega)}\right).$$

Furthermore, if the zero order term $c\,(x)$ in the operator P is negative, then for every $(f,g) \in \Lambda^{s-2}(\Omega)\times \Lambda^{s-\nu}(\partial\Omega)$, there is a unique solution $u \in \Lambda^s(\Omega)$.

(ii) *If $\vec{\ell}$ satisfies case* (**II**) *and the $\mathcal{T}_\nu$ condition on Σ for some $\nu \geq 0$, then for all $1 < s \leq \lambda+2$, there is a subspace $\mathcal{F}$ of finite codimension in $\Lambda^{s-2}(\Omega)\times \Lambda^{s-\nu}(\partial\Omega)\times \Lambda^s(\Gamma)$ such that for $(f,g,h) \in \mathcal{F}$, there exists u satisfying* (1.6) *and $u \in \Lambda^s(\Omega)$. Moreover, if $u \in \Lambda^{s'}(\Omega)$ for some $s' > 1$ satisfies* (1.6) *with f, g, h as above, then $u \in \Lambda^s(\Omega)$, and there is C_s such that*

$$\|u\|_{\Lambda^s(\Omega)} \leq C_s \left(\|f\|_{\Lambda^{s-2}(\Omega)} + \|g\|_{\Lambda^{s-\nu}(\partial\Omega)} + \|h\|_{\Lambda^s(\Gamma)} + \|u\|_{\Lambda^{s-\mu_\infty}(\Omega)}\right).$$

Furthermore, if the zero order term $c\,(x)$ in the operator P is nonpositive, then for every $(f,g,h) \in \Lambda^{s-2}(\Omega)\times \Lambda^{s-\nu}(\partial\Omega)\times \Lambda^s(\Gamma)$, there is a unique solution $u \in \Lambda^s(\Omega)$.

At this point we could immediately obtain a regularity theorem for a semilinear oblique derivative problem. However, in order to handle a fully nonlinear equation with semilinear oblique derivative in the next subsection, we need the following result which establishes an additional gain from g when it is multiplied by a tangential derivative of a.

Theorem 1.2. *Suppose that P, Ω, and $\vec{\ell}$ are as at the beginning of this subsection with $\lambda > 0$, and let $D = \sum\limits_{k=1}^{n+2} b_k\,(x)\,\frac{\partial}{\partial x_k} \in C^{\lambda+2}\left(\overline{\Omega}\right)$ be a vector field tangent to $\partial\Omega$ for $x \in \partial\Omega$. If for some $s > 0$, $u \in \Lambda^s(\Omega)$ and satisfies*

$$\begin{cases} Pu = 0 & \text{in } \Omega, \\ \vec{\ell}\,u = (Da)\,g_1 & \text{on } \partial\Omega \end{cases} \tag{1.11}$$

in case (**I**) *and*

$$\begin{cases} Pu = 0 & \text{in } \Omega, \\ \vec{\ell}\,u = (Da)\,g_1, & \text{on } \partial\Omega, \\ u = 0 & \text{on } \Gamma \end{cases} \tag{1.12}$$

in case **(II)** *with* $g_1 \in \Lambda^{\lambda-\frac{1}{4}+1}(\partial\Omega)$ *as above, then* $u \in \Lambda^{\lambda+1}(\Omega)$, *and there is* C_λ *such that*

$$\|u\|_{\Lambda^{\lambda+1}(\Omega)} \le C_\lambda \left(\|g_1\|_{\Lambda^{\lambda-\frac{1}{4}+1}(\partial\Omega)} + \|u\|_{\Lambda^{\lambda+1-\mu_\infty}(\Omega)} \right).$$

Now we state the Sobolev space analogue.

Theorem 1.3. *Let* $1 < p < \infty$ *and* $\alpha, \nu \ge 0$. *Suppose that* P , Ω, *and* $\vec{\ell}$ *are as at the beginning of this subsection with* $\frac{\lambda}{\lambda+1} > \alpha$.

(A) *Suppose that* $\vec{\ell}$ *satisfies the* $\mathcal{A}^{\mp}_{p,\alpha}$ *condition on* Γ, $\mathcal{A}^{=}_{p,\alpha}$ *on* $(\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, $\mathcal{A}^{\ddagger}_{p,\alpha}$ *on* $(\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$, *and also the* $\mathcal{T}_\nu$ *condition on* Σ. *Then:*

(i) *Assume that* $\vec{\ell}$ *satisfies case* (**I**) *above, and let* $1 + \frac{1}{p} < s < \lambda + 2 - \alpha$. *There is a subspace* $\mathcal{F}$ *of finite codimension in* $H^{s-2+\alpha}_p(\Omega) \times \mathcal{B}^{s-\nu-\frac{1}{p},p}_p(\partial\Omega)$ *such that for* $(f, g) \in \mathcal{F}$, *there exists u satisfying* (1.5) *and* $u \in H^s_p(\Omega)$. *Moreover, if* $u \in H^{s'}_p(\Omega)$ *for some* $s' > 1 + \frac{1}{p}$ *satisfies* (1.5) *with* f, g *as above, then* $u \in H^s_p(\Omega)$, *and there is* C_s *such that*

$$\|u\|_{H^s_p(\Omega)} \le C_s \left(\|f\|_{H^{s-2+\alpha}_p(\Omega)} + \|g\|_{\mathcal{B}^{s-\nu-\frac{1}{p},p}_p(\partial\Omega)} + \|u\|_{H^{s-\mu_p}_p(\Omega)} \right).$$

Furthermore, if the zero order term $c(x)$ *in the operator* P *is negative, then for every* $(f, g) \in H^{s-2+\alpha}_p(\Omega) \times \mathcal{B}^{s-\nu-\frac{1}{p},p}_p(\partial\Omega)$, *there is a unique solution* $u \in \Lambda^s(\Omega)$.

(ii) *Assume that* $\vec{\ell}$ *satisfies case* (**II**) *above and, in addition, the* P_γ *condition on* Γ *for some* $\gamma \ge 0$, *and let* $1 + \frac{1}{p} < s < \lambda + 2 - \alpha$. *There is a subspace* $\mathcal{F}$ *of finite codimension in* $H^{s-2+\alpha}_p(\Omega) \times \mathcal{B}^{s-\nu-\frac{1}{p},p}_p(\partial\Omega) \times \mathcal{B}^{s-\frac{\gamma}{p}-\frac{1}{p},p}_p(\Gamma)$ *such that for* $(f, g, h) \in \mathcal{F}$, *there exists u satisfying* (1.6) *and* $u \in H^s_p(\Omega)$. *Moreover, if* $u \in H^{s'}_p(\Omega)$ *for some* $s' > 1 + \frac{1}{p}$ *satisfies* (1.6) *with* f, g, h *as above, then* $u \in H^s_p(\Omega)$, *and there is* $C_{s,\alpha}$ *such that*

$$\|u\|_{H^s_p(\Omega)} \le C_{s,\alpha} \left(\|f\|_{H^{s-2+\alpha}_p(\Omega)} + \|g\|_{\mathcal{B}^{s-\nu-\frac{1}{p},p}_p(\partial\Omega)} + \|h\|_{\mathcal{B}^{s-\frac{\gamma}{p}-\frac{1}{p},p}_p(\Gamma)} + \|u\|_{H^{s-\mu_p}_p(\Omega)} \right).$$

Furthermore, if the zero order term $c(x)$ *in the operator* P *is nonpositive, then for every* $(f, g, h) \in H^{s-2+\alpha}_p(\Omega) \times \mathcal{B}^{s-\nu-\frac{1}{p},p}_p(\partial\Omega) \times \mathcal{B}^{s-\frac{\gamma}{p}-\frac{1}{p},p}_p(\Gamma)$, *there is a unique solution* $u \in \Lambda^s(\Omega)$.

Finally, the above statements are true when $\alpha = \frac{1}{p(\lambda+3)}$, *since in this case the* $\mathcal{A}^{\mp}_{p,\alpha}$, $\mathcal{A}^{=}_{p,\alpha}$ *and* $\mathcal{A}^{\ddagger}_{p,\alpha}$ *conditions automatically follow from the smoothness of a.*

(B) *Conversely, suppose that* $\vec{\ell}$ *satisfies case* (**I**) *or case* (**II**) *and* $\lambda > 0$ *satisfies* $\mu_p > 0$. *Then:*

(i) *If for some* $1 + \frac{1}{p} < s < \lambda + 2 - \alpha$, $\alpha \ge 0$, $\epsilon > 0$ *and for every* $f \in H^{s-2+\alpha}_p(\Omega)$, *there is an approximate solution* $u \in H^s_p(\Omega)$ (*with* $g = 0$, $h = 0$), *such that*

$$\|Pu - f\|_{H^{s-2+\alpha+\epsilon}_p(\Omega)} \le C_{s,\epsilon} \|f\|_{H^{s-2+\alpha}_p(\Omega)},$$
$$\|\vec{\ell}u\|_{\mathcal{B}^{s-\frac{1}{p},p}_p(\partial\Omega)} \le C_s \left(\|f\|_{H^{s-2+\alpha}_p(\Omega)} + \|u\|_{H^{s-1}_p(\Omega)} \right),$$
$$\|u\|_{\mathcal{B}^{s-\frac{1}{p},p}_p(\Gamma)} \le C_s \left(\|f\|_{H^{s-2+\alpha}_p(\Omega)} + \|u\|_{H^{s-1}_p(\Omega)} \right),$$

then $\vec{\ell}$ *satisfies the* $\mathcal{A}^{\mp}_{p,\alpha}$ *condition on* Γ, $\mathcal{A}^{=}_{p,\alpha}$ *on* $(\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, *and* $\mathcal{A}^{\ddagger}_{p,\alpha}$ *on* $(\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$.

(ii) *If for some* $1 + \frac{1}{p} < s < \lambda + 2 - \alpha$, $\nu \ge 0$, $\epsilon > 0$ *and for every* $g \in \mathcal{B}^{s-\nu-\frac{1}{p},p}_p(\partial\Omega)$, *there is an approximate solution* $u \in H_p^s(\Omega)$ *(with* $f = 0$, $h = 0$*), such that*

$$\|Pu\|_{H_p^{s-2+\frac{1}{p(\lambda+3)}}(\Omega)} \le C_s \left(\|g\|_{\mathcal{B}_p^{s-\nu-\frac{1}{p},p}(\partial\Omega)} + \|u\|_{H_p^{s-1}(\Omega)} \right),$$

$$\|\vec{\ell}u - g\|_{\mathcal{B}_p^{s-\nu-\frac{1}{p}+\epsilon,p}(\partial\Omega)} \le C_{s,\epsilon} \|g\|_{\mathcal{B}_p^{s-\nu-\frac{1}{p},p}(\partial\Omega)},$$

$$\|u\|_{\mathcal{B}_p^{s-\frac{1}{p},p}(\Gamma)} \le C_s \left(\|g\|_{\mathcal{B}_p^{s-\nu-\frac{1}{p},p}(\partial\Omega)} + \|u\|_{H_p^{s-1}(\Omega)} \right),$$

then $\vec{\ell}$ *satisfies the* $\mathcal{T}_\nu$ *condition on* Σ.

(iii) *If for some* $1 + \frac{1}{p} < s < \lambda + 2 - \alpha$, $\gamma \ge 0$, $\epsilon > 0$ *and for every* $h \in \mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)$, *there is an approximate solution* $u \in H_p^s(\Omega)$ *(with* $f = 0$, $g = 0$*), such that*

$$\|Pu\|_{H_p^{s-2+\frac{1}{p(\lambda+3)}}(\Omega)} \le C_s \left(\|h\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)} + \|u\|_{H_p^{s-1}(\Omega)} \right),$$

$$\|\vec{\ell}u\|_{\mathcal{B}_p^{s-\frac{1}{p},p}(\partial\Omega)} \le C_s \left(\|h\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)} + \|u\|_{H_p^{s-1}(\Omega)} \right),$$

$$\|u - h\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p}+\epsilon,p}(\Gamma)} \le C_{s,\epsilon} \|h\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)},$$

then $\vec{\ell}$ *satisfies the* $\mathcal{P}_\gamma$ *condition on* Γ.

Remark 1.2. In part (A) of Theorem 1.3, we can weaken the requirement $\mu_p > 0$ to simply $1 - \delta - \alpha > 0$ (see the remark following Lemma 1.7 in [8]).

Remark 1.3. In Theorems 1.1 and 1.3, the assumptions that $\vec{\ell}$, Γ are $\mathcal{C}^{\lambda+2}$ and $\partial\Omega$ is $\mathcal{C}^{\lambda+3}$ can all be relaxed somewhat. More precisely, if we assume that $\vec{\ell}$, Γ and $\partial\Omega$ are $\mathcal{C}^{\lambda'}$ with $\lambda' + \frac{1}{p} \le \lambda + 2 - \alpha$, and assume that there is a $\mathcal{C}^{\lambda'}$ flow for $\vec{\mathbf{T}}$ in a neighbourhood of $\mathcal{N}$ (the assumption above that $\partial\Omega$ be in $\mathcal{C}^{\lambda+3}$ is used only to guarantee the existence of a $\mathcal{C}^{\lambda+2}$ flow), then Theorem 1.1 remains true for $1 < s \le \lambda'$, and Theorem 1.3 remains true for $1 + \frac{1}{p} < s < \lambda'$, but with the gain of $\frac{1}{p(\lambda+3)}$ from f replaced by $\frac{1}{p(\lambda'+1)}$. The proofs are the same. Furthermore, the above smoothness assumptions on $\vec{\ell}$, Γ and g are needed merely in a neighbourhod of $\mathcal{N}$. Away from $\mathcal{N}$, only the usual elliptic smoothness is required.

Remark 1.4. If $\vec{\ell}$ satisfies the $\mathcal{A}_p^{\mp}$ condition on Γ, etc. as in part (A) of Theorem 1.3, and if $\mathcal{N} = \Gamma$, then the a priori inequalities in part (A) hold with $s = 2$, $\alpha = 0$ and $\mu_p = \frac{1}{2}$ provided $P \in \mathcal{C}^0$, and $\vec{\ell}, \Gamma, \partial\Omega \in \mathcal{C}^2$. This can be proved by locally approximating P by constant coefficient elliptic operators and using $\alpha = 0$.

1.2. Applications to Nonlinear Problems

Here we give two nonlinear applications of Theorem 1.1 — first to the regularity of a fully nonlinear elliptic equation with a semilinear oblique derivative. In particular, we show that if the data and structures are all smooth, then so are the solutions.

Theorem 1.4. *Let* $P(x, u, \xi, \zeta)$ *be a function of the variables* $(x, u, \xi, \zeta) \in R^{n+2} \times R \times R^{n+2} \times R^{(n+2)^2}$, *g be a function on* $\partial\Omega$, *and* $h(x, u)$ *be a function of the variables* $(x, u) \in R^{n+2} \times R$. *Suppose that* $P, \partial_\zeta P \in \mathcal{C}^\lambda$ *and* $g, h \in \mathcal{C}^{\lambda+2}$ *for some noninteger* $\lambda > 0$. *Suppose further that* $u \in \Lambda^s\left(\overline{\Omega}\right)$, $s > 2$, *is such that both* $\partial_\zeta P$ *and* $\partial_u h$ *are elliptic, i.e. for* *some* $C, c > 0$,

$$\begin{aligned} &C|\eta|^2 \geq \sum_{i,j=1}^{n+2} \frac{\partial}{\partial \zeta_{ij}} P\left(x,u,\nabla u,\nabla^2 u\right)\eta_i\eta_j \geq c|\eta|^2, \quad x,\eta \in R^{n+2},\\ &C \geq \partial_u h(x,u) \geq c, \qquad\qquad\qquad\qquad\qquad\qquad x \in R^{n+2}. \end{aligned}$$

(i) *If* $\vec{\ell}$ *satisfies case* (**I**) *above, and if in addition* u *satisfies*

$$\begin{cases} P\left(x,u,\nabla u,\nabla^2 u\right) = 0 & in\ \Omega,\\ \vec{\ell}\, u = g(x) & on\ \partial\Omega, \end{cases}$$

then $u \in \Lambda^{\lambda+2}\left(\overline{\Omega}\right)$.

(ii) *If* $\vec{\ell}$ *satisfies case* (**II**) *above, and if in addition* u *satisfies*

$$\begin{cases} P\left(x,u,\nabla u,\nabla^2 u\right) = 0 & in\ \Omega,\\ \vec{\ell}\, u = g(x) & on\ \partial\Omega,\\ h(x,u) = 0 & on\ \Gamma, \end{cases}$$

then $u \in \Lambda^{\lambda+2}\left(\overline{\Omega}\right)$.

In particular, if the functions P, g, h *and the manifolds* Γ, $\partial\Omega$ *and the vector field* $\vec{\ell}$ *are all smooth, and if* u *is a* Λ^s *solution to* (i) *or* (ii) *as above with* $s > 2$, *then* u *is also smooth. Finally, if* P *is linear in the second order derivatives, i.e.*

$$Pu = \sum_{i,j=1}^{n+2} a_{ij}(x,u,\nabla u)\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x,u,\nabla u),$$

then the above conclusions follow from the weaker assumption $u \in \Lambda^s\left(\overline{\Omega}\right)$, $s > 1$.

Proof. We prove only case (**II**), the other case being similar but simpler. So suppose $u \in \Lambda^s\left(\overline{\Omega}\right)$ with $2 < s < s + \frac{1}{4} \leq \lambda + 2$. We will show that $u \in \Lambda^{s+\frac{1}{4}}\left(\overline{\Omega}\right)$. For this, let $D = \sum_{k=1}^{n+2} b_k(x)\frac{\partial}{\partial x_k} \in C^{\lambda+2}\left(\overline{\Omega}\right)$ be a vector field tangent to $\partial\Omega$ for $x \in \partial\Omega$, and commuting with $\vec{T}$, i.e. $\left[\vec{T}, D\right] = 0$. Applying D to the nonlinear equation above we obtain

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n+2} \frac{\partial}{\partial \zeta_{ij}} P\left(x,u,\nabla u,\nabla^2 u\right)\frac{\partial^2 (Du)}{\partial x_i \partial x_j} & \\ = -DP\left(x,u,\nabla u,\nabla^2 u\right) - \partial_u P\left(x,u,\nabla u,\nabla^2 u\right)Du & \\ \quad -\displaystyle\sum_{k=1}^{n+2} \frac{\partial}{\partial \xi_k} P\left(x,u,\nabla u,\nabla^2 u\right) D\left(\frac{\partial}{\partial x_k}u\right) & \\ \quad +\displaystyle\sum_{i,j=1}^{n+2} \frac{\partial}{\partial \zeta_{ij}} P\left(x,u,\nabla u,\nabla^2 u\right)\left[\frac{\partial^2}{\partial x_i \partial x_j}, D\right]u & \text{in } \Omega,\\ \vec{\ell}\,(Du) = g(x) + \left[\vec{\ell}, D\right]u & \text{on } \partial\Omega,\\ \partial_u h(x,u)(Du) = -Dh(x,u) & \text{on } \Gamma. \end{cases}$$

Since $\left[\vec{\ell}, D\right]u = [a\vec{\mathbf{n}}, D] = -Da(x)\vec{\mathbf{n}}u$, this can be rewritten as

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n+2} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}(Du) = f(x) & \text{in } \Omega,\\ \vec{\ell}\,(Du) = g(x) + Da(x)g_1(x) & \text{on } \partial\Omega,\\ Du = h(x) & \text{on } \Gamma, \end{cases}$$

where $a_{ij} = \frac{\partial}{\partial \zeta_{ij}} P\left(x, u, \nabla u, \nabla^2 u\right) \in \Lambda^{s-2}(\Omega)$, $f \in \Lambda^{s-2}(\Omega)$, $g \in \Lambda^{\lambda+2}(\partial\Omega)$, $g_1 = \vec{\mathbf{n}} u \in \Lambda^{s-1}(\partial\Omega)$ and $h = -\frac{Dh(x,u)}{\partial_u h(x,u)} \in \Lambda^{\min\{\lambda+1,s\}} \subset \Lambda^{s-1+\frac{1}{4}}(\Gamma)$. By Theorem 1.1, there is $w \in \Lambda^{s-1+\frac{1}{4}}\left(\overline{\Omega}\right)$ such that

$$\begin{cases} \sum\limits_{i,j=1}^{n+2} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} w = f(x) & \text{in } \Omega, \\ \vec{\ell}\, w = g(x) & \text{on } \partial\Omega, \\ w = h(x) & \text{on } \Gamma, \end{cases}$$

and by Theorem 1.2, $Du - w \in \Lambda^{s-1+\frac{1}{4}}\left(\overline{\Omega}\right)$. Thus $Du \in \Lambda^{s-1+\frac{1}{4}}\left(\overline{\Omega}\right)$. To obtain $u \in \Lambda^{s+\frac{1}{4}}\left(\overline{\Omega}\right)$, we note that we have proved $\nabla Du \in \Lambda^{s-2+\frac{1}{4}}\left(\overline{\Omega}\right)$ for any tangential derivative D, and now using $P\left(x, u, \nabla u, \nabla^2 u\right) = 0$ together with the implicit function theorem, we obtain $\nabla^2 u \in \Lambda^{s-2+\frac{1}{4}}\left(\overline{\Omega}\right)$, and hence $u \in \Lambda^{s+\frac{1}{4}}\left(\overline{\Omega}\right)$ as required. Finally, if P is linear in the second order derivatives, we need not differentiate the problem, but can immediately apply Theorem 1.1 provided $s > 1$.

Our second application of Theorem 1.1 is the following nonlinear existence theorem for small data.

Theorem 1.5. *Let Ω and $\vec{\ell}$ be as in Theorem 1.1. Suppose that $\lambda > 0$ is nonintegral and that*

$$P(x, u, \xi, \zeta) \in \mathcal{C}^{\lambda}\left(\overline{\Omega} \times R \times R^{n+2} \times R^{(n+2)^2}\right),$$
$$\partial_u P, \partial_\xi P, \partial_\zeta P \in \mathcal{C}^{\lambda}\left(\overline{\Omega} \times R \times R^{n+2} \times R^{(n+2)^2}\right).$$

Moreover, suppose $P(x,0,0,0) = 0$ and that there is $C > 0$ such that

$$C|\eta|^2 \geq \sum_{i,j=1}^{n+2} \frac{\partial}{\partial \zeta_{ij}} P(x,0,0,0)\, \eta_i \eta_j \geq C^{-1}|\eta|^2,$$
$$-C \leq \partial_u P(x,0,0,0) \leq -C^{-1}, \qquad \textit{for all } x \in \overline{\Omega}.$$

(i) *If $\vec{\ell}$ satisfies case* (**I**), *then there is $c_0 > 0$ such that if $\|f\|_{\Lambda^{\lambda}(\Omega)}$ and $\|g\|_{\Lambda^{\lambda+2}(\partial\Omega)} \leq c_0$, then there is $u \in \Lambda^{\lambda+2}(\Omega)$ satisfying*

$$\begin{cases} P\left(x, u, \nabla u, \nabla^2 u\right) = f(x) & \textit{in } \Omega, \\ \vec{\ell}\, u = g & \textit{on } \partial\Omega. \end{cases}$$

(ii) *If $\vec{\ell}$ satisfies case* (**II**), *then there is $c_0 > 0$ such that if $\|f\|_{\Lambda^{\lambda}(\Omega)}$, $\|g\|_{\Lambda^{\lambda+2}(\partial\Omega)}$ and $\|h\|_{\Lambda^{\lambda+2}(\Gamma)} \leq c_0$, then there is $u \in \Lambda^{\lambda+2}(\Omega)$ satisfying*

$$\begin{cases} P\left(x, u, \nabla u, \nabla^2 u\right) = f(x) & \textit{in } \Omega, \\ \vec{\ell}\, u = g & \textit{on } \partial\Omega, \\ u = h & \textit{on } \Gamma. \end{cases}$$

Proof. We prove only part (ii), the first part being similar. Let

$$\mathcal{X} = \left\{ w : w \in \Lambda^{\lambda+2}(\Omega) \text{ and } \vec{\ell}\, w \in \Lambda^{\lambda+2}(\partial\Omega) \right\},$$
$$\mathcal{Y} = \Lambda^{\lambda}(\Omega) \times \Lambda^{\lambda+2}(\partial\Omega) \times \Lambda^{\lambda+2}(\Gamma).$$

Then $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces with the obvious norms. Consider the map $F : \mathcal{X} \to \mathcal{Y}$

given by

$$F(w) = \left(P\left(x, w, \nabla w, \nabla^2 w\right), \vec{\ell} w, w\mid_\Gamma \right).$$

By our hypotheses, $F(0) = 0$ and F has continuous first Fréchet derivatives. Furthermore, since $\partial_u P(x, 0, 0, 0)$ is negative, Theorem 1.1 implies that $F'(0)$ is invertible. Thus the implicit function theorem (see e.g. [12]) shows that F is an open map near $w = 0$, and this completes the proof of the theorem.

In the next section, we reduce (1.2) to a pseudodifferential equation on the boundary, and then use our results on a special class of pseudodifferential operators (see the following paper [8] in this journal) to prove Theorems 1.1 , 1.2 and 1.3.

§2. Transference of the Oblique Derivative Problem to the Boundary and Proofs of Theorems

In this section we reduce the oblique derivative problem to the boundary using the solution to the Neumann problem to consttruct an approximate Poisson operator. This is achieved largely via the method of symbol splitting. For example, if $\sigma \in \mathcal{C}^\lambda S^m_{1,\delta}$, then $\sigma = \sigma^\sharp + \sigma^\flat$ where $\sigma^\sharp \in S^m_{1,\gamma}$ is smooth and $\sigma^\flat \in \mathcal{C}^\lambda S^{m-\lambda(\gamma-\delta)}_{1,\gamma}$ has better order. See Proposition 1.1 in [8]. In particular, this applies to the operator of multiplication by the function $a(x)$, and we will repeatedly interchange a and $a^\sharp$ whenever convenient. We will also make use of the boundedness results for special operators in [8].

2.1. The Neumann Problem and a Change of Variables

We study the oblique derivative problem by reducing it to a pseudodifferential equation on the boundary. In an earlier paper (see Section 3 of [7]), we dealt with this problem in the case of smooth data, following the line of argument in [2]. In order to simplify the equation, we straightened the normal direction with respect to $(g_{i,j})$ (where $P = \sum_{i,j} g_{i,j} \frac{\partial}{\partial_{x_i}} \frac{\partial}{\partial_{x_j}}$ + lower order terms) and then the tangential direction $\vec{\mathbf{T}}$ to $\frac{\partial}{\partial t}$ (where $\vec{\ell} = \vec{\mathbf{T}} + a\vec{\mathbf{n}}$). We shall do the same here, but since the structures are no longer smooth, the process must be modified.

First, in order to simplify the computation of $\vec{\ell}$ on the boundary, we pick a constant c such that the Neumann problem

$$\begin{cases} PF + cF = f & \text{on } \Omega, \\ \vec{\mathbf{n}}F = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution F, which we denote by $F = \mathbf{N}f$. Then, by the theory of elliptic boundary value problems, $\mathbf{N} : \Lambda^s(\Omega) \to \Lambda^{s+2}(\Omega)$ is bounded for $0 \le s \le \lambda$, and $\mathbf{N} : H^s_p(\Omega) \to H^{s+2}_p(\Omega)$ is bounded for $0 \le s < \lambda$. With $v = u - \mathbf{N}f$, the oblique derivative problem reduces to

$$\begin{cases} Pv = c\mathbf{N}f & \text{on } \Omega, \\ \vec{\ell} v = g + (Da)\, g_1 - \vec{\mathbf{T}}\mathbf{N}f & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

or

$$\begin{cases} Pv = c\mathbf{N}f & \text{on } \Omega, \\ \vec{\ell} v = g + (Da)\, g_1 - \vec{\mathbf{T}}\mathbf{N}f & \text{on } \partial\Omega, \\ v = h - \mathbf{N}f & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Since $\mathbf{N} : \Lambda^s(\Omega) \to \Lambda^s(\Omega)$ is compact, our main theorem will follow if we can solve the following equations with the appropriate estimates:

$$\begin{cases} Pv = 0 & \text{on } \Omega, \\ \vec{\ell}\, v = g + (Da)\, g_1 - \vec{\mathbf{T}}\mathbf{N}f & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

and

$$\begin{cases} Pv = 0 & \text{on } \Omega, \\ \vec{\ell}\, v = g + (Da)\, g_1 - \vec{\mathbf{T}}\mathbf{N}f & \text{on } \partial\Omega, \\ v = h - \mathbf{N}f & \text{on } \Gamma. \end{cases} \tag{2.4}$$

As in Section 3 of [7], we now change variables repeatedly to simplify calculations. As indicated there, we want to choose the coordinate charts such that $a(x,t) \neq 0$ on the top and bottom of the t-interval (where $\frac{\partial}{\partial t} = \vec{\mathbf{T}}$) to ensure that singularities will not propagate out of the local charts.

Now $\vec{\ell} = \vec{\mathbf{T}} + a\vec{\mathbf{n}}$. If $a \neq 0$ on the boundary, the problem is elliptic, so the result is well-known. Suppose $\mathcal{N} = \{a = 0\} \neq \phi$. Fix $p \in \mathcal{N}$, and let γ be the integral curve of the vector field $\vec{\mathbf{T}}$ through p with $\gamma(0) = p$. Then γ is $\mathcal{C}^{\lambda+2}$ and, by the finite length restriction (R), there are $s_1 < 0 < s_2$ such that $| a(\gamma(s_1)) | > 0$, $| a(\gamma(s_2)) | > 0$ and $\gamma((s_1, s_2))$ is not self-intersecting. Here, s_1, s_2 can be chosen so that $a(\gamma(s))$ is as small as we wish for $s_1 < s < s_2$ and so that either $\gamma((s_1,s_2)) \cap \Gamma = \emptyset$ or $\gamma((s_1,s_2)) \cap \Gamma$ consists of a single point q and the segment of γ between p and q is completely in $\mathcal{N}$. (Note that γ can't intersect Γ in more than one point since once γ is in $\sum^+$ (resp. $\sum^-$), it must remain there for all future time by assumptions (iii) and (iv) in case (II) of Subsection 1.1. Suppose $\gamma((s_1,s_2)) \cap \Gamma = \{q\}$. Now $\vec{\mathbf{T}}$ is transversal to Γ at q, and so by the theory of ODE, there is a small neighborhood V of q in $\partial\Omega$, and $t_1 < 0 < t_2$ and a $\mathcal{C}^{\lambda+2}$ diffeomorphism $\Phi : I^n \times (t_1, t_2) \longrightarrow V$, such that $\Phi(x, \cdot)$ are the integral curves of the vector field $\vec{\mathbf{T}}$ and $\Phi(0,0) = q$, $\Phi(\cdot, 0) \subset \Gamma$, $| a(\Phi(\cdot, t_1)) | > 0$, $| a((\Phi(\cdot, t_2)) | > 0$. So, $p \in V$ automatically as $\Phi(0,s)$ passes through p. On the other hand, if $\gamma((s_1,s_2)) \cap \Gamma = \phi$, choose any hypersurface H of $\partial\Omega$ through p at which $\vec{\mathbf{T}}$ is transversal. We can then find V and Φ as above with $\Phi(\cdot, 0) \subset H$, but $V \cap \Gamma = \emptyset$.

By introducing geodesic normal coordinates with respect to $\partial\Omega$, and denoting by r the normal variable, then for any local coordinates $y_1, \cdots, y_{n+1}$ in $\partial\Omega$, in particular for the charts (Φ, V) constructed above, we have

$$P = \alpha \frac{\partial^2}{\partial r^2} + \sum_{k=1}^{n+1} b_k \frac{\partial}{\partial y_k}\frac{\partial}{\partial r} + \sum_{i,j=1}^{n+1} g_{ij} \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} + \text{lower order terms}. \tag{2.5}$$

In summary, for every $p \in \partial\Omega$, if $a(p) = 0$, we can find a neighborhood U of p in the boundary which has the following property:

$$U = I^n \times I, \quad I = (-1,1), \quad | a(x,-1) | \neq 0, \quad | a(x,1) | \neq 0, \ \forall x \in I^n,$$
$$\text{either } U \cap \Gamma = \emptyset \ \text{ or } U \cap \Gamma = \{t = 0\}.$$

In the above coordinate chart, P satisfies (2.5) with $\alpha > 0$, and $\vec{\ell} = \frac{\partial}{\partial t} - a\frac{\partial}{\partial r}$. Furthermore, the coefficients in P and $\vec{\ell}$ have the same degree of smoothness ($\mathcal{C}^\lambda$ and $\mathcal{C}^{\lambda+2}$ respectively) as those in the original variables by the theory of ODE. By shrinking U if necessary, we may assume that a is as small as we wish on U. For points $p \in \partial\Omega$, $a(p) \neq 0$, we simply choose a neighbourhood U such that $a \neq 0$ on $\overline{U}$. Since $\partial\Omega$ is compact, there is finite collection $U_1, \cdots, U_k$ of neighbourhoods as above covering $\partial\Omega$ such that each U_j either has the property $(*)$ or $a \neq 0$ on $\overline{U}_j$. Extend U_i to be an open set in R^{n+2} such that $U_j = I^n \times I \times I_\epsilon$, $I_\epsilon = (-\epsilon, \epsilon), U_j \cap \partial\Omega = I^n \times I \times \{0\}$, and $U_j \cap \Omega = \{r > 0\}$. Let $U_0 = \{r > \delta\}$, so that if δ is small enough, then $\overset{k}{\underset{j=0}{\cup}} U_j \supset \overline{\Omega}$.

2.2. Pseudodifferential Equations on the Boundary

On those U (we drop the index j for the convenience) with property $(*)$, equations (2.3) and (2.4) now become

$$\begin{cases} Pv = 0 & \text{in } r > 0, \\ \vec{\ell}\, v = \tilde{g} + (Da)\, g_1 - \vec{\mathbf{T}}F & \text{on } r = 0, \end{cases} \tag{2.6}$$

and

$$\begin{cases} Pv = 0 & \text{in } r > 0, \\ \vec{\ell}\, v = \tilde{g} + (Da)\, g_1 - \vec{\mathbf{T}}F & \text{on } r = 0, \\ v = h - F & \text{on } t = r = 0, \end{cases} \tag{2.7}$$

where

$$P = \frac{\partial^2}{\partial r^2} + \sum_{k=1}^{n+1} b_k \frac{\partial}{\partial y_k}\frac{\partial}{\partial r} + \sum_{i,j=1}^{n+1} g_{ij}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} + \text{lower order terms},$$
$$\vec{\ell} = \frac{\partial}{\partial t} - a\frac{\partial}{\partial r}.$$

Now, if $w_0 = w\,|_{r=0}$ and $F \neq f\,|_{\partial\Omega}$, then w is uniquely determined by w_0 via the Poisson operator $\widetilde{\mathbf{E}}$ of P, namely $w = \widetilde{\mathbf{E}}w_0$. Therefore, to solve (2.6) and (2.7), we need only to calculate $\vec{\ell}\,\widetilde{\mathbf{E}}w_0$ and then solve for w_0 on the boundary. The calculation is standard if the coefficients of P are smooth (see e.g. [2]), but otherwise some care must be exercised. Here, $\widetilde{\mathbf{E}}$ is defined by the theory of the Dirichlet problem for elliptic operators (see e.g. [1]), but we will instead use an approximate version $\mathbf{E}$ defined as follows.

Write

$$P = \left(\frac{\partial^2}{\partial r^2} + 2B(x, D_x)\frac{\partial}{\partial r} + P_1(x, D_x)\right) + B'(r, x, D_x)\frac{\partial}{\partial r} + P_1'(r, x, D_x) + \text{lower order terms},$$

where $B'(0, x, D_x) = P_1'(0, x, D_x) = 0$. Now set $\delta = \max\left\{\frac{1}{\lambda+1}, \frac{1}{2}\right\}$. Using the symbol splitting in Proposition 4 in [8], we define an approximate Poisson operator by

$$(\mathbf{E}u_0)(r, x) = \int e^{ix\cdot\xi} e^{-r\left[\sqrt{-P_1^\sharp(x,i\xi) + B^\sharp(x,i\xi)^2} + B^\sharp(x,i\xi)\right]} \hat{u}_0(\xi) d\xi, \tag{2.8}$$

where $\xi = (\xi_1, \cdots, \xi_{n+1})$ is the dual variable of $x = (x_1, \cdots, x_{n+1})$, and

$$B = B^\sharp + B^\flat, \quad P_1 = P_1^\sharp + P^\flat, \tag{2.9}$$

with

$$B^\sharp = i\sum_{k=1}^{n+1} b_k^\sharp\,(x, \xi)\,\xi_k, \quad P_1^\sharp = -\sum_{i,j=1}^{n+1} g_{i,j}^\sharp\,(x, \xi)\,\xi_i\xi_j, \tag{2.10}$$

where

$$\begin{aligned}&b_k^\sharp,\ g_{i,j}^\sharp \in S_{1,\delta}^0,\\&b_k^\flat = b_k - b_k^\sharp,\ g_{i,j}^\flat = g_{i,j} - g_{i,j}^\sharp \in \mathcal{C}^\lambda S_{1,\delta}^{-\lambda\delta},\end{aligned}$$

and where

$$\widetilde{g}_{i,j}^\sharp = g_{i,j}^\sharp - b_i^\sharp b_j^\sharp \tag{2.11}$$

is positive definite for $|\xi|$ large.

Since $\mathbf{E}$ is a Poisson integral, we have that $\mathbf{E} : \Lambda^s(\partial\Omega) \to \Lambda^s(\Omega)$ for $0 < s \le \lambda$ and $\mathbf{E} : B_p^{s-\frac{1}{p},p}(\partial\Omega) \to H_p^s(\Omega)$ for $0 \le s < \lambda$. We now compute $P\mathbf{E}$:

$$\begin{aligned}P\mathbf{E} &= \left(\frac{\partial^2}{\partial r^2} + 2B(x,D_x)\frac{\partial}{\partial r} + P_1(x,D_x)\right)\mathbf{E}\\&\quad + B'(r,x,D_x)\frac{\partial}{\partial r}\mathbf{E} + P_1'(r,x,D_x)\mathbf{E} + \text{lower order terms}\\&= \left(\frac{\partial^2}{\partial r^2} + 2B^\sharp(x,D_x)\frac{\partial}{\partial r} + P_1^\sharp(x,D_x)\right)\mathbf{E} + B'(r,x,D_x)\frac{\partial}{\partial r}\mathbf{E} + P_1'(r,x,D_x)\mathbf{E}\\&\quad + 2\left(B^\flat(x,D_x)\frac{\partial}{\partial r} + P_1^\flat(x,D_x)\right)\mathbf{E} + \text{lower order terms.}\end{aligned} \tag{2.12}$$

Since $B'(0,x,D_x) = P_1'(0,x,D_x) = 0$, the second, third and last terms in (2.12) are bounded operators $\Lambda^s(\partial\Omega) \to \Lambda^{s-\min\{\lambda,1\}}(\Omega)$ for $0 < s \le \lambda+1$, $B_p^{s-\frac{1}{p},p}(\partial\Omega) \to H_p^{s-\min\{\lambda,1\}}(\Omega)$ for $\frac{1}{p} < s < \lambda+1$. A direct calculation shows that the first term, $\left(\frac{\partial^2}{\partial r^2} + 2B^\sharp\frac{\partial}{\partial r} + P_1^\sharp\right)\mathbf{E}$, is a combination of bounded operators $\Lambda^{s+1+\delta}(\partial\Omega) \to \Lambda^s(\Omega)$ for $0 < s \le \lambda$, $B_p^{s-\frac{1}{p}+1+\delta,p}(\partial\Omega) \to H_p^s(\Omega)$ for $0 \le s < \lambda$. Also $g_{i,j}^\flat\partial_i\partial_j \circ \mathbf{E}$ maps $\Lambda^{s+2-\delta\lambda}(\partial\Omega) \to \Lambda^s(\Omega)$, for $0 < s \le \lambda$, and $B_p^{s+2-\delta\lambda-\frac{1}{p}}(\partial\Omega) \to H_p^s(\Omega)$, for $0 \le s < \lambda$, $1 < p < \infty$ by Proposition 1.1 of [8]. Thus these error terms have order $1+\delta$ and $2-\delta\lambda$ respectively. Note that our choice of δ equalizes these orders for $\lambda \le 1$. To summarize,

$$P\mathbf{E} = B_1, \tag{2.13}$$

where

$$\begin{aligned}&B_1 : \Lambda^{s+1+\delta}(\partial\Omega) \to \Lambda^s(\Omega),\ \text{for } 0 < s \le \lambda,\\&B_1 : B_p^{s+1+\delta-\frac{1}{p}}(\partial\Omega) \to H_p^s(\Omega),\ \text{for } 0 \le s < \lambda,\ 1 < p < \infty.\end{aligned}$$

Now we compute $\overrightarrow{\ell}\mathbf{E}\,|_{\partial\Omega}$. We have

$$\overrightarrow{\ell}\mathbf{E}\,|_{\partial\Omega} = L_1$$

where

$$\begin{aligned}L_1 &= \overrightarrow{T} + a\sqrt{P_1^\sharp(x,D_x) - B^\sharp(x,D_x)^2} + aB^\sharp(x,D_x)\\&= \overrightarrow{T} + a\sqrt{\sum_{i,j=1}^{n+1}\widetilde{g}_{i,j}^\sharp\,(x,\xi)\,\xi_i\xi_j} + ia\sum_{k=1}^{n+1} b_k^\sharp\,(x,\xi)\,\xi_k,\end{aligned} \tag{2.14}$$

by (2.10) and (2.11).

Thus we have reduced the solving of (2.6) and (2.7) to the solution of the pseudodifferential equations

$$L_1 u = \tilde{g} - \overrightarrow{T} F \tag{2.15}$$

and

$$\begin{cases} L_1 u = \tilde{g} - \overrightarrow{T} F, \\ u \mid_{t=0} = h - F. \end{cases} \tag{2.16}$$

2.3. Microlocalization

Note that the pseudodifferential operator L_1 is elliptic on the cone $\{\mid \xi_{n+1} \mid > \lambda' \mid \xi' \mid\}$ for any $\lambda' > 0$, and thus we can easily invert L_1 in this cone. So it remains to investigate L_1 on a cone $\{\mid \xi_{n+1} \mid < \lambda \mid \xi' \mid\}$. As in [7], our goal now is to eliminate ξ_{n+1} from underneath the root sign in (2.14). First we change notation. Set $t = x_{n+1}, \tau = \xi_{n+1}, x = (x_1, \cdots, x_n)$, and $\xi = (\xi_1, \cdots, \xi_n)$ (which used to be denoted by ξ') so that on the cone $\{\mid \tau \mid < \lambda \mid \xi \mid\}$ we have

$$\begin{aligned} &L_1(x,t,\xi,\tau) \\ &= i\tau + a(t,x)\sqrt{\widetilde{g}^{\sharp}_{n+1,n+1}\tau^2 + \sum_{j=1}^{n} \widetilde{g}^{\sharp}_{n+1,j}(x,t,\xi,\tau)\,\xi_i\xi_j + \sum_{i,j=1}^{n+1} \widetilde{g}^{\sharp}_{i,j}(x,t,\xi,\tau)\,\xi_i\xi_j} \\ &\quad + ia(t,x)\sum_{k=1}^{n+1} b^{\sharp}_k(x,t,\xi,\tau)\,\xi_k + ia(t,x)b^{\sharp}_{n+1}(x,t,\xi,\tau)\,\tau. \end{aligned}$$

Now decompose

$$a(x,t,\xi,\tau) = a^{\#}(x,t,\xi,\tau) + a^{\flat}(x,t,\xi,\tau)$$

with $a^{\#} \in S^0_{1,\delta}$ and $a^{\flat} \in \mathcal{C}^{\lambda+2} S^{-\delta(\lambda+2)}_{1,\delta}$ where $\delta = \max\left\{\frac{1}{\lambda+1}, \frac{1}{2}\right\}$, and let

$$\begin{aligned} &\widetilde{L}_1(x,t,\xi,\tau) \\ &= i\tau + a^{\#}(x,t,\xi,\tau)\sqrt{\widetilde{g}^{\#}_{n+1,n+1}\tau^2 + \sum_{j=1}^{n} \widetilde{g}^{\#}_{n+1,j}(x,t,\xi,\tau)\,\xi_i\xi_j + \sum_{i,j=1}^{n+1} \widetilde{g}^{\#}_{i,j}(x,t,\xi,\tau)\,\xi_i\xi_j} \\ &\quad + ia^{\#}(x,t,\xi,\tau)\sum_{k=1}^{n+1} b^{\#}_k(x,t,\xi,\tau)\,\xi_k + ia^{\#}(x,t,\xi,\tau)\,b^{\#}_{n+1}(x,t,\xi,\tau)\,\tau. \end{aligned}$$

Since a is small in U, we may assume that $a^{\#}$ is also small there.

Lemma 2.1. $\widetilde{L}_1(x,t,\xi,\tau) = \widetilde{L}_0(x,t,\xi,\tau)\widetilde{L}(x,t,\xi,\tau)$ *where*

$$\widetilde{L}(x,t,\xi,\tau) = i\tau + ia^{\sharp} Q\left(x,t,\xi,a^{\sharp}\right),$$

with $Q, \left(\frac{\partial}{\partial z}\right)^k Q(x,t,\xi,z) \in S^1_{1,\delta}$, $c\,|\xi| \le ReQ \le c^{-1}\,|\xi|$ *for some* $c > 0$*; and*

$$\widetilde{L}_0(x,t,\xi,\tau) = 1 + a^{\sharp} Q_0\left(x,t,\xi,a^{\sharp}\right),$$

with $Q_0 \in S^1_{1,\delta}$.

Proof. Let $z = a^\sharp$ and write

$$\begin{aligned}
&\widetilde{L}_1(x,t,\xi,\tau)\\
&= i\tau + z\left(\sqrt{\widetilde{g}^\sharp_{n+1,n+1}\tau^2 + \sum_{j=1}^{n}\widetilde{g}^\sharp_{n+1,j}\xi_i\xi_j + \sum_{i,j=1}^{n+1}\widetilde{g}^\sharp_{i,j}\xi_i\xi_j} + iz\sum_{k=1}^{n+1} b^\sharp_k\xi_k + ib^\sharp_{n+1}\tau\right)\\
&= f\left(x,t,\widetilde{\xi},\widetilde{\tau},z\right)|\xi|,
\end{aligned}$$

where $\widetilde{\xi} = \frac{\xi}{|\xi|}, \widetilde{\tau} = \frac{\tau}{|\xi|}$ and $\left|\frac{\partial}{\partial\widetilde{\tau}}f\left(x,t,\widetilde{\xi},\widetilde{\tau},z\right)|_{\widetilde{\tau}=0}\right| \geq 1 - c\,|z|$ for some constant $c > 0$. Since z is small, $\frac{\partial}{\partial\widetilde{\tau}}f\,|_{\widetilde{\tau}=0}$ and so by the Malgrange preparation theorem,

$$f\left(x,t,\widetilde{\xi},\widetilde{\tau},z\right) = f_0\left(x,t,\widetilde{\xi},\widetilde{\tau},z\right)\left(i\widetilde{\tau} + g_0\left(x,t,\widetilde{\xi},z\right)\right), \quad f_0 \neq 0.$$

But $f\left(x,t,\widetilde{\xi},\widetilde{\tau},z\right) = 0$ and so $g_0\left(x,t,\widetilde{\xi},z\right) = zg_1\left(x,t,\widetilde{\xi},z\right)$ and $\left|\left(\frac{\partial}{\partial z}\right)^k g\right| \leq C_k$. Now let

$$\widetilde{L}\left(x,t,\xi,\tau\right) = |\xi|\left(i\widetilde{\tau} + g_0\left(x,t,\widetilde{\xi},a^\sharp\right)\right) = i\tau + a^\sharp\,|\xi|\,g_1\left(x,t,\widetilde{\xi},a^\sharp\right) = i\tau + a^\sharp Q$$

and

$$\widetilde{L}_0\left(x,t,\xi,\tau\right) = f_0\left(x,t,\widetilde{\xi},\widetilde{\tau},a^\sharp\right).$$

A simple computation shows that $\frac{ReQ}{|\xi|}$ is bounded above and below by two positive constants and that $f_0\left(x,t,\widetilde{\xi},\widetilde{\tau},0\right) = 1$. This completes the proof of Lemma 2.1.

We now give the corresponding operator decomposition.

Lemma 2.2. *Let L and L_0 denote the operators with symbols $i\tau + aQ\left(x,t,\xi,a\right)$ and $1 + a^\sharp Q_0\left(x,t,\xi,a^\sharp\right)$ where Q and Q_0 are as in Lemma* 2.1 *respectively. We have*

$$L_1 = L_0 \circ L + aB_\delta + B_{-\delta}, \tag{2.17}$$

where $B_\delta \in \overline{\mathcal{O}}^{\delta}_{(0,\lambda+2)}$ and $B_{-\delta} \in \overline{\mathcal{O}}^{-\delta}_{(0,\lambda+2)}$.

Proof. Since $a^\flat \in \mathcal{C}^{\lambda+2}S^{-\delta-1}_{1,\delta}$, $L_1 - \widetilde{L}_1 \in \overline{\mathcal{O}}^{-\delta}_{(0,\lambda+2)}$. By Lemma 2.1 and Propostion 1.2 in [8],

$$\begin{aligned}
\widetilde{L}_0 \circ \widetilde{L} - \widetilde{L}_1 &= \left(a^\sharp Q_0\right) \circ \left(a^\sharp Q\right) - a^\sharp Q_0 a^\sharp Q\\
&= \left(aQ_0\right) \circ \left(a^\sharp Q\right) - aQ_0 a^\sharp Q + B_{-\delta}\\
&= a\left\{Q_0 \circ \left(a^\sharp Q\right) - Q_0 a^\sharp Q\right\} + B_{-\delta} = aB_\delta + B_{-\delta}.
\end{aligned}$$

On the other hand,

$$\widetilde{L} - L = a^\sharp Q\left(x,t,\xi,a^\sharp\right) - aQ\left(x,t,\xi,a\right) = a^\flat B_1 = B_{-\delta}.$$

If we now write

$$\begin{aligned}
L_1 &= \widetilde{L}_1 + \left(L_1 - \widetilde{L}_1\right) = \widetilde{L}_0 \circ \widetilde{L} + \left(\widetilde{L}_1 - \widetilde{L}_0 \circ \widetilde{L}\right) + \left(L_1 - \widetilde{L}_1\right)\\
&= \widetilde{L}_0 \circ L + \widetilde{L}_0 \circ \left(\widetilde{L} - L\right) + \left(\widetilde{L}_1 - \widetilde{L}_0 \circ \widetilde{L}\right) + \left(L_1 - \widetilde{L}_1\right),
\end{aligned}$$

the lemma follows.

2.4. A Parametrix

An operator K is called a left (respectively right) parametrix of an operator L if $KL = I + S$ (respectively $LK = I + S$), where S is a smoothing operator. In this paper, however, it will be more convenient to call K a left (or right) parametrix of L if KL (or LK) $= I + S$

provided there is $\delta > 0$ such that S is in $\overline{\mathcal{O}}_I^{-\delta}$ for some appropriate interval I. As we shall show, this definition is sufficient to carry out the estimates we need.

In our local setting, the conditions in cases (**I**) and (**II**) in §2 can be restated as follows:

(**I**) $a(x,t) \geq 0$ for $(x,t) \epsilon\ I^n \times I$ or $a(x,t) \leq 0$ for $(x,t) \epsilon\ I^n \times I$; $a(x,\pm 1) \neq 0$ for $x \in I^n$.

(**II**) $a(x,t) \geq 0$ for $t \geq 0$, $x\ \epsilon\ I^n$ and $a(x,t) \leq 0$ for $t \leq 0$, $x\ \epsilon\ I^n$; $a(x,\pm 1) \neq 0$ for $x \in I^n$.

First, we will construct a left parametrix K for L microlocally in the cone $\{|\tau| < \lambda |\xi|\}$ in cases (**I**) and (**II**). Then we use this to obtain a parametrix for L_1, and finally to obtain a parametrix and a priori estimates for the original problem (2.1) and (2.2).

By the choice of the coordinate charts, there exist $c > 0$, $1 > c' > 0$, such that $|a(x,t)| \geq c$ for $|t| \geq c'$. Since $\rho = \rho_j$ in the previous section is compactly supported in $I^n \times I$, we may pick ρ^* and $\tilde{\rho} \in C_c^\infty(I^n \times I)$, $\rho^* = 1$ on supp ρ, and $\tilde{\rho} = 1$ on supp ρ^*. By making the change of variable $t \to -t$ if necessary, we may assume that $a(x,t) \geq 0$ in case (I). Recall that L in Lemma 2.1 is given by

$$L(t,x,\tau,\xi) = i\tau + a(t,x)Q(x,t,\xi,a),$$

where $c|\xi| \leq |ReQ| \leq c^{-1}|\xi|$. Now, let

$$\tilde{A}(x,t,t',\xi) = \int_{t'}^{t} a(x,\theta)Q(x,\theta,\xi,a)d\theta,$$
$$A(x,t,t') = \int_{t'}^{t} a(x,\theta)d\theta,$$
$$K(x,t,t',\xi) = \tilde{\rho}(x,t)e^{-\tilde{A}(x,t,t',\xi)}.$$

Define

$$Ku(x,t) = (2\pi)^{-\frac{n}{2}} \int_{t_0}^{t} \int_{|\xi|\geq 1} e^{ix\cdot\xi} K(x,t,t',\xi) \widetilde{u^*}(\xi,t')d\xi dt',$$
$$Tu(x,t) = (2\pi)^{-\frac{n}{2}} \int_{t_0}^{t} \int_{|\xi|\geq 1} e^{ix\cdot\xi} K(x,t,t',\xi)a(x,t')Q\left(x,t',\xi,a\left(x,t'\right)\right) \widetilde{u^*}(\xi,t')d\xi dt',$$
$$\mathcal{K}u_0(x,t) = -(2\pi)^{-\frac{n}{2}} \int_{|\xi|\geq 1} e^{ix\cdot\xi}\tilde{\rho}(x,t_0)e^{-\tilde{A}(x,t,t_0,\xi)} \widetilde{u_0^*}(\xi)d\xi, \tag{2.18}$$

where $u^* = \rho^* u$, $\widetilde{u^*}(\xi,t') = \int_{R^n} e^{-ix\cdot\xi}u^*(x,t')dx$, $u_0^*(x) = \rho^*(x,0)u_0(x)$, $\widetilde{u_0^*}(\xi) = \int_{R^n} e^{-ix\cdot\xi}u_0^*(x)dx$, and $t_0 = -1$ if a satisfies case (**I**), while $t_0 = 0$ if a satisfies case (**II**).

Remark 2.1. The choice of t_0 is crucial. With the above choice, $a(x,\theta)$ keeps the same sign in (t',t) for $t' \in (t_0,t)$, and $\mathrm{Re}\left(\widetilde{A}(x,t,t',\xi)\right) \geq 0$ and $A(x,t,t') \geq 0$ for $t' \in (t_0,t)$.

Claim 2.1. If $\mu_p = \min\left\{1 - \delta - \frac{1}{p(\lambda+3)}, \frac{\lambda+2}{2(\lambda+3)}\right\} > 0$, then the operator K is a left parametrix for L in Case (**I**), and a left parametrix for L in Case (**II**) for functions u with u vanishing at $t = 0$ (i.e., $KLu = u + Su$ if $u|_{t=0} = 0$).

Proof. We have

$$KL = K \circ \partial_t + K \circ aQ.$$

Performing integration by parts in $K \circ \partial_t$ we get

$$KLu = u - Ru - \mathcal{K}u_0 + \mathcal{O}^{-\infty}_{(-1,\lambda+2)},$$

where $u_0 = u(t_0, x)$ and

$$Ru = -\int\int e^{ix\cdot\xi} a(t', x) Q(t', x, \xi, a) e^{-\tilde{A}(x,t,t',\xi)} \widetilde{u^*}(t', \xi) d\xi dt' + K \circ aQ.$$

By Lemma 1.7 in [8] we have that R is in $\overline{\mathcal{O}}^{-\mu_p}_{(-1,\lambda+2)}$. We get

$$KLu = u + S'u - \mathcal{K} \circ u_0,$$

where S' is in $\mathcal{O}^{-\mu_p}_{(-1,\lambda+2)}$. In Case I, $u_0 = u(x, -1) = 0$ since $\operatorname{supp} u$ is compact. This completes the proof of the claim.

Now we turn to constructing a left parametrix for L_1. Since $L_0 \in \mathcal{C}^{\lambda+2} S^0_{1,\delta}$ is elliptic, there is $L_0^{-1} \in \mathcal{C}^{\lambda+2} S^0_{1,\delta}$ such that $L_0^{-1} \circ L_0 = I + J$ where J is in $\overline{\mathcal{O}}^{-2}_{(0,\lambda+2)}$. Now set $\tilde{K} = K \circ L_0^{-1}$. By the rough ψdo calculus and Lemma 2.2 (since $a \in \mathcal{C}^{\lambda+2}$), we have

$$\begin{aligned}\tilde{K} \circ L_1 u &= K \circ Lu + K \circ aB_\delta u + \overline{\mathcal{O}}^{-\delta}_{(0,\lambda+2)} u \\ &= K \circ Lu + (K \circ a) B_\delta u + \overline{\mathcal{O}}^{-\delta}_{(0,\lambda+2)} u \\ &= u - Ru - \mathcal{K} u_0 + \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)} u,\end{aligned}$$

where $u_0 = u\,|_{t=t_0}$, since $K \circ a \in \overline{\mathcal{O}}^{\frac{1}{(\lambda+3)p}-1}_{(0,\lambda+2)}$ and $\mu_p > 0$. By Lemma 1.7 in [8],

$$\tilde{K} \circ L_1 u = u - \mathcal{K} u_0 + \widetilde{S} u,$$

with $\widetilde{S} \in \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}$. Note that in case (I), u_0 is smooth and so $\tilde{K} \circ L_1 = I + \widetilde{S}$.

Up to this point we have only worked on the cone $\{|\tau| < |\xi'|\}$ for L_1 microlocally. We are now ready to obtain a left parametrix for the model problem in the coordinate chart:

$$L_1 u = g - \partial_t F + (Da) g_1 + aw \tag{2.19}$$

or

$$\begin{cases} L_1 u = g - \partial_t F + (Da) g_1 + aw, \\ u\,|_{t=0} = h - F. \end{cases} \tag{2.20}$$

Proposition 2.1. *If u satisfies either* (2.19) *or* (2.20) *above with the assumption that g, F, g_1, u and w are smooth at $t = -1$ and $t = 1$, then*

$$\begin{aligned}u = {}& \widetilde{K}(\varphi g) + \widetilde{K} \circ (Da) \varphi g + T \circ L_0^{-1}(F) + \widetilde{K} \circ aw \\ &+ \overline{\mathcal{O}}^0_{(0,\lambda+2)}(F) + \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(u) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(w) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(g),\end{aligned}$$

in case (**I**), *and*

$$\begin{aligned}\rho u_b = {}& \widetilde{K}(\varphi g) + \widetilde{K} \circ (Da) \varphi g + T \circ L_0^{-1}(F) + \widetilde{K} \circ aw \\ &+ \mathcal{K}\{h + B_1(g)\,|_{t=0} + B_2(g_1)\,|_{t=0}\} + \overline{\mathcal{O}}^0_{(0,\lambda+2)}(F) \\ &+ \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(u) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(w) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(g),\end{aligned}$$

in case (**II**), *where $B_1, B_2 \in \overline{\mathcal{O}}^{-1}_{(0,\lambda)}(\partial\Omega)$.*

Proof. Let $\widetilde{\psi} \in \mathcal{C}^\infty(R^n)$ be homogeneous of degree zero on $|\xi| \ge 1$, with supp $\widetilde{\psi} \subset \{|\xi_{n+1}| > \lambda' |\xi'|\}$. If we apply $\widetilde{\psi}$ to the above equations and commute $\widetilde{\psi}$ with L_1 we get

$$L_1 \widetilde{\psi} u = \widetilde{\psi} g - \partial_t \widetilde{\psi} F + \widetilde{\psi}(Da) g_1 + \widetilde{\psi} aw + \overline{\mathcal{O}}^0_{(0,\lambda+2)} u.$$

Since L_1 is elliptic on $\{|\xi_{n+1}| > \lambda' |\xi'|\}$, there is $L_1^{-1} \in \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}$ such that $L_1^{-1} \circ L_1 = I + \overline{\mathcal{O}}^{-1}_{(-1,\lambda+2)}$ and $L_1 \circ L_1^{-1} = I + \overline{\mathcal{O}}^{-1}_{(-1,\lambda+1)}$ (again by the rough ψdo calculus). Apply L_1^{-1}

to this equation to obtain

$$\widetilde{\psi}u = L_1^{-1}\widetilde{\psi}g - L_1^{-1}\partial_t\widetilde{\psi}F + L_1^{-1}\widetilde{\psi}\,(Da)\,g_1 + L_1^{-1}\widetilde{\psi}aw + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}u. \tag{2.21}$$

Now pick $\psi, \varphi \in \mathcal{C}^\infty(R^n)$ homogeneous of degree zero on $|\xi| \geq 1$, with

$$\text{supp}\psi \subset \{|\xi_{n+1}| > \lambda' |\xi'|\}$$

and $\psi + \varphi = 1$ on $|\xi| \geq 1$. Replacing $\widetilde{\psi}$ by ψ in (2.21) we obtain

$$\psi u = L_1^{-1}\psi g - L_1^{-1}\partial_t\psi F + L_1^{-1}\psi\,(Da)\,g_1 + L_1^{-1}\psi aw + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}u.$$

Now we wish to identify $\psi u\,|_{t=0}$ in case (II). If we use the equation

$$\partial_t\psi u = -\psi a\widetilde{Q}u + \psi g - \partial_t\psi F + \psi\,(Da)\,g_1 + \psi aw,$$

integrate in the t variable from -1 to 0, and use the fact that evaluation of ψu, ψg, etc. at $t = -1$ is smoothing, we obtain

$$\begin{aligned}
\psi u\,|_{t=0} &= \int_{-1}^0 \psi a\widetilde{Q}u + \int_{-1}^0 \psi g - \psi F\,|_{t=0} + \int_{-1}^0 \psi\,(Da)\,g_1 + \int_{-1}^0 \psi aw + \overline{\mathcal{O}}^{-\infty}_{(0,\lambda+2)}F \\
&= -\int_{-1}^0 \partial_t \circ \partial_t^{-1}\psi a\widetilde{Q}u + \int_{-1}^0 \partial_t \circ \partial_t^{-1}\psi g - \psi F\,|_{t=0} + \int_{-1}^0 \partial_t \circ \partial_t^{-1}\psi\,(Da)\,g_1 \\
&\quad + \int_{-1}^0 \partial_t \circ \partial_t^{-1}\psi aw + \overline{\mathcal{O}}^{-\infty}_{(0,\lambda+2)}F \\
&= -\psi F\,|_{t=0} + \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1)\,|_{t=0} + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(g)\,|_{t=0} \\
&\quad - \int_{-1}^0 \partial_t \circ a\left(\partial_t^{-1} \circ \psi\widetilde{Q}\right)u - \int_{-1}^0 \partial_t \circ \left[\partial_t^{-1}\psi, a\right] \circ \widetilde{Q} \circ \widetilde{\psi}u \\
&\quad + \int_{-1}^0 \partial_t \circ a\partial_t^{-1} \circ \psi w + \int_{-1}^0 \partial_t \circ \left[\partial_t^{-1}\psi, a\right] w + \overline{\mathcal{O}}^{-\infty}_{(0,\lambda+2)}(F, u, g, g_1),
\end{aligned}$$

where $\widetilde{\psi} = 1$ on $\text{supp}\,\psi$, $\text{supp}\,\widetilde{\psi} \subset \{|\xi_{n+1}| > \lambda' |\xi'|\}$.

Now if we substitute (2.21) for $\widetilde{\psi}u$ in the above equation, and note that $a(x,0) = 0$, we get

$$\begin{aligned}
\psi u\,|_{t=0} &= -\psi F\,|_{t=0} + \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1)\,|_{t=0} + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(g)\,|_{t=0} \\
&\quad + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(F)\,|_{t=0} + \overline{\mathcal{O}}^{-2}_{(0,\lambda+2)}(g, g_1, w, u)\,|_{t=0}\,.
\end{aligned} \tag{2.22}$$

On the other hand, applying φ to the equations yields

$$\begin{aligned}
L_1\varphi u &= \varphi g - \partial_t\varphi F + \varphi\,(Da)\,g_1 + \varphi aw + [\varphi, L_1]\,u \\
&= \varphi g - \partial_t\varphi F + \varphi\,(Da)\,g_1 + \varphi aw + \left[\varphi, a\widetilde{Q}\right]u \\
&= \varphi g - \partial_t\varphi F + \varphi\,(Da)\,g_1 + \varphi aw + [\varphi, a] \circ \widetilde{Q}u + a\left[\varphi, \widetilde{Q}\right]u.
\end{aligned}$$

By the rough ψdo calculus Proposition 1.2 in [8], $\left[\varphi, \widetilde{Q}\right] \in \overline{\mathcal{O}}^{\delta}_{(0,\lambda+2)}$. Now apply $\widetilde{K}$ to the above to obtain

$$\begin{aligned}
\varphi u &= \mathcal{K}\left(\varphi u\,|_{t=t_0}\right) + \widetilde{S}u + \widetilde{K}\varphi g - \widetilde{K} \circ \partial_t\varphi F + \widetilde{K}\varphi\,(Da)\,g_1 + \widetilde{K}\varphi aw \\
&\quad + \widetilde{K} \circ [\varphi, a] \circ \widetilde{Q}u + \widetilde{K} \circ a \circ \overline{\mathcal{O}}^{\delta}_{(0,\lambda+2)}(u) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(u)\,.
\end{aligned}$$

Next we commute ∂_t with L_0^{-1}, use integration by parts in $\widetilde{K} \circ \partial_t\varphi F$, commute Da with φ, commute a with L_0^{-1}, and by Proposition 1.4 in [8] use the fact that $Ka \in \overline{\mathcal{O}}^{1-\frac{1}{p(\lambda+3)}}_{(0,\lambda)}$ to arrive at

$$\varphi u = \mathcal{K}\left(\varphi u\mid_{t=t_0}\right) + \widetilde{K}\psi g - \varphi F + T\circ L_0^{-1}\varphi F + \mathcal{K}\left(L_0^{-1}\varphi F\mid_{t=t_0}\right) + \widetilde{K}a\varphi w$$
$$+ \widetilde{K}\circ[\varphi,a]\circ\widetilde{Q}u + \widetilde{K}\circ[\varphi,a]\,w + \widetilde{K}\circ(Da)\,\psi g_1 + \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(u) + \overline{\mathcal{O}}^{0}_{(0,\lambda+2)}(F)$$
$$+ \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(w).$$

In case (**I**), $L_0^{-1}\varphi F\mid_{t=t_0}$ is smoothing, while in case (**II**),

$$L_0^{-1}\varphi F\mid_{t=0} = \varphi F\mid_{t=0} + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(F)\mid_{t=0}.$$

Claim. 2.2. $\widetilde{K}\circ[\varphi,a]\in\overline{\mathcal{O}}^{-\frac{\delta}{4}-1}_{(0,\lambda+2)}$.

Proof. To begin with, note that $[\varphi,a]=\varphi\circ\left(a^\sharp+a^\flat\right)-a\varphi=\left(a^\sharp\right)_x\varphi_\xi+\overline{\mathcal{O}}^{-\frac{\delta}{4}-1}_{(0,\lambda+2)}$. We need to show that $\widetilde{K}\circ\left(a^\sharp\right)_x\in\overline{\mathcal{O}}^{-\frac{\delta}{4}}_{(0,\lambda+2)}$, and for this we write $\widetilde{K}\circ\left(a^\sharp\right)_x=\widetilde{K}\circ a_x-\widetilde{K}\circ\left(a^\flat\right)_x$, where $\widetilde{K}\circ a_x\in\overline{\mathcal{O}}^{-\frac{\delta}{4}}_{(0,\lambda+1)}$ by Proposition 1.4 in [8] and by commuting a_x with L_0^{-1}. We thus get $\widetilde{K}\circ\left(a^\sharp\right)_x\in\overline{\mathcal{O}}^{-\frac{\delta}{4}}_{(0,\lambda+1)}$ since $a^\flat\in\overline{\mathcal{O}}^{-1-\delta}_{(0,\lambda+2)}$. To show $\widetilde{K}\circ\left(a^\sharp\right)_x\in\overline{\mathcal{O}}^{-\frac{\delta}{4}}_{(0,\lambda+2)}$, we apply D_x or D_t to $\widetilde{K}\circ\left(a^\sharp\right)_x$ to get

$$D_x\circ\widetilde{K}\circ\left(a^\sharp\right)_x=\widetilde{K}\circ\left(a^\sharp\right)_x\circ D_x+\overline{\mathcal{O}}^{\frac{1}{2}}_{(0,\lambda+1)},$$
$$D_t\circ\widetilde{K}\circ\left(a^\sharp\right)_x=-KaQ\circ L_0^{-1}\circ\left(a^\sharp\right)_x+L_0^{-1}\circ\left(a^\sharp\right)_x\in\overline{\mathcal{O}}^{\frac{1}{2}}_{(0,\lambda+1)},$$

and this completes the proof of the claim.

Thus we have

$$\varphi u=\widetilde{K}\psi g-\varphi F+T\circ L_0^{-1}\varphi F+\widetilde{K}\,(Da)\,\varphi g_1+\widetilde{K}a\varphi w$$
$$+\overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(u)+\overline{\mathcal{O}}^{0}_{(0,\lambda+2)}(F)+\overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1)+\overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(w),$$

in case (**I**), and

$$\varphi u=\widetilde{K}\psi g-\varphi F+K\,(Da)\,\varphi g_1+\mathcal{K}\left(\varphi u\mid_{t=0}+\varphi F\mid_{t=0}\right)+\widetilde{K}a\varphi w$$
$$+\overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(u)+\overline{\mathcal{O}}^{0}_{(0,\lambda+2)}(F)+\overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1)+\overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(w),$$

in case (**II**). By (2.22),

$$\begin{aligned}(\varphi u+\varphi F)\mid_{t=0}&=(u-\psi u)\mid_{t=0}+\varphi F\mid_{t=0}+\text{smoothing term}\\&=h-\psi F\mid_{t=0}-\psi u\mid_{t=0}+\text{smoothing term}\\&=h+\overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(g)\mid_{t=0}+\overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(g_1)\mid_{t=0}\\&\quad+\overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(F)\mid_{t=0}+\overline{\mathcal{O}}^{-2}_{(0,\lambda+2)}(g,g_1u,w)\mid_{t=0}.\end{aligned}$$

The proposition now follows since $\mathcal{K}\circ\overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}\mid_{t=0}\in\overline{\mathcal{O}}^{0}_{\left(0,\lambda+2-\frac{1}{p}\right)}(\partial\Omega)$ by Proposition 1.4 in [8].

Now we return to the original equations (2.1) and (2.2) and obtain a left parametrix for them. As in Subsection 2.1, $\overline{\Omega}\subset\cup_{j=0}^{k}U_j$ and $\partial\Omega\subset\cup_{j=1}^{k}U_j$. Let $\rho\in\mathcal{C}^{\lambda+2}(\Omega)\cap\mathcal{C}_c^\infty(U_j)$ for some $j\ge1$ (elliptic theory handles the case $j=0$) . Let $U=U_j$.

Proposition 2.2. *Suppose that either $u\in\Lambda^s(\Omega)$ for some $s>1$, or $u\in H_p^s(\Omega)$ for some $s>1+\frac{1}{p}$, where $1<p<\infty$. Let $\rho,\widetilde{\rho}\in\mathcal{C}_c^\infty(U)$ with $\widetilde{\rho}=1$ in a rectangular (relative* *to the coordinate system) subdomain* $\widetilde{U}$ *of* U *with* $\widetilde{U} \supset \operatorname{supp} \rho$*. If* u *satisfies* (2.1)*, then*

$$\begin{aligned}\rho u_b = &\rho \widetilde{K}(\widetilde{\rho} g) + \rho \widetilde{K} \circ (Da) \widetilde{\rho} g + \rho T \circ L_0^{-1}(\widetilde{\rho} N f \mid_{\partial\Omega}) + \overline{\mathcal{O}}^0_{(0,\lambda+2)}(\widetilde{\rho} N f \mid_{\partial\Omega}) \\ &+ \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(\widetilde{\rho} u_b, \rho^* u_b) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(\widetilde{\rho} g_1) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(\widetilde{\rho} g),\end{aligned} \tag{2.23}$$

while if u *satisfies* (2.2)*, then*

$$\begin{aligned}\rho u_b = &\rho \widetilde{K}(\widetilde{\rho} g) + \rho \widetilde{K} \circ (Da) \widetilde{\rho} g + \rho T \circ L_0^{-1}(\widetilde{\rho} N f \mid_{\partial\Omega}) \\ &- \rho \mathcal{K}\left\{\widetilde{\rho} h + \overline{B}_1(\widetilde{\rho} g) \mid_\Gamma + \overline{B}_2(\widetilde{\rho} g_1) \mid_\Gamma\right\} \\ &+ \overline{\mathcal{O}}^0_{(0,\lambda+2)}(\widetilde{\rho} N f \mid_{\partial\Omega}) + \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(\widetilde{\rho} u_b, \rho^* u_b) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(\widetilde{\rho} g_1) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(\widetilde{\rho} g),\end{aligned} \tag{2.24}$$

where $u_b = u \mid_{\partial\Omega}$, $\overline{B}_j \in \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}$, $j = 1, 2$.

Proof. We prove (2.24), the proof for (2.23) being similar but easier. Let $v = u - Nf$, where N denotes the Neumann operator. We have

$$\begin{cases} Pv = 0 & \text{on } \Omega, \\ \vec{\ell} v = g + (Da) g_1 - \vec{\mathbf{T}} \mathbf{N} f \mid_{\partial\Omega} & \text{on } \partial\Omega, \\ v = h - \mathbf{N} f & \text{on } \Gamma. \end{cases}$$

We must prove (2.24) for v. Choose $\rho^* \in \mathcal{C}_c^\infty(U)$ with $\rho^* = 1$ on a rectangular subdomain U^* where $U^* \supset \operatorname{supp} \rho$, $a \neq 0$ on the top and bottom of U^* , and $\widetilde{\rho} = 1$ on $\operatorname{supp} \rho^*$. Then $\rho^* v$ satisfies

$$\begin{cases} P(\rho^* v) = 0 & \text{on } \Omega, \\ \vec{\ell}(\rho^* v) = \rho^* g + (Da) \rho^* g_1 - \rho^* \vec{\mathbf{T}} \mathbf{N} f \mid_{\partial\Omega} - \left(\vec{\ell} \rho^*\right) v & \text{on } \partial\Omega, \\ \rho^* v = \rho^* h - \rho^* \mathbf{N} f & \text{on } \Gamma. \end{cases}$$

Now set $\widetilde{v} = E \rho^* v_b$ so that $\vec{\ell} \widetilde{v} \mid_{\partial\Omega} = L_1(\rho^* v_b)$ by the construction of E.

Meanwhile,

$$\begin{aligned} P(\rho^* v - \widetilde{v}) &= [\rho^*, P] v - B_1(\rho^* v_b) && \text{on } \Omega, \\ \rho^* v - \widetilde{v} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.25}$$

where B_1 is as in (2.13). Solving the Dirichlet problem (2.25), we obtain

$$\rho^* v - \widetilde{v} = G\{[\rho^*, P] v - B_1(\rho^* v_b)\},$$

where $G \in \overline{\mathcal{O}}^{-2}_{(1,\lambda+2)}(\Omega)$, $PG = I$, and $G \mid_{\partial\Omega} = 0$. Since $\vec{\mathbf{T}}(\rho^* v - \widetilde{v}) = 0$ on $\partial\Omega$, we conclude that

$$\vec{\ell}(\rho^* v - \widetilde{v}) \mid_{\partial\Omega} = a \vec{\mathbf{n}} G [\rho^*, P] v - a B_0'(\rho^* v_b),$$

where $B_0' = \left(\vec{\mathbf{n}} G B_1\right) \mid_{\partial\Omega}$. So $B_0' : \Lambda^{s+1-\delta\lambda}(\partial\Omega) \to \Lambda^s(\partial\Omega)$ for $0 < s \leq \lambda + 1$ and $B_0' : \mathcal{B}_p^{s-\frac{1}{p}+1-\delta\lambda,p}(\partial\Omega) \to \mathcal{B}_p^{s-\frac{1}{p},p}(\partial\Omega)$ for $0 \leq s < \lambda + 1$. But, for any $\widetilde{\rho} \in \mathcal{C}_c^\infty(\Omega)$ with $\widetilde{\rho} = 1$ on the support of ρ^*, we get

$$\begin{aligned} &\rho^* g + (Da) \rho^* g_1 - \rho^* \vec{\mathbf{T}} \mathbf{N} f \mid_{\partial\Omega} - \left(\vec{\ell} \rho^*\right) v_b - L_1 v_b \\ &= -a B_0'(\rho^* v_b) + a \widetilde{B}_0(\widetilde{\rho} v_b), \end{aligned}$$

and

$$\vec{\mathbf{n}} G [\rho^*, P] \widetilde{\rho} v \mid_{\partial\Omega} = \widetilde{B}_0(\widetilde{\rho} v_b)$$

with $\widetilde{B}_0 \in \overline{\mathcal{O}}^0_{(0,\lambda+1)}(\partial\Omega)$. By Proposition 2.1, with $w = B_0'(\rho^* v_b) + \widetilde{B}_0(\rho^* v_b)$,

$$\rho^* v_b = \widetilde{K}(\rho^* g) + \widetilde{K} \circ (Da)\rho^* g + T \circ L_0^{-1}(\rho^* N f \mid_{\partial\Omega}) - \widetilde{K}\left(\vec{\ell}\rho^*\right) v_b + \left(\widetilde{K} \circ a\right)\widetilde{B}_0 \widetilde{\rho} v_b$$
$$+ \widetilde{K} \circ a \circ B_0'(\rho^* v_b) - \mathcal{K}\left\{\rho^* h + \overline{B}_1(\rho^* g)\mid_\Gamma + \overline{B}_2(\rho^* g_1)\mid_\Gamma + \overline{B}_1\left(\left(\vec{\ell}\rho^*\right) v_b\right)\mid_\Gamma\right\}$$
$$+ \overline{\mathcal{O}}^0_{(0,\lambda+2)}(\widetilde{\rho} N f \mid_{\partial\Omega}) + \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(\rho^* v_b) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+1)}(\widetilde{\rho} g_1) + \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(\widetilde{\rho} g).$$

If we now multiply the above identity by ρ, commute a with L_0^{-1}, and note that $\widetilde{K} \circ a \in \overline{\mathcal{O}}^{-1+\frac{1}{p(\lambda+3)}}_{(0,\lambda+2)}(\partial\Omega)$, we have $\widetilde{K} \circ a \circ \widetilde{B}_0, \widetilde{K} \circ a \circ B_0' \in \overline{\mathcal{O}}^{-\mu_p}_{(0,\lambda+2)}(\partial\Omega)$ and

$$|x-y| \geq \delta_0 > 0, \quad \int_{t'}^{t} a(x,\theta)\, d\theta \geq \delta_0 > 0,$$

for $(x,t) \in \operatorname{supp}\rho$, $(y,t') \in supp\left(\vec{\ell}\rho^*\right) \cup \operatorname{supp}(\widetilde{\rho} - \rho^*)$. We finally conclude that

$$\rho\widetilde{K}(\widetilde{\rho} - \rho^*), \rho\overline{K} \circ \vec{\ell}\rho^*, \rho\mathcal{K} \circ \left(B_1 \vec{\ell}\rho^*\right)\mid_\Gamma, \rho\mathcal{K} \circ (\widetilde{\rho} - \rho^*) \in \overline{\mathcal{O}}^{-1}_{(0,\lambda+2)}(\partial\Omega),$$

$$\rho T \circ L_0^{-1}(\widetilde{\rho} - \rho^*) \in \overline{\mathcal{O}}^0_{(0,\lambda+2)}(\partial\Omega),$$

and (2.24) now follows easily.

We now consider a priori estimates for the original equations (2.1) and (2.2). Suppose that P, Ω, and $\vec{\ell} = \vec{\mathbf{T}} + a\vec{\mathbf{n}}$ are as at the beginning of Subsection 1.1 with $\lambda > 0$. Consider the following problems:

$$\begin{cases} Pu = f & \text{in } \Omega, \\ \vec{\ell} u = g + (Da) g_1 & \text{on } \partial\Omega, \end{cases} \tag{2.26}$$

and

$$\begin{cases} Pu = f & \text{in } \Omega, \\ \vec{\ell} u = g + (Da) g_1 & \text{on } \partial\Omega, \\ u = h & \text{on } \Gamma. \end{cases} \tag{2.27}$$

Proposition 2.3. *Suppose that P, Ω, and $\vec{\ell}$ are as at the beginning of Subsection 1.1 with $\lambda > 0$, let $\mu_p = \min\left\{1 - \delta + \frac{1}{p(\lambda+3)}, \frac{\lambda+2}{2(\lambda+3)}\right\}$ for $1 < p < \infty$ and $0 < \mu_\infty < \min\left\{1-\delta, \frac{\lambda+2}{2(\lambda+3)}\right\}$, and let $D = \sum_{k=1}^{n+2} b_k(x)\frac{\partial}{\partial x_k} \in C^{\lambda+2}\left(\overline{\Omega}\right)$ be a vector field tangent to $\partial\Omega$ for $x \in \partial\Omega$.*

(i) *Let $1 < s \leq \lambda + 2$. If $\vec{\ell}$ satisfies case* (**I**) *and the $\mathcal{T}_\delta$ condition for some $\delta \geq 0$, and if $u \in \Lambda^{s'}(\Omega)$ for some $s' > 1$ satisfies* (2.27)*, then $u \in \Lambda^s(\Omega)$, and there is C_s such that*

$$\|u\|_{\Lambda^s(\Omega)} \leq C_s\left(\|u\|_{\Lambda^{s-\mu_\infty}(\Omega)} + \|f\|_{\Lambda^{s-2}(\Omega)} + \|g\|_{\Lambda^{s-\delta}(\partial\Omega)} + \|g_1\|_{\Lambda^{s-\frac{1}{4}}(\partial\Omega)}\right). \tag{2.28}$$

(ii) *Let $1 < s \leq \lambda + 2$. If $\vec{\ell}$ satisfies case* (**II**) *and the $\mathcal{T}_\delta$ condition for some $\delta \geq 0$, and if $u \in \Lambda^{s'}(\Omega)$ for some $s' > 1$ satisfies* (2.27)*, then $u \in \Lambda^s(\Omega)$, and there is C_s such that*

$$\|u\|_{\Lambda^s(\Omega)} \leq C_s\left(\|u\|_{\Lambda^{s-\mu_\infty}(\Omega)} + \|f\|_{\Lambda^{s-2}(\Omega)} + \|g\|_{\Lambda^{s-\delta}(\partial\Omega)} + \|g_1\|_{\Lambda^{s-\frac{1}{4}}(\partial\Omega)} + \|h\|_{\Lambda^s(\Gamma)}\right). \tag{2.29}$$

(iii) *Let $\alpha \geq 0, 1 + \frac{1}{p} < s < \lambda + 2 - \alpha$. Suppose that $\vec{\ell}$ satisfies case* (**I**) *above, and in addition satisfies the $\mathcal{A}^{\mp}_{p,\alpha}$ condition on Γ, $\mathcal{A}^{=}_{p,\alpha}$ on $(\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, $\mathcal{A}^{\ddagger}_{p,\alpha}$ on $(\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$*

for some $\alpha \geq 0$ *and* $1 < p < \infty$. *Assume moreover that* $\delta\lambda - \alpha > 0$. *If* $u \in H_p^{s'}(\Omega)$ *for some* $s' > 1 + \frac{1}{p}$ *satisfies* (2.26), *then* $u \in H_p^s(\Omega)$, *and there is* C_s *such that*

$$\|u\|_{H_p^s(\Omega)} \leq C_s \left(\|u\|_{H_p^{s-\mu_p}(\Omega)} + \|f\|_{H_p^{s-2+\sigma}(\Omega)} + \|g\|_{\mathcal{B}_p^{s-\delta-\frac{1}{p},p}(\partial\Omega)} + \|g_1\|_{\mathcal{B}_p^{s-\frac{1}{4}-\frac{1}{p},p}(\partial\Omega)} \right). \tag{2.30}$$

(iv) *Let* $\alpha \geq 0, 1 + \frac{1}{p} < s < \lambda + 2 - \alpha$. *Suppose that* $\vec{\ell}$ *satisfies case* (**II**) *above, and in addition satisfies the* $\mathcal{A}_{p,\alpha}^{\mp}$ *condition on* Γ, $\mathcal{A}_{p,\alpha}^{=}$ *on* $(\Sigma^+ \cap \mathcal{N}) \setminus \Gamma^*$, $\mathcal{A}_{p,\alpha}^{\ddagger}$ *on* $(\Sigma^- \cap \mathcal{N}) \setminus \Gamma^*$ *for some* $\alpha \geq 0$ *and* $1 < p < \infty$, *and also the* P_γ *condition for some* $\gamma \geq 0$. *If* $u \in H_p^{s'}(\Omega)$ *for some* $s' > 1 + \frac{1}{p}$ *satisfies* (2.27), *then* $u \in H_p^s(\Omega)$, *and there is* C_s *such that*

$$\begin{aligned} \|u\|_{H_p^s(\Omega)} \leq C_s \Big(& \|u\|_{H_p^{s-\mu_p}(\Omega)} + \|f\|_{H_p^{s-2+\alpha}(\Omega)} + \|g\|_{\mathcal{B}_p^{s-\delta-\frac{1}{p},p}(\partial\Omega)} \\ & + \|g_1\|_{\mathcal{B}_p^{s-\frac{1}{4}-\frac{1}{p},p}(\partial\Omega)} + \|h\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)} \Big). \end{aligned} \tag{2.31}$$

Proof. Note that

$$\begin{aligned} \widetilde{K}a &= K \circ L_0^{-1} \circ a = K \circ a + K \circ \left[L_0^{-1}, a\right], \\ \left[L_0^{-1}, a\right] &= L_0^{-1} \circ a - aL_0^{-1} = L_0^{-1} \circ a^\sharp - a^\sharp L_0^{-1} + L_0^{-1} \circ a^\flat - a^\flat L_0^{-1} \\ &= \left(L_0^{-1}\right)_\xi \circ \left(a^\sharp\right)_x + \overline{\mathcal{O}}_{(0,\lambda+2)}^{-1-\delta} \in \overline{\mathcal{O}}_{(0,\lambda+2)}^{-1}, \end{aligned}$$

by the improved estimates in Propostion 1.1 in [8]. The Proposition now follows from Proposition 2.2, Theorems 2.1, 2.3, 2.5, 2.6 and Lemmas 1.11, 1.12 in [8], and the fact that the $\mathcal{T}_\delta$ condition implies the $\mathcal{P}_\delta$ condition.

Note that Theorem 2.2 follows from the above proposition.

Proposition 2.4. *If the zero order term* $c(x)$ *in the differential operator* P *is negative* (*respectively nonnegative*) *and* $\mu_p > 0$, *then the solution* u *to* (2.26) (*respectively* (2.27)) *is unique.*

Proof. Suppose u_1 and u_2 are solutions in $H_p^{s'}(\Omega)$ for some $s' > 1+\frac{1}{p}$, and set $u = u_1 - u_2$. Then u satisfies (2.26) or (2.27) with $f = g = h = 0$. Let $1 < s < \min\{\lambda + 2, s' + \mu_p\}$. From the a priori estimates in Proposition 2.3 we have $u \in H_p^s(\Omega)$. By the Sobolev embedding theorem, we now obtain $u \in H_q^{s-\mu_p}(\Omega)$ for $\frac{1}{q} \geq \frac{1}{p} - \frac{\mu_p}{n}$. Applying the a priori estimates in Proposition 2.3 again, but for q in place of p, we conclude that $u \in H_q^s(\Omega)$. Alternating applications of the Sobolev embedding theorem and the a priori estimates a finite number of times leads to $u \in \Lambda^s(\Omega)$. Since $s > 1$, the standard argument using the Hopf lemma as in [7] shows that $u \equiv 0$, so $u_1 = u_2$.

Proposition 2.5. *Suppose* $s > 1 + \frac{1}{p}$ *and that* $P = \sum_{i,j}^{n+2} a_{ij}\partial_i\partial_j + \sum_{k}^{n+2} b_k\partial_k + c$ *where* $c(x) \leq c_0 < 0$ *for some constant* c_0. *Let* $\|P\|_{\Lambda^\lambda} \equiv \sum_{i,j}^{n+2} \|a_{ij}\|_{\Lambda^\lambda} + \sum_{k}^{n+2} \|b_k\|_{\Lambda^\lambda} + \|c\|_{\Lambda^\lambda}$ *and suppose that*

$$\sum_{i,j=1}^{n+2} a_{ij}\xi_i\xi_j \geq c_1 |\xi|^2 \quad \text{for all } \xi \in R^{n+2}.$$

If u *satisfies* (2.26) *or* (2.27), *then the terms*

$$\|u\|_{\Lambda^{s-\mu_\infty}(\Omega)} \quad \text{and} \quad \|u\|_{H_p^{s-\mu_p}(\Omega)}$$

can be omitted from the inequalities (2.28) *to* (2.31) *in Proposition* 2.3 *if we replace* C_s *by* $C\left(c_0, c_1, \|P\|_{\Lambda^\lambda}, \Omega, \overrightarrow{\ell}\right)$.

Proof. If for example

$$\|u\|_{H_p^s(\Omega)} \le C\left(\|f\|_{H_p^{s-2+\alpha}(\Omega)} + \|g\|_{\mathcal{B}_p^{s-\delta-\frac{1}{p},p}(\partial\Omega)} + \|h\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)}\right) \tag{2.32}$$

fails, then there are sequences $\{P_k\}$, $\{u_k\}$, $\{f_k\}$, $\{g_k\}$, and $\{h_k\}$ such that

$$\|P_k\|_{\Lambda^\lambda(\Omega)} \le C' \text{ and } P_k \text{ is uniformly elliptic for all } k,$$
$$\|u_k\|_{H_p^{s-\mu_p}(\Omega)} = 1 \text{ for all } k,$$
$$\|f_k\|_{H_p^{s-2+\alpha}(\Omega)} + \|g_k\|_{\mathcal{B}_p^{s-\delta-\frac{1}{p},p}(\partial\Omega)} + \|h_k\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)} \to 0 \text{ as } k \to \infty.$$

From (2.31) we obtain that $\|u_k\|_{H_p^s(\Omega)} \le C$ for all k, and so by compactness there is a subsequence, which we continue to denote by $\{u_k\}$, such that u_k converges in $H_p^{s-\epsilon}(\Omega)$ to some function $u \in H_p^{s-\epsilon}(\Omega)$. Also, there is an elliptic operator P such that $P_k \to P$ in $\Lambda^{\lambda-\epsilon}$. However since $P_k \in \Lambda^\lambda$, a straightforward computation with a difference operator shows that $P \in \Lambda^\lambda$. Now $P_k u_k \to Pu$ in $H_p^{s-\epsilon-2}(\Omega)$, $\vec{\ell} u_k\mid_{\partial\Omega} \to \vec{\ell} u\mid_{\partial\Omega}$ in $\mathcal{B}_p^{s-\epsilon-1-\frac{1}{p},p}(\partial\Omega)$ and $u_k\mid_\Gamma \to u\mid_\Gamma$ in $\mathcal{B}_p^{s-\epsilon-\frac{2}{p},p}(\Gamma)$. Thus u satisfies (1.6) with $f = g = g_{jk} = h \equiv 0$ and $\|u\|_{H_p^{s-\mu_p}(\Omega)} = 1$, contradicting Proposition 2.4. Thus (2.32) holds and the proof is complete.

Now we can give the proof of Theorems 1.1 and 1.3.

Proof (of Theorems 1.1 and 1.3). The proof of Theorem 1.1 is similar to the proof of part (A) of Theorem 1.3 given below. First suppose that $P(x,0) < 0$. Fix $1 + \frac{1}{p} < s < \lambda + 2 - \alpha$ and $f \in H_p^{s-2+\alpha}(\Omega)$, $g \in \mathcal{B}_p^{s-\delta-\frac{1}{p},p}(\partial\Omega)$ and $h \in \mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)$ (again, we consider only case (**II**)). Choose $f_k \in \mathcal{C}^\infty(\Omega)$, $g_k \in \mathcal{C}^\infty(\partial\Omega)$, $h_k \in \mathcal{C}^\infty(\Gamma)$ and $P_k \in \mathcal{C}^\infty(\Omega)$ such that

$$\|f - f_k\|_{H_p^{s-2+\alpha}(\Omega)} + \|g - g_k\|_{\mathcal{B}_p^{s-\delta-\frac{1}{p},p}(\partial\Omega)} + \|h - h_k\|_{\mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma)} \to 0 \text{ as } k \to \infty,$$
$$\|P - P_k\|_{\Lambda^\lambda(\Omega)} \to 0 \text{ as } k \to \infty.$$

Then by a theorem in [15], there is $u_k \in \Lambda^{\lambda+1}(\Omega)$ satisfying

$$\begin{cases} P_k u_k = f_k & \text{in } \Omega, \\ \overrightarrow{\ell} u_k = g_k & \text{on } \partial\Omega, \\ u_k = h_k & \text{on } \Gamma. \end{cases}$$

But the two earlier propositions, together with the a priori estimates in Proposition 2.3, show that

$$\|u_k\|_{H_p^s(\Omega)} \le C, \quad \text{for all } k. \tag{2.33}$$

Thus given $\epsilon > 0$ with $1 + \frac{1}{p} < s - \epsilon$, there is a subsequence, which we continue to denote by $\{u_k\}$, that converges to u in $H_p^{s-\epsilon}(\Omega)$. It follows upon letting $k \to \infty$ that

$$\begin{cases} Pu = f & \text{in } \Omega, \\ \overrightarrow{\ell} u = g & \text{on } \partial\Omega, \\ u = h & \text{on } \Gamma. \end{cases}$$

Finally, by (2.33), we obtain $\|u\|_{H_p^s(\Omega)} \le C$ also.

For general P, choose a constant c sufficiently negative that the coefficient of the zero order term of $P+c<0$. By the result just proved, we can solve the problem

$$\begin{cases}(P+c)\,u=f & \text{in } \Omega,\\ \vec{\ell}\,u=g & \text{on } \partial\Omega,\\ u=h & \text{on } \Gamma\end{cases}$$

for $u\in H_p^s(\Omega)$ given that

$$f\in H_p^{s-2+\alpha}(\Omega),\quad g\in \mathcal{B}_p^{s-\delta-\frac{1}{p},p}(\partial\Omega)\ \text{ and }\ h\in \mathcal{B}_p^{s-\frac{\gamma}{p}-\frac{1}{p},p}(\Gamma).$$

But then $Pu=f-cu$ where cu is smoother than f and functional analysis now completes the proof.

Finally, we consider the necessary assertions in part (B) of Theorem 1.3 . If for some $\epsilon>0$ there is, for every $f\in H_p^{s-2+\alpha}(\Omega)$, a function $u\in H_p^s(\Omega)$ with $Pu-f\in H_p^{s+\epsilon}(\Omega)$,

$$\left(\vec{\ell}+b\right)u\,|_{\partial\Omega}\equiv g\in \mathcal{B}_p^{s-\frac{1}{p},p}(\partial\Omega)$$

and

$$u\,|_{\Gamma}\equiv h\in \mathcal{B}_p^{s-\frac{1}{p},p}(\partial\Omega),$$

then for any $c\in R$, we have

$$\begin{cases}(P+c)\,u=f+cu+(Pu-f) & \text{in } \Omega,\\ \vec{\ell}\,u=g & \text{on } \partial\Omega,\\ u=h & \text{on } \Gamma.\end{cases}$$

Now choose c sufficiently negative and let $\mathbf{N}$ denote the Neumann operator for $P+c$. Then by Proposition 2.2,

$$\rho T\circ L_0^{-1}\left(\widetilde{\rho}\mathbf{N}\left(f+cu+(Pu-f)\right)\right)\in \mathcal{B}_p^{s-\frac{1}{p},p}(\partial\Omega),$$

since the remaining terms in (2.24) are automatically in this Besov space. Since T maps $\mathcal{B}_p^{t,p}(\partial\Omega)$ to $\mathcal{B}_p^{t-\frac{1}{(3+\lambda)p},p}(\partial\Omega)$ by Theorem 2.1 in [8], we conclude that

$$\rho T\circ L_0^{-1}\left(\widetilde{\rho}\mathbf{N}f\right)\in \mathcal{B}_p^{s+\epsilon-\frac{1}{(3+\lambda)p}-\frac{1}{p},p}(\partial\Omega)$$

if $\epsilon\le\frac{1}{(3+\lambda)p}-\alpha$ since both u and $Pu-f$ are smoother than f by order ϵ. Thus, since the Neumann operator is surjective (for c sufficiently negative), and L_0^{-1} is elliptic on the cone $\{|\tau|<\lambda|\xi|\}$, we see that $\rho\varphi\circ T\circ\varphi\widetilde{\rho}$ is bounded from $\mathcal{B}_p^{s-\frac{1}{p},p}(U)$ to $\mathcal{B}_p^{s+\epsilon-\frac{1}{(3+\lambda)p}-\frac{1}{p},p}(\partial\Omega)$ if $\epsilon\le\frac{1}{(3+\lambda)p}-\alpha$. So by Theorem 2.1 and Lemma 1.11 in [8],

$$\begin{aligned}\vec{\ell}\ \text{satisfies}\ \mathcal{A}_{p,\beta}^{\mp}\ &\text{ on } \Gamma,\\ \mathcal{A}_{p,\beta}^{=}\ &\text{ on } \left(\Sigma^+\cap\mathcal{N}\right)\setminus\Gamma^*,\\ \mathcal{A}_{p,\beta}^{\ddagger}\ &\text{ on } \left(\Sigma^-\cap\mathcal{N}\right)\setminus\Gamma^*,\end{aligned}\tag{2.34}$$

with $\beta=\frac{1}{(3+\lambda)p}-\epsilon$. Now repeat this argument to obtain that $\vec{\ell}$ satisfies (2.34) with $\beta=\frac{1}{(3+\lambda)p}-2\epsilon$ (provided of course that $\beta\le\alpha$). In this way we finally see that $\vec{\ell}$ satisfies (2.34) with $\beta=\alpha$. This completes the proof of part (B)(i) of Theorem 1.3. The proofs of the two other assertions in part (B) are similar if we make use of Theorems 2.3 and 2.5 in [8] in place of Theorem 2.1. We do not repeat the details.

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